

Numerical Solution for Riesz Fractional Diffusion Equation via Fractional Centered Difference Scheme

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Abstract

In this paper, a mixed matrix transform method with fractional centered difference scheme for solving fractional diffusion equation with Riesz fractional derivative was examined. It was obtained that the numerical scheme was unconditionally stable and feasible using the matrix analysis method. Numerical experiments were, then, carried out to support the theoretical predictions.

Keywords: Fractional centered difference, Padé approximation, Riesz fractional diffusion equations [2,0], Stability and feasible

Introduction

Fractional partial differential equations have received considerable interest in recent years and have been extensively investigated and applied for many real problems which are modeled in various areas, e.g., viscoelasticity, diffusion, control, relaxation processes, and so forth [1-4]. These new fractional-order models are more adequate than the previously used integer-order models because fractional-order derivatives and integrals enable the description of the memory and hereditary properties of different substances [4]. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which such effects are neglected.

In the area of physics, fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle spreads at a rate inconsistent with the classical Brownian motion model [5]. The regularity criterion is of particular importance for the diffusion equation. In this case, Gala and Ragusa [6] studied the regularity criterion in terms of the homogeneous Besov space for the incompressible Boussinesq equations. In particular, the Riesz fractional derivative includes a left Riemann-Liouville derivative and a right Riemann-Liouville derivative that allows the modeling of flow regime impacts from either side of the domain [7]. It is well known that the analytical solutions to the fractional differential equations are usually difficult to derive and (if luckily obtained) always contain some infinite series which make evaluation very expensive. Therefore, we resort to some numerical methods for fractional differential equations such as finite difference method [8-11], finite element method [12,13], spectral method [14], radial basis function collocation method [15], spectral Tau algorithm [16], a hybrid of Lagrange operational matrix and Tau-collocation method [17] and other analytical methods (e.g., variational iteration method [18], homotopy perturbation method [19], a domain decomposition method [20]).

The finite difference method is accepted as one of the most popular numerical methods for fractional differential equations because it is direct and convenient to use. But Ortigueira and Trujillo presented fractional centered difference for estimation of Riesz type fractional derivatives [21,22]. In this paper, the focus is on diffusion equations with integer time derivative and Riesz fractional space

derivatives. Models of this form are indeed common in many applications; thus, it requires effective numerical methods for their resolution. Many numerical methods have been developed for the efficient solution of the Riesz space fractional diffusion equation and related problems. Typical techniques are the applications of finite difference method based on fractional centered difference scheme [23,24], improved matrix transform method [25], the mixture of L1/L2-approximation method, standard/shifted Grünwald method, and matrix transform method [26], variational iteration method [27], McCormack method [28], Galerkin finite element method [29]. Also, some recent research work in numerical methods can be found in [30-32]. In this paper, we consider the following space Riesz fractional diffusion equations derived from the aforementioned model as follows [26,33].

$$\frac{\partial u(x, t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}, \tag{1}$$

in a bounded domain with the initial value and boundary conditions given by;

$$u(x, 0) = f(x), \tag{2}$$

$$u(0, t) = u(L, t) = 0. \tag{3}$$

Here x and t are the space and time variables, and K_α is a positive constant, $1 < \alpha \leq 2$.

This paper aims to combine the Padé approximation method with a fractional centered difference scheme to design the high order finite difference scheme for the Riesz fractional diffusion Eqs. (1) - (3). The numerical scheme will be theoretically proven and numerically verified to be unconditionally stable.

The outline of this paper is organized as follows. In section 2, some preliminary materials are provided. Then, section 3 presents the padé approximation method coupled with a fractional centered difference scheme for solving Riesz fractional diffusion equation. In Section 4, the stability and feasible analysis are given for the proposed algorithm. In Section 5, numerical examples are carried out to demonstrate the theoretical results and verify the efficiency of our method. Finally, conclusions are drawn in section 6.

Preliminaries and Basic Lemmas

The space fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}$ is Riesz space-fractional derivatives of order α , which is defined by Gorenflo and Mainardi [33] in Definition 1.

Definition 1 (see [33]) The Riesz fractional operator for $n \in \mathbb{N}$, $n - 1 < \alpha \leq n$, on a finite interval $0 \leq x \leq L$ is defined as;

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -C_\alpha ({}^R_0D_x^\alpha + {}^R_xD_L^\alpha)u(x, t),$$

where the coefficient $C_\alpha = \frac{1}{2 \cos \frac{\alpha\pi}{2}}$, $\alpha \neq 1$, and;

$${}^R_0D_x^\alpha u(x, t) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_0^x \frac{u(\xi, t) d\xi}{(x - \xi)^{\alpha-n+1}},$$

$${}^R_xD_L^\alpha u(x, t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_x^L \frac{u(\xi, t) d\xi}{(x - \xi)^{\alpha-n+1}},$$

are the left-side and right-side Riemann-Liouville fractional derivatives, respectively.

