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Prediction of a Semi-Exact Analytic Solution of a Convective Porous Fin with Variable Cross Section by Different Methods

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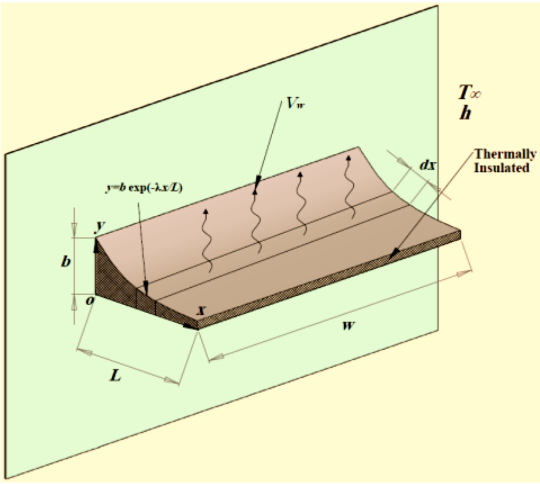
Abstract

In the present study, the problem of nonlinear equations arising in a convective porous fin with a variable cross section is investigated using a Collocation Method (CM). The obtained results from this method are compared with the Homotopy Perturbation Method (HPM), Variation Iteration Method (VIM), and those from a numerical solution, namely the Boundary Value Problem method (BVP), to verify the accuracy of the proposed method. It is found that the CM can achieve suitable results in predicting the solutions of such problems.

Keywords: Porous fin, variable cross section, Collocation Method, Homotopy Perturbation Method, Variation Iteration Method

Introduction

Heat transfer rate enhancement in fins, with reductions in size and cost, is the aim of many researchers in engineering applications. To achieve this goal, convective heat transfer coefficient, surface area available, and temperature difference between surface and surrounding fluid are ways that can be used. Most problems and scientific phenomena, such as heat transference, are inherently ones of nonlinearity. In most cases, these problems do not admit analytical solution, so the associated equations should be solved using special techniques. In recent years, much attention has been devoted to newly developed methods to construct an analytic solution of the equations; such methods include the Perturbation techniques. Perturbation techniques are too strongly dependent upon so-called ‘‘small parameters’’ [1]. Many other different methods have been introduced to solve nonlinear equations, such as the homotopy perturbation [2-9] and homotopy analysis methods [10-13], the variational iteration method [14-19], and the Exp-Function method [20]. Recently, a new analytical technique called the homotopy perturbation sumudu transform method was presented in [21-24]. In this paper, analytical solution of nonlinear equations arising in fin problems [25] has been studied by the Collocation Method (CM). Obtaining the analytical solutions of the models and comparing with the Homotopy Perturbation Method (HPM), Variation Iteration Method (VIM), and numerical results reveal the capability, effectiveness and convenience of CM. This method gives successive approximations of highly accurate solutions.



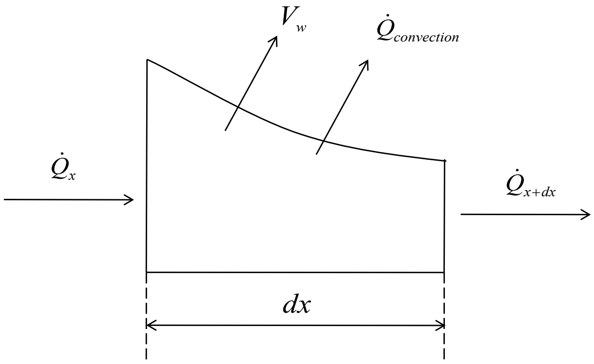


Figure 1 Schematic diagram of porous fin under investigation.

Figure 2 Control volume for thermal analysis.

Materials and methods

We consider a longitudinal porous fin with an exponential function profile, as shown in **Figure 1**, which extends into a fluid of temperature T_∞ and where the base is maintained at a constant temperature T_b. Let the fin length be L, width W, and its thicknesses at the base b. This fin is porous, to allow the flow of infiltrate through it [26]. With the assumption of one-dimensional heat conduction along the fin, steady-state operation, and also considering the fact that the porous medium is isotropic and saturated with single-phase fluid, an energy balance applied to a differential element according to **Figure 2** yields;

$$\dot{Q}_x = \dot{Q}_{x+dx} + \dot{Q}_{convection} + \dot{m}C_p(T - T_\infty) \tag{1}$$

where $\dot{m}$ accounts for the mass flow rate of the fluid passing through the porous material, and can be written as;

$$\dot{m} = \rho W V_w dx \tag{2}$$

From the Darcy model we have;

$$V_w = \frac{gK\beta}{\upsilon}(T - T_\infty) \tag{3}$$

With the use of standard expressions for conduction, convection, and the energy balance, which can be written as;

$$\frac{d}{dx}(k_{eff}A(x)\frac{dT}{dx})-hW(T-T_{\infty})-\frac{\rho gK\beta C_pW}{\upsilon}(T-T_{\infty})^2=0 \tag{4}$$

