Solution of Fifth Order Caudrey-Dodd-Gibbon-Sawada-Kotera Equation by the Alternative (G'/G)-Expansion Method with Generalized Riccati Equation

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Abstract

In the present paper, the alternative (G'/G)-expansion is used to find new and precise solutions of Caudrey-Dodd-Gibbon-Sawada-Kotera equation with the assist of symbolic computation Maple, in which the generalized Riccati equation is used as an auxiliary equation. Plentiful traveling wave solutions including; exponential, hyperbolic and trigonometric functions are successfully accomplished by the proposed method with capricious parameters. It is revealed that the proposed method is straightforward, constructive and many nonlinear evolution equations in mathematical physics are solved by this method.

Keywords: Alternative (G'/G)-expansion method, exact solutions, traveling wave solutions, nonlinear evolution equation

Introduction

The nonlinear evolution equations are widely used as model to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and biology. One of the basic physical problems for those models is to obtain their traveling wave solutions. Particularly, various methods have been utilized to explore different kinds of solutions of physical models depicted by nonlinear partial differential equations.

In recent years, the exact solutions of nonlinear partial differential equations have been investigated by many researchers who are anxious with nonlinear physical phenomena and many powerful and proficient methods have been presented by them. Among non-integrable nonlinear differential equations, there is a wide class of equations that are referred to as partially integrable, because these equations become integrable for some values of their parameters. There are many different methods that can be used to look for the exact solutions of these equations. The most famous algorithms are Jacobi elliptic function expansion method [1] and tanh-function method [2,3]. For integrable nonlinear differential equations, inverse scattering transform method [4], Hirota method [5], Bäcklund transform method [6], Exp-function method [7-9], truncated Painlevé expansion method [10], extended tanh-method [11], and homogeneous balance method [12] are used for searching the exact solutions.

A straightforward and brief method called (G'/G)-expansion was introduced by Wang *et al*. [13] to obtain exact traveling wave solutions of nonlinear evolution equations, whereas the second order ordinary differential equation $G'' + \lambda G' + \mu G = 0$ is used as an auxiliary equation. The better perceptive of (G'/G)-expansion method are given in [14-17]. In order to instigate the efficiency and trustworthiness of the (G'/G)-expansion method and to enlarge the possibility of its application, further research has been

conceded out by many authors. For instance, an improvement of (*G'/G*)-expansion method was made by Zhang et al. [18] to search for further general traveling wave solutions. Another extension of (*G'/G*)-expansion method was given by Zayed [19] to attain new exact solutions, whereas Jacobi elliptic equation $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$ is used as an auxiliary equation and e_2, e_1, e_0 are arbitrary constants. Again an alternative approach of (*G'/G*)-expansion method was given by Zayed [20], in which the nonlinear Riccati equation $G'(\xi) = AG(\xi) + BG^2(\xi)$ is used as an auxiliary equation.

In this article, an alternative approach, called the alternative (*G'/G*) expansion method together with the generalized Riccati equation $G' = r + p\,G + q\,G^2$, introduced by Akbar *et al.* [14] is used to discover the exact traveling wave solutions of the CDGSK equation.

The alternative (*G'/G*)-expansion method

Consider the following nonlinear differential equation in the form;

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots) = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown function and P is a polynomial in $u = u(x, t)$. Moreover, it involves highest order derivatives and nonlinear terms. The main steps of the alternative (*G'/G*)-expansion method are:

Step 1: The wave transformation;

$$u\,(x,t) = u(\xi),\ \xi = x - V\,t, \tag{2}$$

where V is the speed of the traveling wave, that converts Eq. (1) into an ODE in the form;

$$Q(u, u', u'', u''', \cdots) = 0, \tag{3}$$

where primes stand for the ordinary derivative with respect to ξ.

Step 2: Integrate Eq. (3) term by term, if possible.