To obtain fractional centered difference we proceed as in [21,34]. Divide the fractional differences by h^α ($h \in \mathbb{R}^+$) and let $h \rightarrow 0$. For the Riesz fractional derivative for the case of $1 < \alpha \leq 2$, and assuming that $\alpha > -1$, we obtain;

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = \lim_{h \rightarrow 0} \frac{\Delta_h^\alpha u(x, t)}{h^\alpha} = - \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{i=-\infty}^{+\infty} \frac{(-1)^i \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - i + 1) \Gamma(\frac{\alpha}{2} + i + 1)} u(x - ih, t). \quad (4)$$

Lemma 1 Let $\omega_{k,\alpha} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1) \Gamma(\frac{\alpha}{2}+k+1)}$ be the coefficients of the centered finite difference approximation (4) for $k = 0, \mp 1, \mp 2, \dots$ and $\alpha > -1$. Then;

- 1) $\omega_{0,\alpha} \geq 0$
- 2) $\omega_{-k,\alpha} = \omega_{k,\alpha} \leq 0$ for $|k| \geq 1$,

Proof. [24]

Lemma 2 If the real part of z is positive, then $\left| \frac{2}{2+2z+z^2} \right| < 1$.

Proof. Let $z = a + ib, i = \sqrt{-1}$, following [35,36], it can be proved in the following manner.

$$\left| \frac{2}{2+2z+z^2} \right| < 1,$$

$$\begin{aligned} 4 &< |2 + 2z + z^2|^2, \\ 4 &< (2 + 2z + z^2)(2 + 2\bar{z} + \bar{z}^2), \\ 0 &< 4(z + \bar{z}) + 2(z + \bar{z})^2 + 2z\bar{z}(z + \bar{z}) + |z|^4, \\ 0 &< 16a + 4a(a^2 + b^2) + (a^2 + b^2)^2, \end{aligned}$$

This is always true if the real part of z (i.e, a) is positive. Hence, the required result is obtained.

Lemma 3 Suppose $\rho(M) < 1$, then $\rho(M) \leq 1 + C\tau$ for some non-negative C is a necessary and sufficient condition for stability of the difference scheme;

$$U_{k+1} = MU_k$$

concerning the matrix 2-norm, where $\rho(M)$ denote the spectral radius of the matrix M . [37].

Implementation of the numerical method

Consider Riesz space fractional diffusion equation of the form;

$$\frac{\partial u(x, t)}{\partial t} = K_\alpha \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}, 0 \leq t \leq T, 0 < x < L, \quad (5)$$

with the initial value and 0 boundary conditions given by;

$$u(x, 0) = f(x), \quad (6)$$

$$u(0, t) = u(L, t) = 0, \quad (7)$$

where $u(x, t)$ and $f(x)$ are both real-valued and sufficiently well-behaved functions.

Here we obtain a new difference scheme for the solution of Eqs. (5) - (7) which is based upon the matrix transform method [37].

Let $x_i = ih, i = 0, 1, 2, \dots, m - 1$ and $t_j = jk, j = 0, 1, 2, \dots, n$, where $h = \frac{L}{m}$ and $k = \frac{T}{n}$ are space and time steps, respectively.

Values of the finite difference approximations of $u(x, t)$ at the grid are denoted by;

$$u_{i,j} = u(x_i, t_j). \tag{8}$$

Assume that $u(x, t)$ is a sufficiently smooth function and replace the fractional partial derivatives in (5) with respect to x by the fractional centered difference estimate:

$$\frac{\partial^\alpha u(x_i, t)}{\partial |x|^\alpha} = -\frac{1}{h^\alpha} \sum_{s=-m+i}^i \omega_{s,\alpha} u(x_{i-s}, t) + O(h^2), \tag{9}$$

where the coefficients $\omega_{s,\alpha}$ are defined by $\omega_{0,\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+1)^2}$, $\omega_{s+1,\alpha} = \left(1 - \frac{\alpha+1}{\frac{\alpha}{2}+s+1}\right) \omega_{s,\alpha}$, for $0, +1, \mp 2, \dots$.

Let $u_i(t) = u(x_i, t)$ for $i = 0, 1, 2, \dots, m - 1$, and
 $U(t) = [u_1(t), u_2(t), \dots, u_{m-1}(t)]^T$,
 $U_0 = [u_1(0), u_2(0), \dots, u_{m-1}(0)]^T$

