WALAILAK JOURNAL
http://wjst.wu.ac.th Applied Mathematics

Series Solution for Painlevé Equation II

Fazle MABOOD[1,*], Waqar Ahmad KHAN[2], Ahmad Izani Md ISMAIL[1] and Ishak HASHIM[3]

[1] *School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia*
[2] *Department of Engineering Sciences, National University and Technology, Karachi, Pakistan*
[3] *School of Mathematical Sciences, Universiti Kebangsaan Malaysia, Malaysia*

([*] Corresponding author's e-mail: mabood1971@yahoo.com)

Received: 23 July 2013, Revised: 28 November 2013, Accepted: 29 December 2013

Abstract

The Painlev'e equations are second order ordinary differential equations which can be grouped into six families, namely Painlev'e equation I, II,…, VI. This paper presents the series solution of second Painlevé equation via optimal homotopy asymptotic method (OHAM). This approach is highly efficient and it controls the convergence of the approximate solution. Comparison of the obtained solution via OHAM is provided with those obtained by Homotopy Perturbation Method (HPM), Adomian Decomposition Method (ADM), Sinc-collocation and Runge-Kutta 4 methods. It is revealed that there is an excellent agreement between OHAM and other published data which confirm the effectiveness of the OHAM.

Keywords: Optimal homotopy asymptotic method, Painlevé equation, Nonlinear ODE

Introduction

As is well known, the 6 Painlevé equations were formulated by Paul Painlevé and colleagues in relation to their study of nonlinear second-order ordinary differential equations in the late nineteenth century. The nonlinear Painlevé equations and their solutions are important and arise in several areas of mathematics and physics.

In recent years, several approximate analytical and numerical techniques have been used by a number of researchers for Painlevé equations. For instance, Dehghan and Shakeri [1] used the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Legendre Tau Method for the approximate solution of Painlevé equation II, Ellahi *et al.* [2] used the Homotopy Analysis Method (HAM) and made comparisons with ADM and HPM for the same equation, whilst Saadatmandi [3] utilized the Sinc-collocation and Variational Iteration Method (VIM) with Padé for the study of Painlevé equation II. Behzadi [4] used a variety of approximate analytical methods and their modifications for Painlevé equation I, while Esmail and Peyrovi [5] applied the Variational Iteration and Homotopy Perturbation Methods for the approximate solution of Painlevé equation I. Esmail and Peyrovi [6] conducted a comparative study of Painlevé equation II by using the HPM, Analytic Continuation Extension (ACE) and Chebyshev Series Method (CSM). Bassom *et al.* [7], Clenshaw and Norton [8], and Dastidar and Majumdar [9] used CSM for numerical studies of the Painlevé equation IV.

The Painlevé equation II is given as;

$$\frac{d^2u}{dx^2} = 2u^3 + xu + \mu \tag{1}$$

with the given initial conditions being $u(0)=1,\ u'(0)=0$, where μ is a parameter.

Generally, Painlevé equations have arisen in applications including statistical mechanics, plasma physics, nonlinear waves, and others [1]. In particular, Painlevé equation II has attracted much interest, because of reduction of the soliton equations, such as the Korteweg-de Vries, Boussinesq, and nonlinear Schrödinger equations into Painlevé equation II [1]. Mazzocco and Mo [10] cite papers which study Painlevé II in areas of applications such as Hele-Shaw geometry, nonlinear optics, and random matrix theory.

Optimal homotopy asymptotic method (OHAM), proposed by Marinca *et al.* [11], is a powerful method and has been effectively employed by numerous researchers on various nonlinear problems [12,13]. The objective of this paper is to propose an accurate procedure to nonlinear differential equation namely Painlevé equation II using OHAM. OHAM is applied in this study to derive highly accurate analytical expressions of the solution. OHAM does not depend upon any small or large parameters as other known methods in the literature [6]. The main advantage of OHAM is the control of the convergence of the approximate solutions in a very rigorous way. A very good agreement was found between our approximate analytical solution and other numerical solution, which proves that our method is very effective and accurate.

The paper is structured as follows: first, in Section 2, we discuss the basic principles of OHAM. The Painlevé equation II is obtained via OHAM in Section 3. Section 4 is reserved for results and discussion. Conclusions are drawn in Section 5.