Step 3: Presume the solution of Eq. (3) can be articulated as follows;

$$u(\xi) = \sum_{i=0}^{n} a_i \left(G'/G\right)^i \tag{4}$$

where G satisfies the nonlinear generalized Riccati equation;

$$G' = r + p\,G + q\,G^2 \tag{5}$$

where $a_i\ (i = 1, 2, 3, \cdots, n)$, p, q and r are arbitrary constants to be determined later.

The generalized Riccati Eq. (5) has twenty-seven individual solutions (see Zhu, [21] for details).

Step 4: The positive integer n in Eq. (4) can be calculated by balancing the highest order nonlinear term with the highest order linear term appearing in Eq. (3).

Step 5: Inserting Eq. (4) into Eq. (3) and utilizing Eq. (5), we obtain a system of algebraic equations in G^i and G^{-i}. Suppose with the assist of symbolic computation software such as Maple, the unknown constants can be found and substituting these values into Eq. (4), we obtain new and general exact traveling wave solutions of the nonlinear partial differential Eq. (1).

Some new traveling wave solutions of the CDGSK equation

In this section, the alternating $(G′/G)$-expansion method, together with the generalized Riccati equation, is employed to construct some new traveling wave solutions for the Caudrey-Dodd-Gibbon-Sawada-Kotera equation [22];

$$u_t + u_{xxxxx} + 5\left(u\,u_{xxx} + u_x\,u_{xx} + u^2\,u_x\right) = 0. \tag{6}$$

Now, using the traveling wave variable (2) in Eq. (6), we have;

$$-V\,u' + u^{(5)} + 5\left(u\,u''' + u'\,u'' + u^2\,u'\right) = 0, \tag{7}$$

where $u^{(5)}$ denotes the fifth order derivative of u with respect to ξ. Integrating Eq. (7), we obtain;

$$C - V\,u + u^{(4)} + 5u\,u'' + \frac{5}{3}u^{\,3} = 0. \tag{8}$$

where C is constant of integration. According to Step 3, the solution of Eq. (8) can be expressed by a polynomial in $(G′/G)$ as follows;

$$u(\xi) = a_0 + a_1\left(G'/G\right) + a_2\left(G'/G\right)^2 + ... + a_n\left(G'/G\right)^n, \tag{9}$$

where $a_i, (i = 0, 1, 2, 3, \cdots n)$ are constants to be determined, and $G = G(\xi)$ satisfies the generalized Riccati Eq. (5). Considering the homogeneous balance between the highest order derivative and the nonlinear terms in Eq. (8), we obtain $n = 2$.

Therefore, the solution of Eq. (9) takes the form;

$$u(\xi) = a_0 + a_1\left(G'/G\right) + a_2\left(G'/G\right)^2. \tag{10}$$

Using Eq. (5), Eq. (10) can be rewritten as;

$$u(\xi) = a_0 + a_1\left(p + r\,G^{-1} + q\,G\right) + a_2\left(p + r\,G^{-1} + q\,G\right)^2. \tag{11}$$

Substituting Eq. (11) into Eq. (8), we obtain the following polynomials in G^i and G^{-i}, $(i = 0, 1, 2, 3, \cdots)$. Setting each coefficient of these resulted polynomial to zero, we obtain a set of simultaneous algebraic equations for a_0, a_1, a_2, p, q, r and V (which are omitted here for simplicity):

Solving the above system of algebraic equations, we acquire;

$$a_0 = -p^2 + 4rq,\ a_1 = 6p,\ a_2 = -6,\ V = p^4 - 8p^2qr + 16q^2r^2,$$

$$C = 32p^2q^2r^2 - 8p^4qr - \frac{128}{3}q^3r^3 + \frac{2}{3}p^6,$$

(12)

where p, q and r are arbitrary constants.

Now on the basis of the solutions of Eq. (5), we obtain the following families of solutions of Eq. (6).