Then above discretization results in an initial-value problem of the form:

$$\begin{cases} \frac{dU(t)}{dt} = -AU(t), \\ U(0) = U_0, \end{cases} \tag{10}$$

where A is the symmetric Toeplitz matrix of order $m - 1$ as follows:

$$A = \begin{bmatrix} \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{1,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{2,\alpha}}{h^\alpha} & \dots & \frac{K_\alpha \omega_{m-2,\alpha}}{h^\alpha} \\ \frac{K_\alpha \omega_{1,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{1,\alpha}}{h^\alpha} & \dots & \frac{K_\alpha \omega_{m-3,\alpha}}{h^\alpha} \\ \frac{K_\alpha \omega_{2,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{1,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha} & \dots & \frac{K_\alpha \omega_{m-4,\alpha}}{h^\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{K_\alpha \omega_{m-2,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{m-3,\alpha}}{h^\alpha} & \frac{K_\alpha \omega_{m-4,\alpha}}{h^\alpha} & \dots & \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha} \end{bmatrix}$$

By Duhamel's principle [38], the exact solution of (10) can be written as;

$$U(t) = \exp(-tA)U_0, \tag{11}$$

Replacing t by $t + k$, we can write the exact solution (11) as;

$$U(t + k) = \exp(-(t + k)A)U_0 = \exp(-kA)U(t), \tag{12}$$

which satisfies the recurrence formula;

$$U(t_{j+1}) = \exp(-kA)U(t_j). \tag{13}$$

Recurrence formula (13) is the basis of different time-stepping schemes depending upon how we approximate the matrix exponential functions. We shall use [2,0] Padé approximations of the matrix exponential functions $\exp(-kA)$ to construct a family of time-stepping schemes. Then we obtain the following numerical scheme for solving (5) - (7);

$$V_{n+1} = \left(I + kA + \frac{1}{2}k^2A^2 \right)^{-1} V_n, \tag{14}$$

V is the approximate solution of U.

Stability analysis

In this section, we demonstrate that the mixed [2,0] Padé approximation-fractional centered difference scheme for the fractional initial-boundary value problem (1) - (3) is unconditionally stable.

Lemma 4 (Gerschgorin theorem [39]). Let $B = (b_{i,j})$ be a complex matrix of order $M - 1$, and;

$$R_i = \sum_{k \neq i} |b_{i,k}|, i = 1, 2, \dots, M - 1,$$

Let D_i be the closed disk centered at $b_{i,i}$ with radius $R_i: D_i = \{z \in \mathbb{C} : |z - b_{i,i}| \leq R_i\}$.

Then, all the eigenvalues of matrix B are in $\cup_{i=1}^{M-1} D_i$.

Theorem 1 Numerical algorithm (14) is unconditionally stable.

Proof. According to the Gerschgorin theorem the eigenvalues of matrix A lie in the union of the $m - 1$ circles centered at $A_{i,i}$, with radius $R_i = \sum_{k \neq i} |A_{i,k}|, i = 1, 2, \dots, m - 1$. Now from the definition of A we have:

$$A_{i,i} = \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha} \text{ and } R_i = \sum_{s=1}^{m-2} \left| \frac{K_\alpha \omega_{s,\alpha}}{h^\alpha} \right|$$

Hence, If λ be the eigenvalue of matrix A, Then;

$$\left| \lambda - \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha} \right| \leq \sum_{s=1}^{m-2} \left| \frac{K_\alpha \omega_{s,\alpha}}{h^\alpha} \right| < \frac{K_\alpha \omega_{0,\alpha}}{h^\alpha}.$$

We can see that these Gerschgorin disks are within the right half of the complex plane. Therefore, the eigenvalues of matrix A have positive real parts.

$$\text{Let } T = \left(I + kA + \frac{1}{2}k^2A^2 \right)^{-1},$$

The spectral radius of the matrix T is given by;

$$\rho(T) = \max|\mu_i|, i = 1, 2, \dots, m - 1,$$

where μ_i are the eigenvalues of the matrix $\left(I + kA + \frac{1}{2}k^2A^2\right)^{-1}$.

We easily know that the eigenvalues of the matrix $\left(I + kA + \frac{1}{2}k^2A^2\right)^{-1}$, are given by;

$$\mu_i = \frac{2}{2+2k\lambda_i(A)+k^2(\lambda_i(A))^2}, i = 1, 2, \dots, m - 1.$$

By using the Lemma 2, we have;

$$|\mu_i| < 1, i = 1, 2, \dots, m - 1,$$

i.e.,

$$\rho(T) < 1, i = 1, 2, \dots, m - 1.$$

Therefore, from Lemma 3, it is very easy to find that the difference scheme (14) is unconditionally stable.

Numerical experiments

In this section, we exhibit numerical results for a particular fractional diffusion equation with a known solution.

Example 1 We consider the following Riesz fractional diffusion equation;

$$\frac{\partial u(x, t)}{\partial t} = 0.25 \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}, 0 \leq t \leq T, 0 < x < 1, \tag{15}$$

$$u(x, 0) = \sin(2\pi x), 0 \leq x \leq 1, \tag{16}$$

$$u(0, t) = u(1, t) = 0, 0 \leq t \leq T. \tag{17}$$

The analytic solution of Eqs. (15) - (17) is given by [26] $u(x, t) = \sin(2\pi x) \exp(-0.25(2\pi)^\alpha t)$.

