GDDs with two associate classes and with one group of size 1 and m groups of size n and $\lambda_1 = 3, \lambda_2 = 1$

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Abstract

A group divisible design GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, \lambda_1, \lambda_2)$ is an ordered pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is an $(1 + mn)$-set of symbols and $\mathcal{B}$ is a collection of 3-subsets (called blocks) of $\mathcal{V}$ satisfying the following properties: the $(1 + mn)$-set is divided into 1 group of size 1 and m groups of size n; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$. In this paper we find necessary and sufficient conditions for the existence of a GDD$(v = 1 + n+n+...+n, 1+m, 3, 3, 1$, where $m,n \geq 3$.

Keywords: Group divisible designs Graph decomposition

Introduction

A group divisible design or GDD$\left(v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2\right)$, is an ordered triple $(\mathcal{V}, \mathfrak{G}, \mathcal{B})$, where $\mathcal{V}$ is a v-set of symbols, $\mathfrak{G}$ is a partition of $\mathcal{V}$ into g sets of size $v_1, v_2, \ldots, v_g$, and $\mathcal{B}$ is a collection of k-subsets (called blocks) of $\mathcal{V}$ such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks. Elements occurring together in the same group are called *first associates*, and elements occurring in different groups are called *second associates*. It is clear that if the indices λ_1 and λ_2 are equal, then the design is a BIBD. The existence problem of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs (Bose & Shimamoto, 1952). More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (Colbourn & Dinitz, 2007), and the designs here are called partially balanced incomplete block designs} (PBIBDs) of group divisible type in (Fu, Rodger, & Sarvate, 2000). The existence question for $k = 3$ has been solved by Sarvate, Fu and Rodger (Fu & Rodger, 1998; Fu, Rodger & Sarvate, 2000) when all groups are the same size. Recently, the necessary and sufficient conditions for the existence of a GDD $(v = 1 + n + n, 3, 3, \lambda_1, \lambda_2)$, were found by Lapchinda, Punnim and Pabhapote (Lapchinda et al. 2013; Lapchinda et al. 2014).

In this paper, we consider the problem of determining necessary conditions for the existence of a GDD$(v =$ $1+n+n+...+n, 1+m, 3, 3, 1$, and prove that the conditions are sufficient. We will see that necessary conditions for the existence of a GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, \lambda_1, \lambda_2)$ can be easily obtained by describing it graphically as follows.

Let G and H be multigraphs. A *G-decomposition* of H is a partition of the edges of H such that each element of the partition induces a copy of G. We write G|H if there exists a G-decomposition of H.

Let λk_v denote the multigraph on v vertices in which each pair of distinct vertices is joined by λ edges. Let G_1 and G_2 be vertex disjoint graphs. Then $G_1 \vee_\lambda G_2$ is the graph obtained from the union of G_1 and G_2 and by joining each vertex in G_1 to each vertex in G_2 with λ edges. Thus the existence of a GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, \lambda 1, \lambda 2$ is easily seen to be equivalent to the existence of a k-decomposition of $\lambda_1 K_1 \vee_{\lambda_2} \lambda_1 K_n \vee_{\lambda_2} \dots \vee_{\lambda_2} \lambda_1 K_n$($m$ copies of K_n)

The graph $\lambda_1 K_1 \vee_{\lambda_2} \lambda_1 K_n \vee_{\lambda_2} \dots \vee_{\lambda_2} \lambda_1 K_n$ is of order $1 + mn$ and size $\lambda_1 m \binom{n}{2} + \lambda_2 \left[mn + \binom{m}{2} n^2\right]$.