Family 1: when $p^2 - 4qr < 0$ and $pq \neq 0$ (or $qr \neq 0$), the periodic form solutions of Eq. (6) are;

$$u_1 = -p^2 + 4rq + 6p\left(\frac{2\Psi^2 \sec^2(\Psi\xi)}{-p + 2\Psi \tan(\Psi\xi)}\right) - 6\left(\frac{2\Psi^2 \sec^2(\Psi\xi)}{-p + 2\Psi \tan(\Psi\xi)}\right)^2,$$

(13)

where $\Psi = \frac{1}{2}\sqrt{4qr - p^2}$, $\xi = x - \left(p^4 - 8p^2qr + 16q^2r^2\right)t$, p, q, r are arbitrary constants.

$$u_2 = -p^2 + 4rq - 6p\left(\frac{2\Psi^2 \csc^2(\Psi\xi)}{p + 2\Psi \cot(\Psi\xi)}\right) - 6\left(\frac{2\Psi^2 \csc^2(\Psi\xi)}{p + 2\Psi \cot(\Psi\xi)}\right)^2.$$

(14)

$$u_3 = -p^2 + 4rq + 6p\left(\frac{4\Psi^2 \sec(2\Psi\xi)(1 \pm \sin(2\Psi\xi))}{-p\cos(2\Psi\xi) + 2\Psi \sin(2\Psi\xi) \pm 2\Psi}\right) - 6\left(\frac{4\Psi^2 \sec(2\Psi\xi)(1 \pm \sin(2\Psi\xi))}{-p\cos(2\Psi\xi) + 2\Psi \sin(2\Psi\xi) \pm 2\Psi}\right)^2.$$

(15)

$$u_4 = -p^2 + 4rq - 6p\left(\frac{4\Psi^2 \csc(2\Psi\xi)(1 \pm \cos(2\Psi\xi))}{p\sin(2\Psi\xi) + 2\Psi \cos(2\Psi\xi) \pm 2\Psi}\right) - 6\left(\frac{4\Psi^2 \csc(2\Psi\xi)(1 \pm \cos(2\Psi\xi))}{p\sin(2\Psi\xi) + 2\Psi \cos(2\Psi\xi) \pm 2\Psi}\right)^2.$$

(16)

$$u_5 = -p^2 + 4rq - 6p\left(\frac{2\Psi^2 \csc(\Psi\xi)}{p\sin(\Psi\xi) + 2\Psi \cos(2\Psi\xi)}\right) - 6\left(\frac{2\Psi^2 \csc(\Psi\xi)}{p\sin(\Psi\xi) + 2\Psi \cos(2\Psi\xi)}\right)^2.$$

(17)

$$u_6 = -6p\left(\frac{4A\Psi^2\left\{\sqrt{A^2 - B^2}\cos(2\Psi\xi) - B\sin(2\Psi\xi) - A\right\}\left\{A\sin(2\Psi\xi) + B\right\}}{\left\{A^2\cos^2(2\Psi\xi) - A^2 - B^2 - 2AB\sin(2\Psi\xi)\right\}\left\{pA\sin(2\Psi\xi) + 2A\Psi\cos(2\Psi\xi) + pB - 2\Psi\sqrt{A^2 - B^2}\right\}}\right)$$

$$-6\left(\frac{4A\Psi^2\left\{\sqrt{A^2 - B^2}\cos(2\Psi\xi) - B\sin(2\Psi\xi) - A\right\}\left\{A\sin(2\Psi\xi) + B\right\}}{\left\{A^2\cos^2(2\Psi\xi) - A^2 - B^2 - 2AB\sin(2\Psi\xi)\right\}\left\{pA\sin(2\Psi\xi) + 2A\Psi\cos(2\Psi\xi) + pB - 2\Psi\sqrt{A^2 - B^2}\right\}}\right)^2$$

$$-p^2 + 4rq.$$

(18)