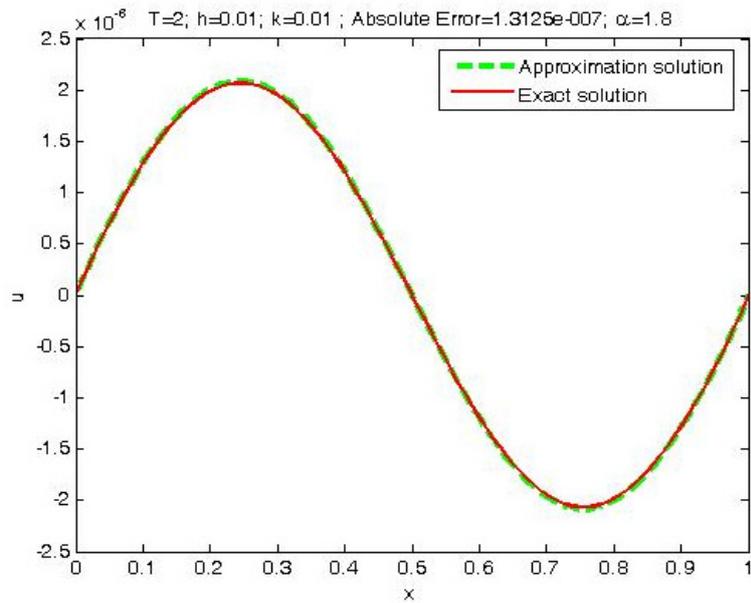


Figure 1 Comparison of the numerical solutions with the analytic solution at $t = 2$ for the RFDE (15) - (17) with $h = 0.01$, $k = 0.01$, and $\alpha = 1.8$.

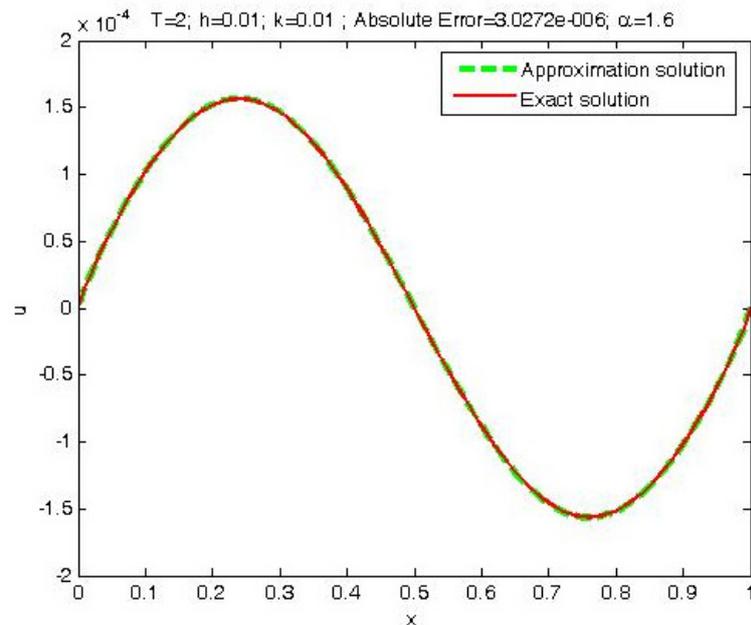


Figure 2 Comparison of the numerical solutions with the analytic solution at $t = 2$ for the RFDE (15) - (17) with $h = 0.01$, $k = 0.01$, and $\alpha = 1.6$.

Figures 1 and 2 show the analytic solution and numerical solution obtained by the proposed method for $h = 0.01$, $k = 0.01$ and $\alpha = 1.8$ and $\alpha = 1.6$ respectively. From **Figures 1 and 2**, it can be seen that the numerical proposed method is in good agreement with the analytic solution.

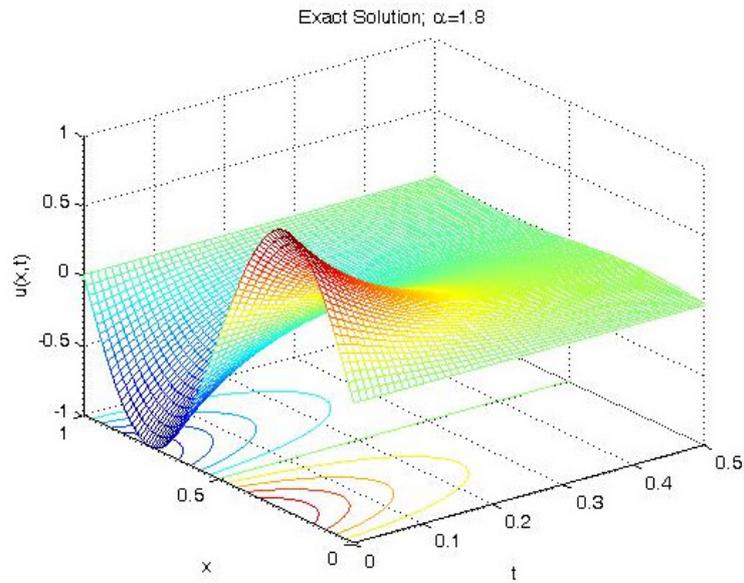


Figure 3 3D plot of analytical solution for the RFDE (15) - (17) when $\alpha = 1.8$.