The fin profile is defined according to variation of the fin thickness along its extended length. For example, the cross section area of the fin may vary as;

$$A(x)=Wt(x) \tag{5}$$

Where W is the width of the fin, and $t(x)$ is the fin thickness along the length. The $t(x)$ for this profile can be defined as follows;

$$t(x)=be^{-\lambda\frac{x}{L}} \tag{6}$$

By introducing $X=\frac{x}{L},\theta=\frac{T-T_{\infty}}{T_b-T_{\infty}}$, and with some manipulation, we have;

$$\frac{d^2\theta}{dx^2}-\lambda\frac{d\theta}{dx}-e^{\lambda X}(S_h\theta^2+m^2\theta)=0 \tag{7}$$

Where, $S_h=\frac{D_axRa}{k_r}\left(\frac{L}{b}\right)^2$ is the porous parameter and $m=\left(\frac{hL^2}{k_{eff}b}\right)^{1/2}$ is the convective parameter of the fin. Note that the temperature at the base of the fin is uniform, T_b, and that there is no heat transfer from the tip of the fin; boundary conditions for Eq. (7) can be written as;

$$\theta(1)=1 \tag{8a}$$

And

$$\theta'(0)=0 \tag{8b}$$

Application of collocation method

Suppose we have a differential operator D acting on a function u to produce a function p;

$$D(u(x))=p(x) \tag{9}$$

We wish to approximate u by a function of $\tilde{u}$, which is a linear combination of basic functions chosen from a linearly independent set. That is;

$$u \cong \tilde{u} = \sum_{i=0}^{n} c_i \varphi_i \tag{10}$$

Now, when substituted into the differential operator, D, the result of the operations is not, in general, $p(x)$. Hence, an error or residual will exist;

$$E(x) = R(x) = D(\tilde{u}(x)) - p(x) \neq 0 \tag{11}$$

The notion in the collocation is to force the residual to zero in some average sense over the domain. That is;

$$\int_x R(x) W_i(x) = 0 \qquad i = 0, 1, 2......n \tag{12}$$

Where the number of weight functions W_i are exactly equal to the number of unknown constants c_i in $\tilde{u}$. The result is a set of n algebraic equations for the unknown constant c_i. For the collocation method, the weighting functions are taken from the family of Dirac δ functions in the domain, that is, $W_i(x) = \delta(x - x_i)$. The Dirac δ function property means that;

$$\delta(x - x_i) = \begin{cases} 1 & If \;\; x = x_i \\ 0 & If \;\; x \neq x_i \end{cases} \tag{13}$$

And the residual function in Eq. (11) must be forced to be zero at specific points. Accordingly, consider the trial function as;

$$\begin{aligned} \theta(X) = 1 &+ c_1\left(1 - X^2\right) + c_2\left(1 - X^3\right) + c_3\left(1 - X^4\right) + c_4\left(1 - X^5\right) \\ &+ c_5\left(1 - X^6\right) + c_6\left(1 - X^7\right) \end{aligned} \tag{14}$$

Which satisfies the boundary condition in Eqs. (8a) and (8b), and set it into Eq. (11). The residual function, $R(c_1, c_2, c_3, c_4, c_5, c_6, X)$, is found as;

$$\begin{aligned} &R(c_1, c_2, c_3, c_4, c_5, c_6, X) = -e^{\lambda X} m^2 c_3 + 4\lambda c_3 X^3 + 6\lambda c_5 X^5 + 7\lambda c_6 X^6 \\ &+5\lambda c_4 X^4 + 3\lambda c_2 X^2 + 2e^{\lambda X} S_h c_4 X^5 c_6 - 2e^{\lambda X} S_h c_4 X^{12} c_6 + \cdots + 2e^{\lambda X} S_h c_5 c_6 X^7 \\ &+2e^{\lambda X} S_h c_5 X^6 c_6 - 2e^{\lambda X} S_h c_5 X^{13} c_6 - 2e^{\lambda X} S_h c_4 X^{11} c_5 - 2e^{\lambda X} S_h c_3 X^{11} c_6 \\ &+2e^{\lambda X} S_h c_4 c_5 X^6 + 2e^{\lambda X} S_h c_4 c_6 X^7 + 2e^{\lambda X} S_h c_4 X^5 c_5 - 2c_1 = 0 \end{aligned} \tag{15}$$