$$u_7 = -6p\left(\frac{4A\Psi^2\left\{\sqrt{A^2-B^2}\cos(2\Psi\xi)+B\sin(2\Psi\xi)+A\right\}\{A\sin(2\Psi\xi)+B\}}{\left\{A^2\cos^2(2\Psi\xi)-A^2-B^2-2AB\sin(2\Psi\xi)\right\}\left\{pA\sin(2\Psi\xi)-2A\Psi\cos(2\Psi\xi)+pB-2\Psi\sqrt{A^2-B^2}\right\}}\right)$$

$$-6\left(\frac{4A\Psi^2\left\{\sqrt{A^2-B^2}\cos(2\Psi\xi)+B\sin(2\Psi\xi)+A\right\}\{A\sin(2\Psi\xi)+B\}}{\left\{A^2\cos^2(2\Psi\xi)-A^2-B^2-2AB\sin(2\Psi\xi)\right\}\left\{pA\sin(2\Psi\xi)-2A\Psi\cos(2\Psi\xi)+pB-2\Psi\sqrt{A^2-B^2}\right\}}\right)^2 \quad (19)$$

$$-p^2+4rq,$$

wherever A and B are non-zero constants and gratifying the composition $A^2-B^2>0$.

$$u_8 = -p^2+4rq-6p\left(\frac{2\Psi^2\sec(\Psi\xi)\{p\cos(\Psi\xi)+2\Psi\sin(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)+4\Psi^2}\right)$$

$$-6\left(\frac{2\Psi^2\sec(\Psi\xi)\{p\cos(\Psi\xi)+2\Psi\sin(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)+4\Psi^2}\right)^2. \quad (20)$$

$$u_9 = -p^2+4rq+6p\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)$$

$$-6\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)^2. \quad (21)$$

$$u_{10} = -p^2+4rq-6p\left(\frac{2\Psi^2\sec(2\Psi\xi)\{1\pm\sin(2\Psi\xi)\}\{p\cos(2\Psi\xi)+2\Psi\sin(2\Psi\xi)\pm2\Psi\}}{(p^2-2rq)\cos^2(2\Psi\xi)+2\Psi\{1\pm\sin(2\Psi\xi)\}\{2\Psi\pm p\cos(2\Psi\xi)\}}\right)$$

$$-6\left(\frac{2\Psi^2\sec(2\Psi\xi)\{1\pm\sin(2\Psi\xi)\}\{p\cos(2\Psi\xi)+2\Psi\sin(2\Psi\xi)\pm2\Psi\}}{(p^2-2rq)\cos^2(2\Psi\xi)+2\Psi\{1\pm\sin(2\Psi\xi)\}\{2\Psi\pm p\cos(2\Psi\xi)\}}\right)^2. \quad (22)$$

$$u_{11} = -p^2+4rq\pm6p\left(\frac{2\Psi^2\csc(2\Psi\xi)\{-p\sin(2\Psi\xi)+2\Psi\cos(2\Psi\xi)\pm2\Psi\}}{(2rq-p^2)\cos(2\Psi\xi)-2p\{1\pm\sin(2\Psi\xi)\}\{2p\Psi\sin(2\Psi\xi)\pm2rq\}}\right)$$

$$-6\left(\frac{2\Psi^2\csc(2\Psi\xi)\{-p\sin(2\Psi\xi)+2\Psi\cos(2\Psi\xi)\pm2\Psi\}}{(2rq-p^2)\cos(2\Psi\xi)-2p\{1\pm\sin(2\Psi\xi)\}\{2p\Psi\sin(2\Psi\xi)\pm2rq\}}\right)^2. \quad (23)$$

$$u_{12} = -p^2+4rq+6p\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)$$

$$-6\left(\frac{2\Psi^2\csc(\Psi\xi)\{p\sin(\Psi\xi)-2\Psi\cos(\Psi\xi)\}}{2(p^2-2rq)\cos^2(\Psi\xi)+4p\Psi\sin(\Psi\xi)\cos(\Psi\xi)-p^2}\right)^2. \quad (24)$$