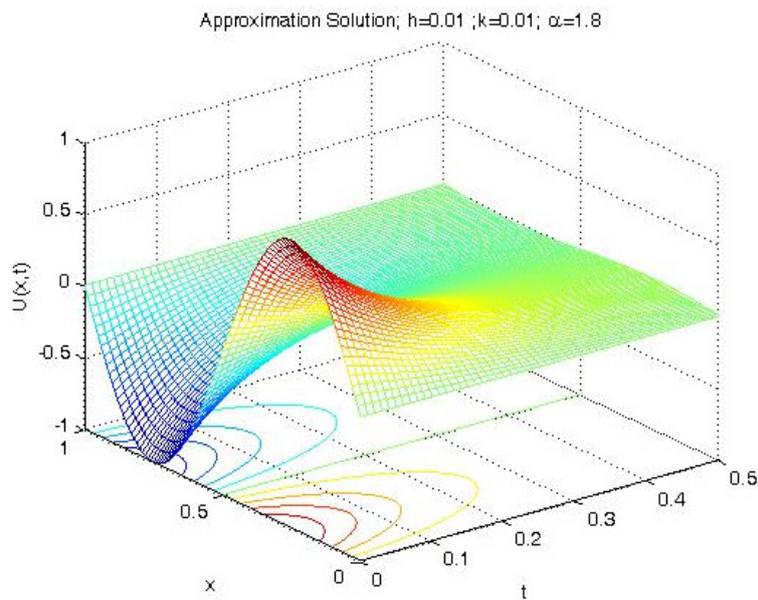


Figure 4 3D plot of approximation solution for the RFDE (15) - (17) with $h = 0.01$, $k = 0.01$ when $\alpha = 1.8$.

Figures 3 and 4 present analytical solution and numerical solution of the Riesz fractional diffusion Eqs. (15) - (17) for $0 \leq t \leq 0.5$ when $\alpha = 1.8$.

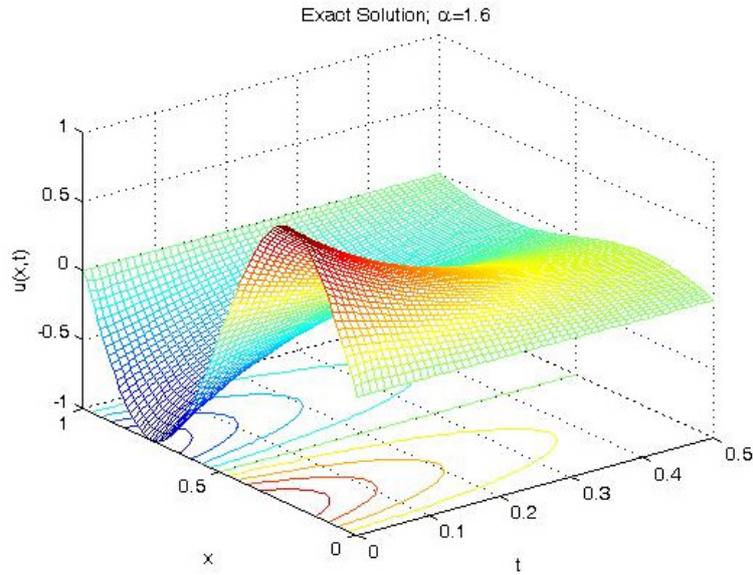


Figure 5 3D plot of analytical solution for the RFDE (15) - (17) when $\alpha = 1.6$.