This residual vanishes only with the exact solution for the problem. Now, the problem of finding the approximate solution of the problem in the interval $0 < X < 1$ becomes one of adjusting the values of

$c_1, c_2, \ldots$ so that the residual stays close to zero throughout the interval $0 < X < 1$. The basic assumption is that the residual does not deviate much from zero between the collocation locations.

$$R(\frac{1}{7}),\ R(\frac{2}{7}),\ R(\frac{3}{7}),\ R(\frac{4}{7}),\ R(\frac{5}{7}),\ R(\frac{6}{7}) \tag{16}$$

$$\begin{aligned} R(\frac{1}{7}) = e^{0.14285714 29\lambda} \Big(& -0.9999975 S_h c_6^2 - 0.99999150 m^2 c_5 - 0.99994050 m^2 c_4 \\ & -0.999998785 m^2 c_6 - 0.99916718 S_h c_3^2 - 1.994169096 S_h c_2 - 0.999983 S_h c_5^2 \\ & -1.95918367 S_h c_1 + \cdots - 1.9999975 S_h c_6 - 0.9970845481 m^2 c_2 - 0.9596001 S_h c_1^2 \\ & -1.9999830 S_h c_5 - 0.994177550 S_h c_2^2 - 0.99958332 m^2 c_3 - 0.99988145 S_h c_4^2 \Big), \end{aligned} \tag{17}$$

$$\begin{aligned} R(\frac{2}{7}) = e^{0.2857142 85\lambda} \Big(& -0.9996891752 S_h c_6^2 - 0.996195614 S_h c_4^2 - 1.99968914021 S_h c_6 \\ & -0.993336110 m^2 c_3 - 1.9961920631 S_h c_4 - 1.9989120181 S_h c_5 - 0.9867166270 S_h c_3^2 \\ & -0.998912319 S_h c_5^2 + \cdots - 0.953896760 S_h c_2^2 - 1.98667222 S_h c_3 - 0.999844574 m^2 c_6 \\ & -0.84339858 S_h c_1^2 - 1.836449218 S c_1 c_6 - 1.940335842 S c_2 c_3 - 1.949633647 S c_2 c_4 \Big) \end{aligned} \tag{18}$$

$$\begin{aligned} R(\frac{3}{7}) = e^{0.4285714286\lambda} \Big(& -0.9855417380 m^2 c_4 - 0.921282790 m^2 c_2 - 0.6663894100 S_h c_1^2 \\ & -0.9973444010 m^2 c_6 - 0.8163265306 m^2 c_1 - 0.9662640566 m^2 c_3 - 1.994688802 S_h c_6 \\ & -1.932528113 S_h c_3 + \cdots - 1.971083477 S_h c_4 - 0.971292518 S_h c_4^2 - 0.987645599 S_h c_5^2 \\ & -0.9938036022 m^2 c_5 - 0.9946958541 S_h c_6^2 - 1.842565598 S_h c_2 - 0.8487619954 S_h c_2^2 \Big) \end{aligned} \tag{19}$$

$$\begin{aligned} R(\frac{4}{7}) = e^{0.5714285714\lambda} \Big(& -0.9801054711 m^2 c_6 - 0.9651845745 m^2 c_5 - 0.4535610162 S_h c_1^2 \\ & -0.6734693878 m^2 c_1 - 1.626822157 S_h c_2 - 0.9606067345 S_h c_6^2 - 0.8818581093 S_h c_4^2 \\ & -0.8134110787 m^2 c_2 + \cdots - 1.87814601 S_h c_4 - 1.93036914 S_h c_5 - 0.798123820 S_h c_3^2 \\ & -0.9315812628 S_h c_5^2 - 0.8933777593 m^2 c_3 - 0.6616375830 S_h c_2^2 - 1.786755519 S_h c_3 \Big) \end{aligned} \tag{20}$$