It contains 1 vertex of degree $\lambda_2 mn$ and mn vertices of degree $\lambda_1(n-1) + \lambda_2[1 + (m-1)n]$. Thus the existence of a K_n-decomposition of $\lambda_1 K_1 \vee_{\lambda_2} \lambda_1 K_n \vee_{\lambda_2} \dots \vee_{\lambda_2} \lambda_1 K_n$ implies;

$$3 \,\Big|\, \left\{\lambda_1 m \binom{n}{2} + \lambda_2 \left[mn + \binom{m}{2} n^2\right]\right\} \tag{1}$$

$$2 | \lambda_2 mn \text{ and } 2 | \{\lambda_1(n-1) + \lambda_2[1 + (m-1)n]\} \tag{2}$$

Preliminary Results

In this section, we will review some known results concerning triple designs that will be used in the sequel, most of which are taken from (Lindner & Rodger, 2009).

A balanced incomplete block design BIBD(v, b, r, k, λ) is a set S of v elements together with a collection of v k-subset of S, called *blocks*, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks.

Note that in a BIBD(v, b, r, k, λ), the parameters must satisfy the necessary conditions;

$$vr = bk$$
$$\lambda(v-1) = r(k-1)$$

With these conditions a BIBD(v, b, r, k, λ) is usually written as BIBD(v, k, λ).

The following results on the existence of λ-fold triple systems are well known (Lindner & Rodger, 2009).

Theorem 2.1 Let n be a positive integer. Then a BIBD$(v, 3, \lambda)$ exists if and only if λ and n are in one of the following cases:

(a) $\lambda \equiv 0$ (mod 6) and $n \neq 2$,
(b) $\lambda \equiv 1$ or 5 (mod 6) and $n \equiv 1$ or 3 (mod 6),
(c) $\lambda \equiv 2$ or 4 (mod 6) and $n \equiv 0$ or 1 (mod 3),
(d) $\lambda \equiv 3$ (mod 6) and n is odd.

The following Results are found in handbook of combinatorial designs (Colbourn & Dinitz, 2007, p. 255).

Theorem 2.2 The necessary and sufficient conditions for the existence of a GDD$(v = n + n + \cdots + n, m, 3, 0, \lambda$ are:

1. $m \geq 3$
2. $\lambda(m-1)n \equiv 0$ (mod 2)
3. $\lambda m(m-1)n^2 \equiv 0$ (mod 6)

Theorem 2.3 The necessary and sufficient conditions for the existence of a GDD$(v = n + n + \cdots + n, m, 3, \lambda 1, \lambda 2$ are:,

1. $\lambda_1(n-1) + \lambda_2(m-1)n \equiv 0$ (mod 2), and
2. $\lambda_1 mn(n-1) + \lambda_2 m(m-1)n^2 \equiv 0$ (mod 3)

Theorem 2.4 Let m, n and t be nonnegative integers. There exists a GDD$(v = t + n + \cdots + n, 1 + m, 3, 0, 1$ if and only if the following conditions are satisfied:,

1. If $n > 0$, then $m \geq 3$ or $m = 2$ and $t = n$ or $m = 1$ and $t = 0$ or $m = 0$,
2. $t \leq n(m-1)$ or $mn = 0,$
3. $n(m-1) + t \equiv 0$ (mod 2) or $mn = 0$
4. $mn \equiv 0$ (mod 2) or $t = 0$
5. $\frac{1}{2}n^2 m(m-1) + mnt \equiv 0$ (mod 3)

The following notations will be used throughout the paper for our constructions.