Family 2: when $p^2 - 4qr > 0$ and $pq \neq 0$ (or $qr \neq 0$), the soliton solutions of Eq. (6) are;

$$u_{13} = -p^2 + 4rq + 6p\left(\frac{2\Delta^2 \sec h^2 (\Delta\xi)}{p + 2\Delta \tanh(\Delta\xi)}\right) - \frac{3}{2}\left(\frac{2\Delta^2 \sec h^2 (\Delta\xi)}{p + 2\Delta \tanh(\Delta\xi)}\right)^2,$$

(25)

where $\Delta = \frac{1}{2}\sqrt{p^2 - 4qr}$, $\xi = x - \left(p^4 - 8p^2qr + 16q^2r^2\right)t$ and p, q, r are arbitrary constants.

$$u_{14} = -p^2 + 4rq - 6p\left(\frac{2\Delta^2 \csc h^2 (\Delta\xi)}{p + 2\Delta \coth(\Delta\xi)}\right) - 6\left(\frac{2\Delta^2 \csc h^2 (\Delta\xi)}{p + 2\Delta \coth(\Delta\xi)}\right)^2.$$

(26)

$$u_{15} = -p^2 + 4rq + 6p\left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i \sinh(2\Delta\xi))}{p\cosh(2\Delta\xi) + 2\Delta \sinh(2\Delta\xi) \pm i2\Delta}\right) - 6\left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i \sinh(2\Delta\xi))}{p\cosh(2\Delta\xi) + 2\Delta \sinh(2\Delta\xi) \pm i2\Delta}\right)^2.$$

(27)

$$u_{16} = -6p\left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{p\sinh(2\Delta\xi) + 2\Delta \cosh(2\Delta\xi) \pm 2\Delta}\right) - 6\left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{p\sinh(2\Delta\xi) + 2\Delta \cosh(2\Delta\xi) \pm 2\Delta}\right)^2$$
$$- p^2 + 4rq.$$

(28)

$$u_{17} = -p^2 + 4rq - 6p\left(\frac{\Delta^2 \sec h^2 (\Delta\xi/2)}{2\left\{\cosh^2 (\Delta\xi/2) - 1\right\}\left\{p + \Delta\left(\tanh(\Delta\xi/2) + \coth(\Delta\xi/2)\right)\right\}}\right)$$
$$- 6\left(\frac{\Delta^2 \sec h^2 (\Delta\xi/2)}{2\left\{\cosh^2 (\Delta\xi/2) - 1\right\}\left\{p + \Delta\left(\tanh(\Delta\xi/2) + \coth(\Delta\xi/2)\right)\right\}}\right)^2.$$

(29)

$$u_{18} = -6p\left(\frac{4A\Delta^2 \left\{A - B\sinh(2\Delta\xi) - \sqrt{A^2 + B^2}\cosh(2\Delta\xi)\right\}}{\left(A\sin(2\Delta\xi) + B\right)\left\{pA\sinh(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 + B^2} + 2A\Delta\cosh(2\Delta\xi)\right\}}\right)$$
$$- 6\left(\frac{4A\Delta^2 \left\{A - B\sinh(2\Delta\xi) - \sqrt{A^2 + B^2}\cosh(2\Delta\xi)\right\}}{\left(A\sin(2\Delta\xi) + B\right)\left\{pA\sinh(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 + B^2} + 2A\Delta\cosh(2\Delta\xi)\right\}}\right)^2$$
$$- p^2 + 4rq.$$