Basic principles of OHAM

We now review the basic principles of OHAM, as expounded by Marinca *et al.* [11] and other papers, in the following steps:

(i) Let us consider the following differential equation;

$$A\left[\mathrm{w}\left(\tau\right)\right]+h\left(\tau\right)=0, \qquad \tau\in\Omega, \tag{2}$$

where Ω is problem domain, $A\left(\mathrm{w}\right)=L\left(\mathrm{w}\right)+N\left(\mathrm{w}\right)$, where L, and N are linear and nonlinear operators respectively, $\mathrm{w}\left(\tau\right)$ is an unknown function, and $h\left(\tau\right)$ is a known function,

(ii) We construct an optimal homotopy equation as follows:

$$\left(1-r\right)\left[L\left(\phi\left(\tau;r\right)\right)+h\left(\tau\right)\right]-H\left(r\right)\left[A\left(\phi\left(\tau;r\right)\right)+h\left(\tau\right)\right]=0, \tag{3}$$

where $0\le r\le 1$ is an embedding parameter, and $H\left(r\right)=\sum_{t=1}^{s} r^t C_t$ is an auxiliary function on which the convergence of the solution is dependent. The auxiliary function $H\left(r\right)$ serves also to adjust the convergence domain, as well as control the convergence region.

(iii) If $\phi\left(\tau;r,C_j\right)$ is expanded in a Taylor's series about r, the following approximate solution is obtained;

$$\phi\left(\tau;r,C_j\right)=\mathrm{w}_0\left(\tau\right)+\sum_{t=1}^{\infty}\mathrm{w}_k\left(\tau,C_j\right)r^t, \qquad j=1,2,3,..., \tag{4}$$

It has been observed by previous researchers that the convergence of the series Eq. (4) depends upon C_j, $\left(j=1,2,...\right)$. If it is convergent, then;

$$\tilde{\mathrm{w}}=w_0\left(\tau\right)+\sum_{t=1}^{s}\mathrm{w}_k\left(r,C_j\right), \tag{5}$$

is obtained.

(iv) Substituting Eq. (5) in Eq. (2), results in the following residual;

$$R(\tau;C_j)=L(\tilde{\mathrm{w}}(\tau;C_j))+h(\tau)+N(\tilde{\mathrm{w}}(\tau;C_j)). \tag{6}$$

If $R(\tau;C_j)=0$, then $\tilde{\mathrm{w}}$ will be the exact solution although, for nonlinear problems, this will not usually be the case. For determining C_j, $(j=1,2,...)$ methods such as Galerkin's method, or the method of least squares, can be utilized.

(v) Substitution of these constants into Eq. (5) results in the approximate solution.

OHAM solution of Painlevé equation II

According to OHAM, we obtain:
Zeroth order problem with initial conditions;

$$u_0''(x)-\mu=0, \qquad u_0(0)=1,\ u_0'(0)=0. \tag{7}$$

Its solution is;

$$u_0(x)=1+\frac{\mu x^2}{2}. \tag{8}$$

First order problem with initial conditions;

$$u_1''(x,C_1)+\mu+\mu C_1+xC_1u_0(x)+2C_1u_0^3(x)-(1+C_1)u_0''(x)=0, \qquad u_1(0)=0,\ u_1'(0)=0. \tag{9}$$

Its solution is;

$$u_1(x,C_1)=\frac{x^2C_1}{3360}\left(-3360-560x-840x^2\mu-84x^3\mu-168x^4\mu^2-15x^6\mu^3\right). \tag{10}$$

Second order problem with initial conditions;

$$\begin{aligned}&u_2''(x,C_1,C_2)+\mu C_2+xC_2u_0(x)+2C_2u_0^3(x)+xC_1u_1(x,C_1)\\&\quad+6C_1u_0^2(x)u_1(x,C_1)-C_2u_0''(x)+(1+C_1)u_1''(x,C_1)=0, \qquad u_2(0)=0,\ u_2'(0)=0.\end{aligned} \tag{11}$$

Its solution is;

$$u_2(x,C_1,C_2)=\frac{1}{201801600}\left(\begin{array}{l}201801600x^2C_1+33633600x^3C_1+50450400x^4\mu C_1+5045040x^5\mu C_1\\+10090080x^6\mu^2C_1+900900x^8\mu^3C_1+201801600x^2C_1^2+33633600x^3C_1^2\\+100900800x^4C_1^2+20180160x^5C_1^2+1121120x^6C_1^2+50450400x^4\mu C_1^2\\+5045040x^5\mu C_1^2+50450400x^6\mu C_1^2+6726720x^7\mu C_1^2+90090x^8\mu C_1^2\\+10090080x^6\mu^2C_1^2+11891880x^8\mu^2C_1^2+1261260x^9\mu^2C_1^2+900900x^8\mu^3C_1^2\\+1573572x^{10}\mu^3C_1^2+76986x^{11}\mu^3C_1^2+155610x^{12}\mu^4C_1^2+7425x^{14}\mu^5C_1^2\\-201801600x^2C_2-33633600x^3C_2-50450400x^4\mu C_2-5045040x^5\mu C_2\\-10090080x^6\mu^2C_2-900900x^8\mu^3C_2\end{array}\right). \tag{12}$$