$$\begin{aligned} R(\frac{5}{7}) = e^{0.7142857143\lambda} \Big(& -0.8192702622 S_h c_6^2 - 1.2711370260 S_h c_2 - 0.4897959184 m^2 c_1 \\ & -0.2399000416 S_h c_1^2 - 1.734379383 S_h c_5 - 1.8102709880 S_h c_6 - 0.90513549380 m^2 c_6 \\ & -0.6627027489 S_h c_4^2 + \cdots - 1.628131136 S_h c_4 - 0.5471439517 S c_3^2 - 1.47938359 S_h c_3 \\ & -0.7520179608 S_h c_5^2 - 1.S - 1.m^2 - 0.4039473349 S_h c_2^2 - 0.9402496287 S_h c_2 c_3 \Big) \end{aligned} \tag{21}$$

$$R(\frac{6}{7}) = e^{0.8571428571\lambda}\left(-0.6600833229m^2c_6 - 0.36412842070S_hc_5^2 - 0.53061224491S_hc_1\right.$$
$$-0.4602249063m^2c_3 - 0.26530612241m^2c_1 - 0.5373356340m^2c_4 - 1.0746712680S_hc_4$$
$$-0.9204498126S_hc_3 - 0.07038733861S_hc_1^2 - 0.3702623907m^2c_2 - 0.2118069644S_hc_3^2 \quad (22)$$
$$-0.7405247813S_hc_2 + \cdots - 0.6034305434m^2c_5 - 1.320166646S_hc_6 - 0.1370942379S_hc_2^2$$
$$\left.-0.4357099932S_hc_6^2\right)$$

Thus, we can obtain a coefficient for the different value of parameters that shows graphically.

Application of Variational Iteration Method

First, we construct a correction functional which reads;

$$\theta_{n+1}(X) = \theta_n(X) + \int_0^X \lambda'\left\{\theta''(\tau) - \lambda\theta'(\tau) - e^{\lambda\tau}\left[S_h\theta^2(\tau) + m^2\theta(\tau)\right]\right\}d\tau \quad (23)$$

Where λ is General Lagrange multiplier. To make the above correction functional stationary, we obtain the following stationary conditions;

$$\lambda''(t) = 0, \quad 1 - \lambda'(t)\Big|_{t=x} = 0, \quad \lambda(t)\Big|_{t=x} = 0 \quad (24)$$

The Lagrange multiplier, therefore, can be identified as;

$$\lambda' = -X + \tau; \quad (25)$$

As a result, we obtain the following iteration formula;

$$\theta_{n+1}(X) = \theta_n(X) + \int_0^X \lambda'\left\{\theta''(\tau) - \lambda\theta'(\tau) - e^{\lambda\tau}\left[S_h\theta^2(\tau) + m^2\theta(\tau)\right]\right\}d\tau \quad (26)$$

Now we start with an arbitrary initial approximation that satisfies the initial condition;

$$\theta_0(X) = 1, \quad (27)$$

Using the above variational formula (26), after some simplifications, we have;

$$\theta_1(X) = \frac{\lambda^2 - S_h\lambda X - S_h + S_he^{\lambda X} - m^2\lambda X - m^2 + m^2e^{\lambda X}}{\lambda^2} \quad (28)$$

$$
\begin{aligned}
\theta_2(X)=&\frac{1}{36\lambda^6}\Big(9m^4e^{2\lambda x}\lambda^2+18S_h^{\ 2}e^{2\lambda x}\lambda^2-18XS_h^{\ 3}e^{2\lambda x}\lambda+27S_hm^2e^{2\lambda x}\lambda^2+4m^4S_he^{3\lambda x}-45m^4\lambda^2\\
&+8S_h^{\ 2}m^2e^{3\lambda x}216S_h^{\ 2}m^2e^{\lambda X}+108m^4S_he^{\lambda X}+108S_h^{\ 3}e^{\lambda X}-72XS_hm^4e^{\lambda X}\lambda-144XS_h^{\ 2}m^2e^{\lambda X}\lambda\\
&-108XS_hm^2e^{\lambda X}\lambda^3+72S_h^{\ 2}m^2e^{\lambda X}\lambda^2X^2+36m^4S_he^{\lambda X}\lambda^2X^2-18xS_hm^4e^{2\lambda x}\lambda-36\lambda^3XS_h^{\ 2}\\
&-18X^2S_h\lambda^6-18X^2m^2\lambda^6+36m^4e^{\lambda x}\lambda^2+72S_he^{\lambda X}\lambda^4+72m^2e^{\lambda X}\lambda^4+72S_h^{\ 2}e^{\lambda X}\lambda^2-112S_h^{\ 3}\\
&-36xS_h^{\ 2}e^{2\lambda x}m^2\lambda+4S_h^{\ 3}e^{\lambda X}-18\lambda^3Xm^4+36\lambda^6-72XS_h^{\ 3}e^{\lambda X}\lambda-72XS_h^{\ 2}e^{\lambda X}\lambda^3-36xm^4e^{\lambda X}\lambda^3\\
&+108S_hm^2e^{\lambda X}\lambda^2+36x^2S_h^{\ 3}e^{\lambda X}\lambda^2-72\lambda^4S_h-72\lambda^4m^2-90\lambda S_h^{\ 2}-224S_h^{\ 2}m^2-112m^4S_h\\
&-135\lambda^2S_hm^2-54\lambda^3XS_hm^2-60\lambda XS_h^{\ 2}m^2-30\lambda Xm^4S_h-30\lambda XS^3-72\lambda^5XS_h-72\lambda^5Xm^2
\end{aligned}
\tag{29}
$$