(30)

$$u_{19} = -6p\left(\frac{4A\Delta^2 \left\{A - B\sinh(2\Delta\xi) + \sqrt{A^2 + B^2}\cosh(2\Delta\xi)\right\}}{\left(A\sin(2\Delta\xi) + B\right)\left\{pA\sinh(2\Delta\xi) + pB + 2\Delta\sqrt{A^2 + B^2} + 2A\Delta\cosh(2\Delta\xi)\right\}}\right)$$
$$- 6\left(\frac{4A\Delta^2 \left\{A - B\sinh(2\Delta\xi) + \sqrt{A^2 + B^2}\cosh(2\Delta\xi)\right\}}{\left(A\sin(2\Delta\xi) + B\right)\left\{pA\sinh(2\Delta\xi) + pB + 2\Delta\sqrt{A^2 + B^2} + 2A\Delta\cosh(2\Delta\xi)\right\}}\right)^2$$
$$- p^2 + 4rq,$$

(31)

wherever A and B are non-zero constants and gratifying the composition $A^2 - B^2 < 0$.

$$u_{20} = -p^2 + 4rq - 6p\left(\frac{2\Delta^2 \sec h(\Delta\xi)}{2\Delta\sinh(\Delta\xi) - p\cosh(\Delta\xi)}\right) - 6\left(\frac{2\Delta^2 \sec h(\Delta\xi)}{2\Delta\sinh(\Delta\xi) - p\cosh(\Delta\xi)}\right)^2. \tag{32}$$

$$u_{21} = -p^2 + 4rq - 6p\left(\frac{2\Delta^2 \csc h(\Delta\xi)}{2\Delta\cosh(\Delta\xi) - p\sinh(\Delta\xi)}\right) - 6\left(\frac{2\Delta^2 \csc h(\Delta\xi)}{2\Delta\cosh(\Delta\xi) - p\sinh(\Delta\xi)}\right)^2. \tag{33}$$

$$u_{22} = 6p\left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i\sinh(2\Delta\xi))}{p\cosh(2\Delta\xi) - 2\Delta\sinh(2\Delta\xi) \mp i2\Delta}\right) - 6\left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i\sinh(2\Delta\xi))}{p\cosh(2\Delta\xi) - 2\Delta\sinh(2\Delta\xi) \mp i2\Delta}\right)^2$$
$$- p^2 + 4rq. \tag{34}$$

$$u_{23} = 6p\left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{-p\sinh(2\Delta\xi) + 2\Delta\cosh(2\Delta\xi) \pm 2\Delta}\right) - 6\left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{-p\sinh(2\Delta\xi) + 2\Delta\cosh(2\Delta\xi) \pm 2\Delta}\right)^2$$
$$- p^2 + 4rq. \tag{35}$$

$$u_{24} = -p^2 + 4rq + 6p\left(\frac{2\Delta^2 \csc h(2\xi)}{2\Delta\cosh(\Delta\xi) - p\sinh(\Delta\xi)}\right) - 6\left(\frac{2\Delta^2 \csc h(2\xi)}{2\Delta\cosh(\Delta\xi) - p\sinh(\Delta\xi)}\right)^2. \tag{36}$$

Family 3: when $r = 0$ and $pq \neq 0$, the solutions of Eq. (6) are;

$$u_{25} = -p^2 + 4rq + 6p\left(\frac{p(\cosh(p\xi) - \sinh(p\xi))}{d + \cosh(p\xi) - p\sinh(p\xi)}\right) - 6\left(\frac{p(\cosh(p\xi) - \sinh(p\xi))}{d + \cosh(p\xi) - p\sinh(p\xi)}\right)^2. \tag{37}$$

$$u_{26} = -p^2 + 4rq + 6p\left(\frac{pd}{d + \cosh(p\xi) + p\sinh(p\xi)}\right) - 6\left(\frac{pd}{d + \cosh(p\xi) + p\sinh(p\xi)}\right)^2, \tag{38}$$

where d is an arbitrary constant.