1. Let $\mathcal{V}$ be a v-set. We use $K(\mathcal{V})$ for the complete graph K_v on the vertex set $\mathcal{V}$.
2. Let $\mathcal{V}$ be a v-set. BIBD$(\mathcal{V}, 3, \lambda)$ can be defined as
   BIBD$(\mathcal{V}, 3, \lambda)$ = { $\mathcal{B}$ | $(\boldsymbol{\mathcal{V}}, \mathcal{B})$ is a BIBD$(v, 3, \lambda)$}.
3. We denote (X,Y,Z;$\mathcal{B}$) for a GDD$(v = n_1 + n_2 + n_3, 3, 3, \lambda_1, \lambda_2)$ if X, Y, and Z are n_1-set, n_2-set, and n_3-set, respectively.
4. We denote (X,Y_1, Y_2, …, Y_m;$\mathcal{B}$) for a GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, \lambda_1, \lambda_2)$ if | Y_i|=n.
5. Let X, Y and Z be disjoint sets of cardinality n_1, n_2 and n_3, respectively. We define GDD(X, Y, Z; λ_1, λ_2) as
   GDD(X, Y, Z; λ_1, λ_2) ={ $\mathcal{B}$ | (X, Y, Z; $\mathcal{B}$) is a GDD$(v = n_1 + n_2 + n3, 3, 3, \lambda 1, \lambda 2$ }
6. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v -set $\boldsymbol{\mathcal{V}}$, $\mathcal{B}$ may contain repeated blocks. Thus ``$\bigcup$" in our construction will be used for the union of multisets.

Result

By (1) and (2) we have the following theorem.

Theorem 3.1 Let m, n be positive integers and $m \geq 3$. If a GDD$(v = 1 + n + \cdots + n, 1 + m, 3, 3, 1)$ exists, then

1. $m \equiv 0, 2, 3$ or 5 (mod 6) and $n \equiv 0$ or 4 (mod 6),
2. $m \equiv 0$ (mod 6) and $n \equiv 1$ or 3 (mod 6),
3. $m \equiv 0$ or 3 (mod 6) and $n \equiv 2$ (mod 6),
4. $m \equiv 0$ (mod 6) and $n \equiv 5$ (mod 6),
5. $m \equiv 1$ (mod 6) and $n \equiv 0$ (mod 6),
6. $m \equiv 4$ (mod 6) and $n \equiv 0$ or 3 (mod 6).

We will prove that the necessary conditions in Theorem 3.1 are sufficient.

Let $m \equiv 0, 2, 3$ or 5 (mod 6) and $n \equiv 0$ or 4 (mod 6) Let X be a singleton set and $Y_1, Y_2, \ldots, Y_m$ be n-sets. Since $mn + 1 \equiv 1$ or 3 (mod 6), by Theorem 2.1(b), there exists a BIBD$(1 + mn, 3,1)$ Let (X $\cup \bigcup_1^m Y_i$, $\mathcal{B}_0$) be a BIBD(X $\cup \bigcup_1^m Y_i$, $\mathcal{B}$, 3, 1). Since $n \equiv 0$ or 4 (mod 6). By Theorem 2.1(c), there exists a BIBD$(n, 3,1)$. Let (,Y_i $\mathcal{B}_i$) be a BIBD(Y_i, 3, 1) where $i = 1, 2, \ldots, m$. Put $\mathcal{B}$ = $\mathcal{B}_0 \cup (\bigcup_1^m \mathcal{B}_i)$ forms a GDD$(v = 1 + n +$ *…+n, 1+m, 3, 3, 1*. Thus we have the following lemma

Lemma 3.1 If $m \equiv 0, 2, 3$ or 5 (mod 6) and $n \equiv 0$ or 4 (mod 6), then there exists a GDD$(v = 1 + n +$ *…+n, 1+m, 3, 3, 1*.

Using a similar construction as above, it is true for the case when $m \equiv 0$ (mod 6) and $n \equiv 1$ or 3 (mod 6), $m \equiv 1$ (mod 6) and $n \equiv 0$ and $m \equiv 4$ (mod 6) and $n \equiv 0$ or 3 (mod 6) Thus we have the following lemmas.

Lemma 3.2 If $m \equiv 0$ (mod 6) and $n \equiv 1$ or 3 (mod 6), then there exists a GDD$(v = 1 + n + \cdots +$ *n, 1+m, 3, 3, 1*.

Lemma 3.3 If $m \equiv 1$ (mod 6) and $n \equiv 0$ (mod 6), then there exists a GDD$(v = 1 + n + \cdots + n, 1 +$ *m, 3, 3, 1*.