Application of Homotopy Perturbation Method

In this section, we employ HPM to solve Eq. (7), subject to boundary conditions in Eqs. (8a) and (8b). We can construct the homotopy function of Eq. (7) as described in [2];

$$H(\theta,\ p)=(1-P)\left[\ \theta''(X)\ \right]+P\left[\ \theta''(X)-\lambda\,\theta'(X)-e^{\lambda X}\left[S_h\,\theta^{\,2}(X)+m^2\,\theta(X)\right]\right] \tag{30}$$

Where $p\in[0,1]$ is an embedding parameter. For $p=0$ and $p=1$ we have;

$$\theta(X,0)=\theta_0(X)\qquad,\qquad \theta(X,1)=\theta(X) \tag{31}$$

Note that when p increases from 0 to 1, $\theta(X\ ,p)$ varies from $\theta_0(X)$ to $\theta(X)$. By substituting;

$$\theta(X)=\theta_0(X)+p\,\theta_1(X)+p^2\theta_2(X)+\cdots=\sum_{i=0}^{n}p^i\theta_i(X), \tag{32}$$

Into Eq. (30) and rearranging the result based on powers of p-terms, we have;

$$
\begin{aligned}
P^0:&\qquad \frac{d^2}{dx^2}\theta_0(x)=0\\
&\qquad \theta_0(1)=1,\qquad \theta_0(0)=0
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
P^1:&\quad -\lambda\left(\frac{d}{dx}\theta_0(x)\right)+\frac{d^2}{dx^2}\theta_1(x)-e^{\lambda x}S_h\theta_0(x)^2-e^{\lambda x}m^2\theta_0(x)\\
&\qquad \theta_1(1)=1,\qquad \theta_1(0)=0
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
P^2:&\quad -\lambda\left(\frac{d}{d}\theta_1(x)\right)-e^{\lambda x}m^2\theta_1(x)+\frac{d^2}{dx^2}\theta_2(x)-2e^{\lambda x}S_h\ \theta_0(x)\theta_1(x)\\
&\qquad \theta_2(1)=0,\qquad \theta_2(0)=0
\end{aligned}
\tag{35}
$$

Solving Eqs. (33) - (35) with boundary conditions, we have;

$$\theta_0\left(X\right)=1 \tag{36}$$

$$\theta_1\left(X\right)=\frac{e^{\lambda x}\left(S_h+m^2\right)}{\lambda^2}-\frac{e^{\lambda}\left(S_h+m^2\right)X}{\lambda}-\frac{S_h+m^2}{\lambda^2} \tag{37}$$

$$\begin{aligned}\theta_2\left(X\right)=&-\frac{2\left(S_h+m^2\right)}{\lambda^3}\frac{\left(0.5m^2+S_h\right)\left(e^{\lambda X+\lambda}\left(\lambda X+\lambda\right)-2e^{\lambda X+\lambda}-e^{\lambda X+\lambda}\lambda\right)}{\lambda}\\
&+-\frac{2\left(S_h+m^2\right)}{\lambda^3}\frac{1}{2}\frac{\left(-0.5S_h-0.25m^2\right)e^{2\lambda X}}{\lambda}+\frac{\left(-0.5\lambda^2+S_h+0.5m^2\right)e^{\lambda X}}{\lambda}+\frac{1}{4}\lambda^3e^{\lambda}X^2\\
&+\frac{1}{2}\frac{\left(6S_he^{2\lambda}m^2\lambda-9S_he^{2\lambda}m^2+4e^{2\lambda}\lambda S_h{}^2-6e^{2\lambda}S_h{}^2+4e^{\lambda}S_h{}^2-2\lambda^2e^{\lambda}S_h\right)X}{\lambda^3}\\
&+\frac{1}{2}\frac{\left(6S_he^{\lambda}m^2+2S_h\lambda^3e^{\lambda}+2m^4e^{2\lambda}\lambda-3m^4e^{2\lambda}-2\lambda^2e^{\lambda}m^2+2m^4e^{\lambda}+2m^2\lambda^3e^{\lambda}\right)X}{\lambda^3}\\
&-\frac{1}{4}\frac{24S_he^{\lambda}m^2+16e^{\lambda}S_h{}^2-9m^2S_h-6S_h{}^2+4\lambda^2S_h+8m^4e^{\lambda}-3m^4+4\lambda^2m^2}{\lambda^4}\end{aligned} \tag{38}$$