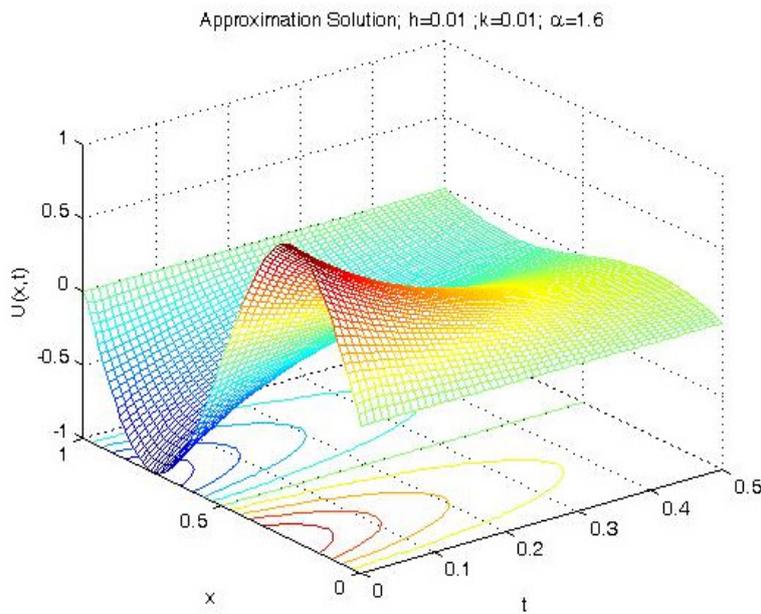


Figure 6 3D plot of approximation solution for the RFDE (15) - (17) with $h = 0.01$, $k = 0.01$ when $\alpha = 1.6$.

Figures 5 and 6 display analytical solution and numerical solution of the Riesz fractional diffusion equation (15) - (17) for $0 \leq t \leq 0.5$ when $\alpha = 1.6$.

Table 1 Absolute error for mixed [2,0] Padé approximation - finite difference method with different h, k and α in example 1.

$h = \Delta x$	$k = \Delta t$	Diffusion with $\alpha = 1.8$	Diffusion with $\alpha = 1.6$
0.10	0.10	4.2952e - 006	8.0479e - 005
0.10	0.05	1.0942e - 006	2.1877e - 005
0.10	0.01	4.8580e - 008	9.6819e - 007
0.05	0.10	5.2528e - 006	1.0922e - 004
0.05	0.05	1.3311e - 006	2.9678e - 005
0.05	0.01	5.9047e - 008	1.3139e - 006
0.01	0.10	2.3660e - 004	2.5229e - 004
0.01	0.05	3.2097e - 006	6.8396e - 005
0.01	0.01	1.3125e - 007	3.0272e - 006

Table 1 shows the magnitude of the maximum error, at $t = 2$ between the exact solution and numerical solution for different values of h, k, and α .

Figures 3 - 6 and **Table 1** show that the approximation solutions obtained using the proposed method are in excellent agreement with those obtained using the exact solution.

Example 2 We consider the following Riesz fractional diffusion equation;

$$\frac{\partial u(x, t)}{\partial t} = 0.25 \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}, 0 \leq t \leq T, 0 < x < 1, \tag{18}$$

$$u(x, 0) = x(1 - x), 0 \leq x \leq 1, \tag{19}$$

$$u(0, t) = u(1, t) = 0, 0 \leq t \leq T. \tag{20}$$

The analytic solution of Eqs. (18) - (20) is given by [26];

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3} [(-1)^{n+1} + 1] \sin(n\pi x) \exp(-0.25(n\pi)^\alpha t) \tag{21}$$

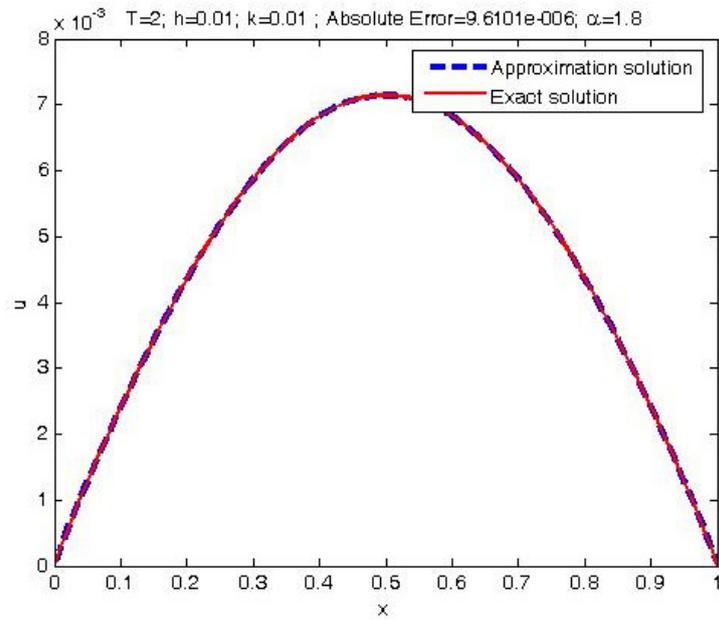


Figure 7 Comparison of the numerical solutions with the analytic solution at $t = 2$ for the RFDE (18) - (20) with $h = 0.01$, $k = 0.01$, and $\alpha = 1.8$.