Family 4: when $q \neq 0$ and $r = p = 0$, the solutions of Eq. (6) are;

$$u_{27} = -p^2 + 4rq - 6p\left(\frac{q}{q\xi + d_1}\right) - 6\left(\frac{q}{q\xi + d_1}\right)^2, \tag{39}$$

where d_1 is an arbitrary constant.

Graphical presentation

Graphs are influential tools for communication, and depict coherently the solutions of problems. Consequently, some graphs of the solutions are given (**Figures 1 - 6**).
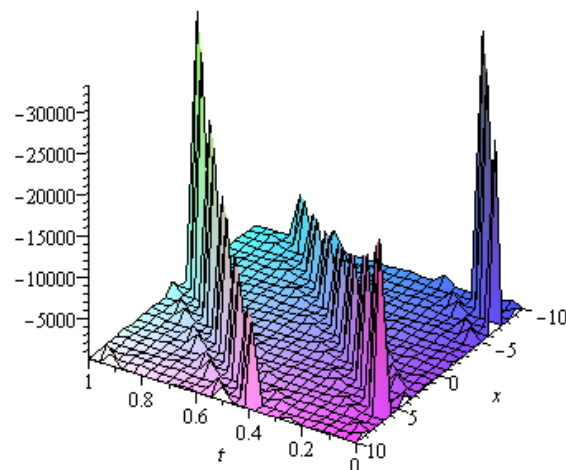
Figure 1 Represents the periodic traveling wave solution corresponding to u_1 for $p = q = r = 1$, $-10 \leq x \leq 10$ and $0 \leq t \leq 1$.
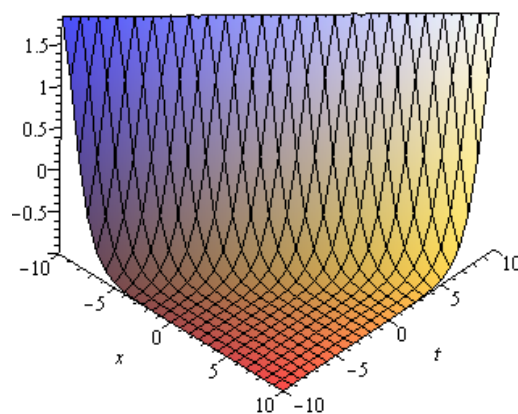

Figure 2 Represents soliton solution analogous to u_{13} for $p = 3, \ q = 2, \ r = 1$ and $-10 \leq x, t \leq 10$.

The soliton solution is a spatially confined solution, hence $u'(\xi), u''(\xi), u'''(\xi) \to 0$ as $\xi \to \pm\infty, \ \xi = x - Vt$. An amazing property of solitons is that it maintains its individuality upon interacting with other solitons.

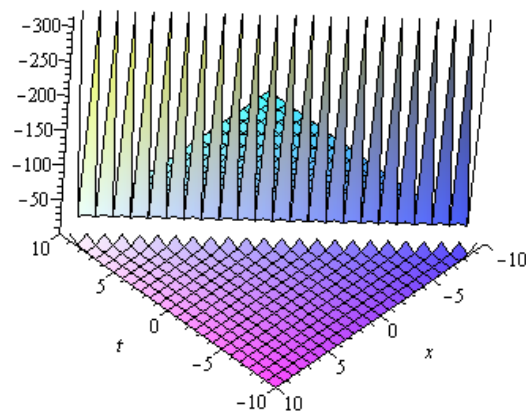**Figure 3** Represents the singular soliton solution to u_{14} for $p = 3, q = 2, r = 1$ and $-10 \le x, t \le 10$.
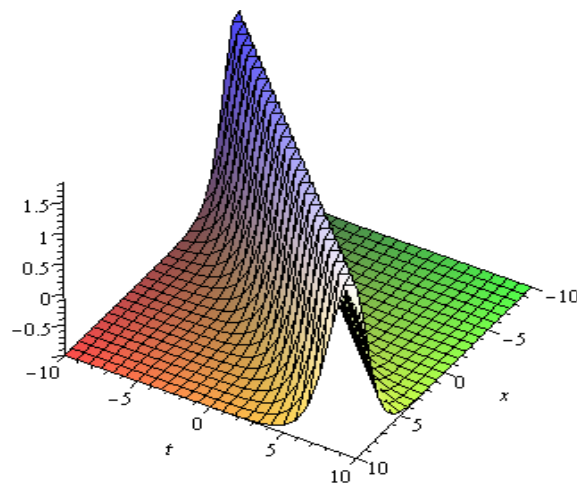