The solution of this equation, when $p\to 1$, will be as follows;

$$\begin{aligned}\theta\left(X\right)=&\sum_{i=0}^{2}\lim_{p\to 1}\ p^i\,\theta_i\left(X\right)=1+\frac{e^{\lambda X}\left(S_h+m^2\right)}{\lambda^2}-\frac{e^{\lambda}\left(S_h+m^2\right)X}{\lambda}-\frac{S_h+m^2}{\lambda^2}\\
&+\frac{1}{4}\lambda^3e^{\lambda}X^2-\frac{2\left(S+m^2\right)}{\lambda^3}\frac{\left(0.5m^2+S\right)\left(e^{\lambda X+\lambda}\left(\lambda X+\lambda\right)-2e^{\lambda X+\lambda}-e^{\lambda X+\lambda}\lambda\right)}{\lambda}\\
&+-\frac{2\left(S+m^2\right)}{\lambda^3}\frac{1}{2}\frac{\left(-0.5S-0.25m^2\right)e^{2\lambda X}}{\lambda}+\frac{\left(-0.5\lambda^2+S+0.5m^2\right)e^{\lambda X}}{\lambda}\\
&+\frac{1}{2}\frac{\left(6S_he^{2\lambda}m^2\lambda-9S_he^{2\lambda}m^2+4e^{2\lambda}\lambda S_h{}^2-6e^{2\lambda}S_h{}^2+4e^{\lambda}S_h{}^2-2\lambda^2e^{\lambda}S_h\right)X}{\lambda^3}\\
&+\frac{1}{2}\frac{\left(6S_he^{\lambda}m^2+2S_h\lambda^3e^{\lambda}+2m^4e^{2\lambda}\lambda-3m^4e^{2\lambda}-2\lambda^2e^{\lambda}m^2+2m^4e^{\lambda}+2m^2\lambda^3e^{\lambda}\right)X}{\lambda^3}\\
&-\frac{1}{4}\frac{24S_he^{\lambda}m^2+16e^{\lambda}S_h{}^2-9m^2S_h-6S_h{}^2+4\lambda^2S_h+8m^4e^{\lambda}-3m^4+4\lambda^2m^2}{\lambda^4}\end{aligned} \tag{39}$$

Results and discussion

In the present study the analytical methods are applied to obtain an explicit analytic solution for nonlinear equations arising in porous fin problem (**Figure 1**). A numerical procedure is used for solving temperature distribution of a porous fin subjected to insulated tip case. The numerical solution is performed using the algebra package Maple 15.0 to solve the present case. The package uses a fourth-fifth order Runge-Kutta-Fehlberg procedure to solve the nonlinear boundary value problem (BVP). The algorithm is proved to be precise and accurate aimed at solving a wide range of mathematical and engineering problems, especially heat transfer cases [26]. A Comparison between the results obtained from the numerical method, CM, HPM and VIM for different values of active parameters is shown in **Figures 3 - 6** and **Table 1**. In this table, the %Error is defined as;

$$\% Error = \left| \theta(X)_{NUM} - \theta(X)_{Analytical} \right| \tag{40}$$

Figure 3 shows the approximate solutions for the governing equation using CM, HPM, and VIM setting the parameters, $\lambda = 1$, $S_h = 0.8$, $m = 0.5$. By changing the porosity and convective parameters with constant value for Lagrange multiplier as $\lambda = 1$, $S_h = 0.9$, $m = 0.4$, the approximate solutions are presented in **Figure 4**. **Figures 5 - 6** exhibit the analytic solutions for the proposed equation applying CM, HPM, and VIM varying pororsity and convective parameters with constant the lagrange multiplier at 1.5 as $\lambda = 1.5$, $m = 0.3$ and $S_h = 0.6$. With respect to the plots (**Figures 3 - 6**) it is found that the analytical methods have a good capability to solve such heat transfer equations. Through these figures it is clear that CM has a good agreement with the numerical approach, even though, VIM and HPM show some deviations toward numeric method. All the values pertaining to the analytical and numerical approaches and the errors as well are addressed in **Table 1**.