Lemma 3.4 If $m \equiv 4$ (mod 6) and $n \equiv 0$ or 3(mod 6), then there exists a GDD$(v = 1 + n + \cdots + n, 1 +$ *m, 3, 3, 1*.

Let $m \equiv 0$ or 3 (mod 6) and $n \equiv 2$ (mod 6). Let X be a singleton set and $Y_1, Y_2, \ldots, Y_m$ be n-sets. Since $1 + n \equiv 3$ (mod 6), by Theorem 2.1(b), there exists a BIBD$(1 + n, 3,1)$ Let (X $\cup$,Y_i, $\mathcal{B}_i$) be a BIBD(X $\cup$ Y_i, 3, 1). Since $m \equiv 0$ or 3 (mod 6) and $n \equiv 2$ (mod 6). By Theorem 1.2, there exists a GDD$(v = n + \cdots +$ *n, m, 3, 2, 1*.. Let ($_i$ $Y_1, Y_2, \ldots, Y_m$ $\mathcal{B}0$) be a GDD *v=n+n+…+n, m, 3, 2, 1* where *i=1, 2, …, m*. Put $\mathcal{B}$ = $\mathcal{B}_0 \cup (\bigcup_1^m \mathcal{B}_i)$ forms a GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, 3, 1)$.

Thus we have the following lemma

Lemma 3.5 If $m \equiv 0$ or 3 (mod 6) and $n \equiv 2$ (mod 6), then there exists a GDD$(v = 1 + n + \cdots + n, 1 +$ *m, 3, 3, 1*.

Let $m \equiv 0$ (mod 6) and $n \equiv 5$ (mod 6) . Let X be a singleton set and $Y_1, Y_2, \ldots, Y_m$ be n-sets. Since $n \equiv 5$ (mod 6), by Theorem 2.1(d), there exists a BIBD$(n, 3{,}3)$ Let (Y_i, $\mathcal{B}_i$) be a BIBD(Y_i, 3, 3). Since $m \equiv 0$ (mod 6) and $n \equiv 5$ (mod 6). By Theorem 2.4, there exists a GDD$(v = 1 + n + \cdots + n, 1 + m, 3, 0, 1)$.. Let (X $Y_1, Y_2, \ldots, Y_m$ $\mathcal{B}_0$) be a GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, 0, 1)$ where $i = 1, 2, \ldots, m$. Put $\mathcal{B}$ = $\mathcal{B}_0 \cup (\bigcup_1^m \mathcal{B}_i)$ forms a GDD$(v = 1 + n + n + \cdots + n, 1 + m, 3, 3, 1)$.

Thus we have the following lemma

Lemma 3.6 If $m \equiv 0$ (mod 6) and $n \equiv 5$ (mod 6) , then there exists a GDD$(v = 1 + n + \cdots + n, 1 + m, 3, 3, 1$.

Combining Theorem 3.1 and Lemmas 3.1-3.6, we have finally the following theorem.

Theorem 3.2 Let m, n be positive integers and $m, n \geq 3$. A GDD$(v = 1 + n + .. + n, 1 + m, 3, 3, 1)$ exists, if and only if

1. $m \equiv 0, 2, 3$ or 5 (mod 6) and $n \equiv 0$ or 4 (mod 6),
2. $m \equiv 0$ (mod 6) and $n \equiv 1$ or 3 (mod 6),
3. $m \equiv 0, 2$ or 3 (mod 6) and $n \equiv 2$ (mod 6),
4. $m \equiv 0$ (mod 6) and $n \equiv 5$ (mod 6),
5. $m \equiv 1$ (mod 6) and $n \equiv 0$ (mod 6),
6. $m \equiv 4$ (mod 6) and $n \equiv 0$ or 3 (mod 6).

We have already found necessary and sufficient conditions for the existence of a GDD$(v = 1 + n + n + \ldots + n, 1 + m, 3, 3, 1$, where $m, n \geq 3$ by Theorem 3.2.

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