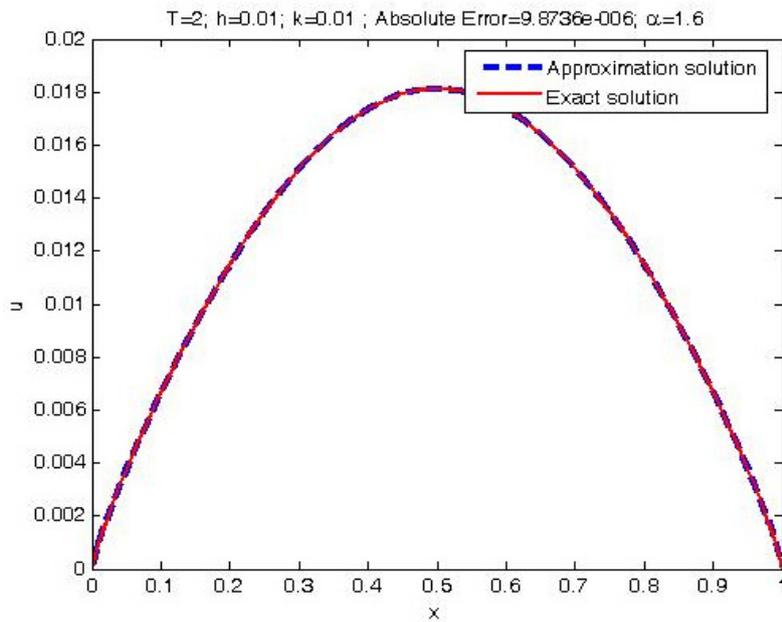


Figure 8 Comparison of the numerical solutions with the analytic solution at $t = 2$ for the RFDE (18) - (20) with $h = 0.01$, $k = 0.01$, and $\alpha = 1.6$.

Figures 7 and 8 show the analytic solution and numerical solution obtained by the proposed method for $h = 0.01$, $k = 0.01$ and $\alpha = 1.8$ and $\alpha = 1.6$, respectively. From Figures 7 and 8, it can be seen that the numerical proposed method is in good agreement with the analytic solution obtained via (21).

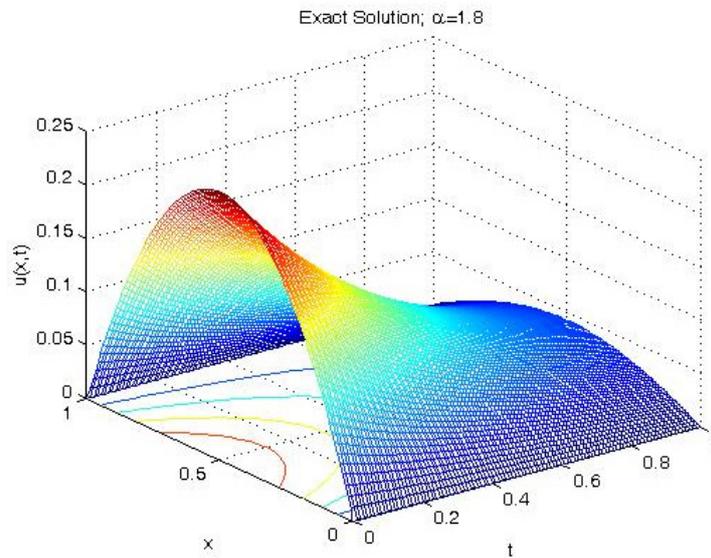


Figure 9 3D plot of analytical solution for the RFDE (18) - (20) when $\alpha = 1.8$.

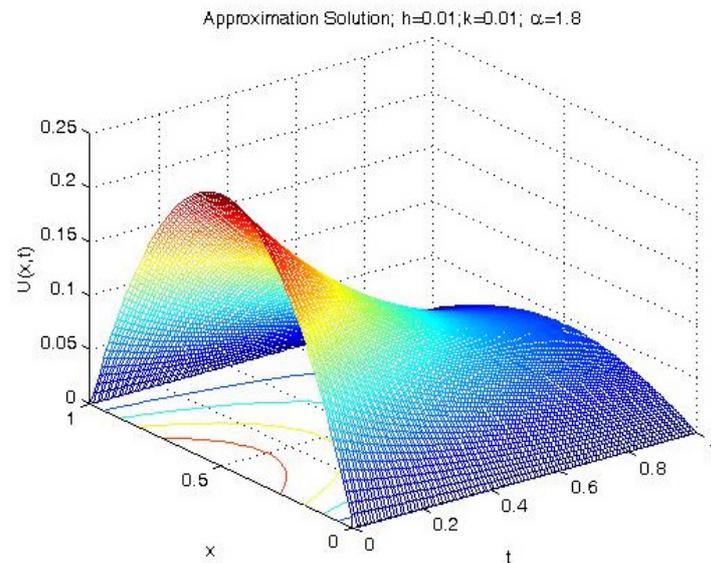


Figure 10 3D plot of approximation solution for the RFDE (18) - (20) with $h = 0.01$, $k = 0.01$ when $\alpha = 1.8$.

Figures 9 and 10 present analytical solution and numerical solution of the Riesz fractional diffusion Eqs. (18) - (20) for $0 \leq t \leq 1$ when $\alpha = 1.8$.