Table 1 The results of CM, VIM, HPM, and numerical methods for $\theta(X)$ for $\lambda = 1$, $S_h = 0.8$, $m = 0.1$.

X	CM	HPM	VIM	NUM	Error of CM	Error of HPM	Error of VIM
0.00	0.626278894	0.474010068	0.526081341	0.625638812	0.000640081	0.151628744	0.099557472
0.10	0.627946436	0.477806272	0.528360736	0.627347877	0.000598559	0.149541606	0.098987141
0.20	0.633507782	0.489627174	0.535853323	0.632973403	0.000534379	0.143346230	0.097120080
0.30	0.643868135	0.510164954	0.549695946	0.643395422	0.000472714	0.133230468	0.093699475
0.40	0.660127020	0.540192135	0.571279775	0.659715517	0.000411503	0.119523382	0.088435743
0.50	0.683682292	0.580614157	0.602326840	0.683332074	0.000350218	0.102717917	0.081005234
0.60	0.716338846	0.632568432	0.644994453	0.716048999	0.000289847	0.083480567	0.071054546
0.70	0.760465325	0.697597451	0.702018664	0.760236391	0.000228934	0.062638940	0.058217727
0.80	0.819242103	0.777936979	0.776912394	0.819074571	0.000167532	0.041137592	0.042162177
0.90	0.897043857	0.876979448	0.874240154	0.896935608	0.000108249	0.019956160	0.022695454
1.00	1.000000000	1.000000000	1.000000000	1.000000000	0.000000000	0.000000000	0.000000000

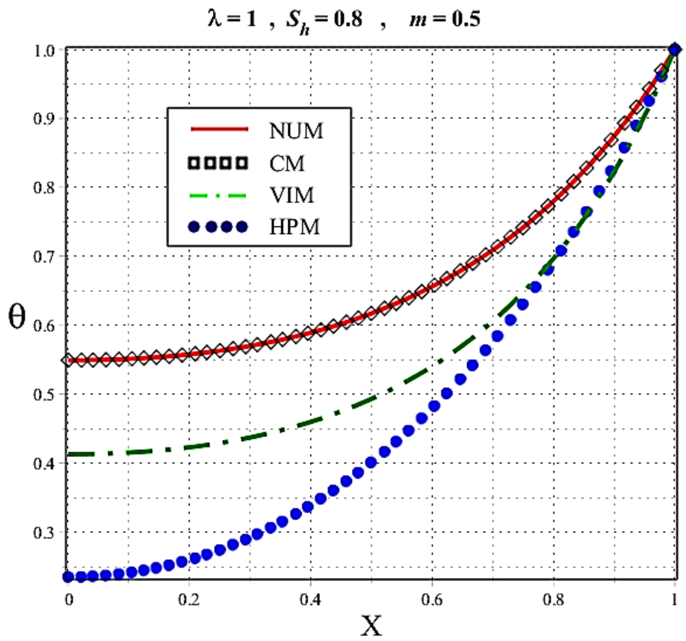


Figure 3 Comparison between the CM, VIM, HPM, and numerical solutions for $\lambda = 1$, $S_h = 0.8$, $m = 0.5$.