Figure 4 Represents **s**oliton solution corresponding to u_{20} for $p = 3, \; q = 1, \; r = 2$ and $-10 \le x, t \le 10$. The soliton solution is a spatially restricted solution, hence $u'(\xi), u''(\xi), u'''(\xi) \to 0$ as $\xi \to \pm\infty$.
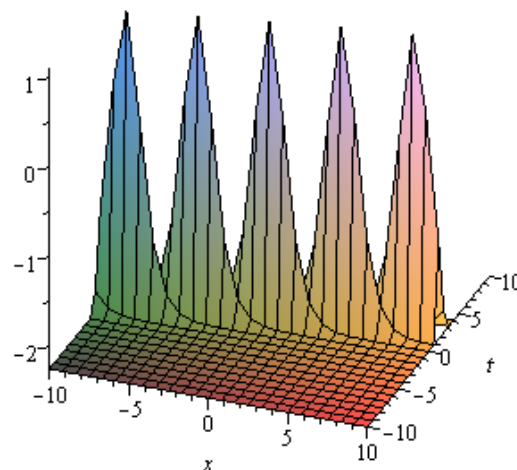
Figure 5 Represents soliton solution corresponding to solution u_{26} for $p = 1.5,\ q = 1,\ r = 0$ and $-10 \le x, t \le 10$.
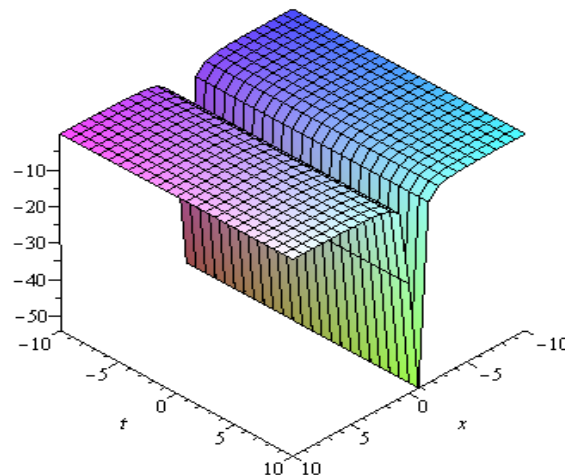

Figure 6 Represents the singular soliton solution corresponding to solution u_{27} for $p = 0,\ q = 1,\ r = 0$ and $-10 \le x, t \le 10$.

Discussion

In this article, an alternative (*G'/G*)-expansion method together with the generalized Riccati equation is suggested and applied to construct the copious new exact traveling wave solutions of CDGSK equation. It is obvious that the obtained solutions are more general with more arbitrary constants. The copious traveling wave solutions including; trigonometric function solutions, hyperbolic function solutions, and rational solutions are obtained in the form of four different families. The projected method is relatively resourceful and virtually well-matched to be applied in finding exact solutions of nonlinear evolution equations.

Conclusions

The alternative (*G'/G*)-expansion method, along with the generalized Riccati equation, was used in this work for searching abundant exact traveling wave solutions to the equation, with the help of symbolic computation, such as Maple. We assured the correctness of our solutions by putting them back into the original Eq. (6). The new type of traveling wave solutions found in this technique may be applicable to other kinds of NLEEs in mathematical physics.

References

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