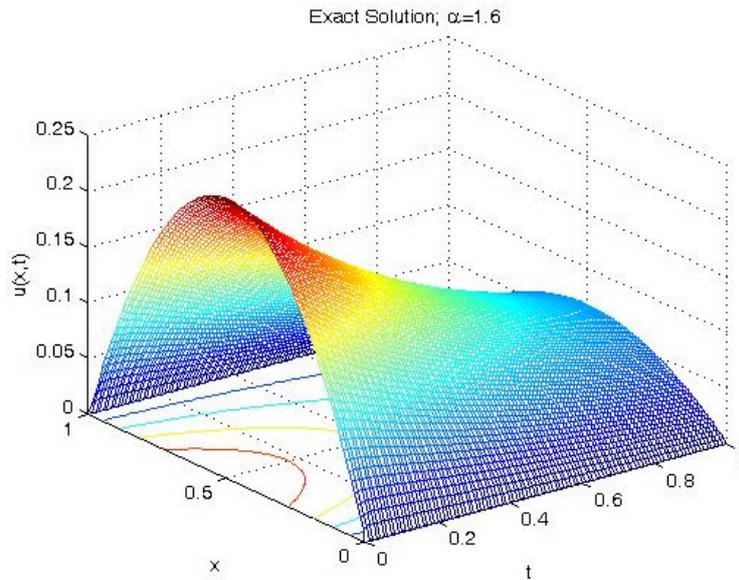


Figure 11 3D plot of analytical solution for the RFDE (18) - (20) when $\alpha = 1.6$.

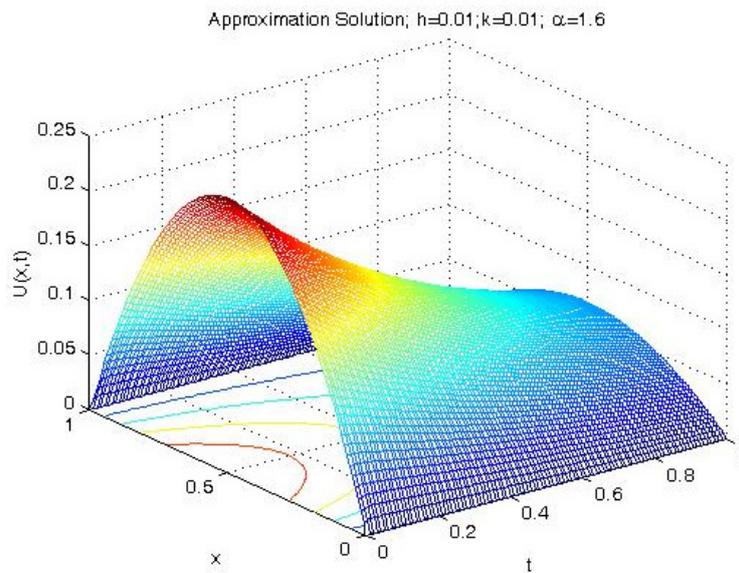


Figure 12 3D plot of approximation solution for the RFDE (18) - (20) with $h = 0.01$, $k = 0.01$ when $\alpha = 1.6$.

Figures 11 and 12 display analytical solution and numerical solution of the Riesz fractional diffusion equation (18)–(20) for $0 \leq t \leq 1$ when $\alpha = 1.6$.

Table 2 Absolute error for mixed [2,0] Padé approximation - finite difference method with different h, k and α in example 2.

$h = \Delta x$	$k = \Delta t$	Diffusion with $\alpha = 1.8$	Diffusion with $\alpha = 1.6$
0.10	0.10	2.7040e – 004	2.8832e – 004
0.10	0.05	7.1812e – 005	7.5562e – 005
0.10	0.01	3.0247e – 006	3.1433e – 006
0.05	0.10	3.8312e – 004	4.0684e – 004
0.05	0.05	1.0174e – 004	1.0658e – 004
0.05	0.01	4.2845e – 006	4.4321e – 006
0.01	0.10	8.6041e – 004	9.0709e – 004
0.01	0.05	2.2825e – 004	2.3753e – 004
0.01	0.01	9.6101e – 006	9.8736e – 006

Table 2 shows the magnitude of the maximum error, at $t = 2$ between the exact solution and numerical solution for different values of h, k, and α .

Figures 9 - 12 and **Table 2** show that the approximation solutions obtained using the proposed method are in excellent agreement with those obtained using the exact solution (21).

Conclusions

In the present work, a mixed [2,0] Padé approximation-fractional centered difference scheme has been developed for obtaining the solution of a special family of fractional initial-boundary value problems. We give the stability analysis and prove that it is unconditionally stable by using the matrix method. Numerical experiments were carried out to support the theoretical results and indicate the efficiency of the proposed method. The presented method in this paper can be upgraded for 2-dimensional partial differential equations of this type.

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