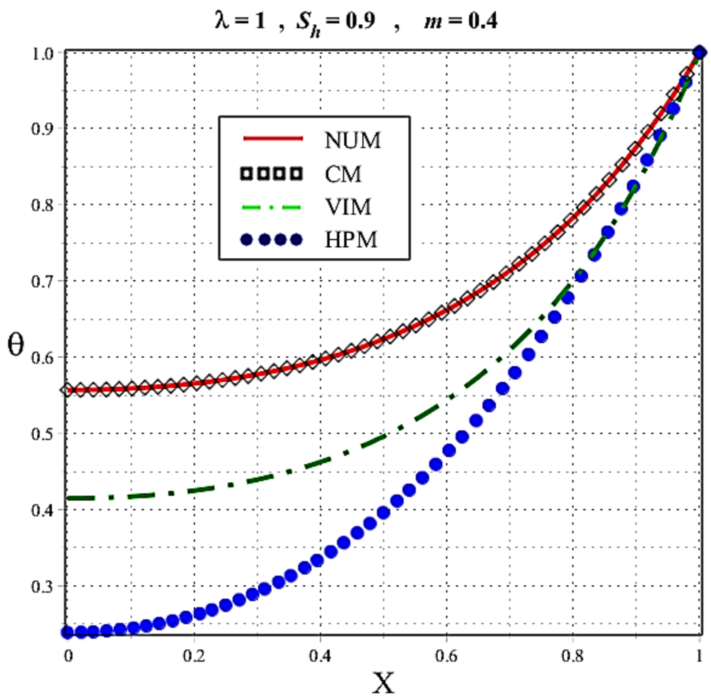


Figure 4 Comparison between the CM, VIM, HPM, and numerical solutions for $\lambda = 1$, $S_h = 0.9$, $m = 0.4$.

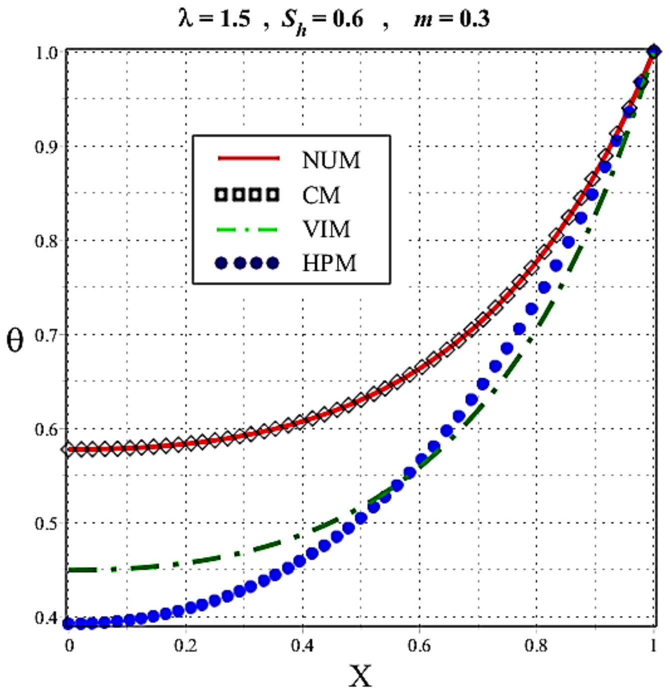


Figure 5 Comparison between the CM, VIM, HPM, and numerical solutions for $\lambda = 1.5$, $S_h = 0.6$, $m = 0.3$.

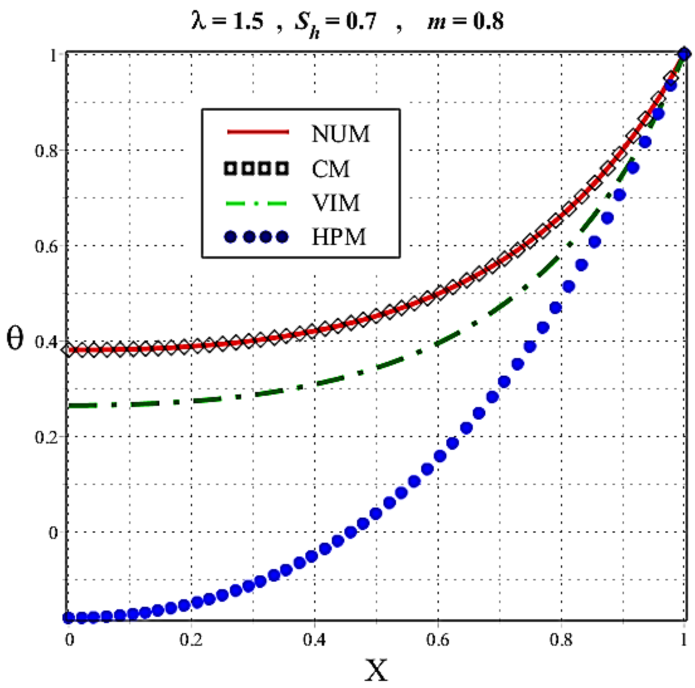


Figure 6 Comparison between the CM, VIM, HPM, and numerical solutions for $\lambda = 1.5$, $S_h = 0.7$, $m = 0.8$.

Conclusions

In this paper, we have studied the nonlinear equations arising in a convective porous fin with a variable cross section by applying CM, HPM, and VIM. The figures and table clearly show that the results by CM are in excellent agreement with the results of HPM, VIM, and the numerical solution. The results show that this scheme provides some excellent approximations to the solution of the nonlinear equation with high accuracy.

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