Third order problem with initial conditions;

$$u_3''\left(x,C_{1,},C_2,C_3\right)+\mu C_3+xC_3u_0\left(x\right)+2C_3u_0^3\left(x\right)+xC_2u_1\left(x,C_1\right)+6C_2u_0^2\left(x\right)u_1\left(x,C_1\right)$$
$$+6C_1u_0\left(x\right)u_1^2\left(x,C_1\right)+xC_1u_2\left(x,C_1,C_2\right)+6C_1u_0^2\left(x\right)u_2\left(x,C_1,C_2\right)-C_3u_0''\left(x\right) \tag{13}$$
$$-C_2u_1''\left(x,C_1\right)-\left(1+C_1\right)u_2''\left(x,C_1,C_2\right)=0, \qquad u_3(0)=0,\ u_3'(0)=0.$$

Its solution in a long expression with few terms is;

$$u_3\left(x,C_{1,},C_2,C_3\right)=\frac{1}{93861960192000}\begin{pmatrix}-93861960192000x^2C_1-15643660032000x^3C_1\\-23465490048000x^4\mu C_1-2346549004800x^5\mu C_1\\+...+6907032000x^{14}\mu^5C_1C_2-419026608000x^8\mu^3C_3\end{pmatrix}. \tag{14}$$

The four terms solution via OHAM for $p=1$ is;

$$\tilde{u}(x,C_1,C_2,C_3)=u_0(x)+u_1(x,C_1)+u_2(x,C_1,C_2)+u_3(x,C_1,C_2,C_3). \tag{15}$$

We use the method of least squares to obtain the values of C_1, C_2, C_3, the unknown convergent constants in $\tilde{u}$. The values of C_1, C_2, C_3 for $\mu=1$ are;

$C_1=-1.0203159608885928,\ C_2=0.0414450982244707,\ C_3=0.04146020791680742.$

By substituting these constants in Eq. (15), we obtain the OHAM third order approximate solution.

Results and discussion

In order to show the feasibility of the mentioned method, the obtained series solution is compared with existing solutions. Comparison of the solution via OHAM is made with those obtained by HPM, ADM [1] and the Sinc-collocation method [3] for $\mu=1$. It is clear that the solution using OHAM is in good agreement with published results for Painlevé equation II. In **Table 1**, the comparison of results obtained by OHAM and the solution obtained by HPM, ADM and the Sinc-collocation method for $\mu=1$, **Table 2** compares the solution obtained by OHAM with those obtained by ACM and RK4 [6] for $\mu=5$ confirming the validity of the proposed method.

Figure 1 shows that the present solutions are exactly the same as those obtained by Esmail and Peyrovi [6] for $\mu=2$. The results indicate to us that OHAM is a highly effective tool for solving a mathematical model in the form of Eq. (1).

Table 1 Comparison of solutions using OHAM with other published data [1,3] for $\mu = 1$.

x	ADM & HPM [1]	Sinc collocation [3]	OHAM
0.05	1.003775569	1.003775662	1.003775589
0.10	1.015243537	1.015243802	1.015243588
0.15	1.034708876	1.034708564	1.034708856
0.20	1.062614651	1.062615730	1.062614821
0.25	1.099567603	1.099567603	1.099567581
0.30	1.146376034	1.146376034	1.146376031
0.35	1.204104480	1.204104479	1.204104477
0.40	1.274152285	1.274152278	1.274152281
0.45	1.358367366	1.358367333	1.358367341
0.50	1.459213448	1.459213319	1.459213374
0.60	1.725375546	1.725374098	1.725374318
0.70	2.118443462	2.118431139	2.118435214

Table 2 Comparison of solutions using OHAM with other published data [6] for $\mu = 5$.

x	OHAM	ACE [6]	RK4 [6]
-0.7	3.6785	3.6952	3.6987
-0.6	2.6349	2.6385	2.6352
-0.5	2.0139	2.0157	2.0139
-0.4	1.6053	1.6062	1.6053
-0.3	1.3263	1.3268	1.3263
-0.2	1.1416	1.1418	1.1416
-0.1	1.0350	1.0351	1.0351
0.0	1.0000	1.0000	1.0000
0.1	1.0353	1.0353	1.0353
0.2	1.1444	1.1444	1.1442
0.3	1.3365	1.3366	1.3366
0.4	1.6322	1.6323	1.6323
0.5	2.0757	2.0757	2.0757
0.6	2.7700	2.7705	2.7705
0.7	3.9850	4.0021	4.0022

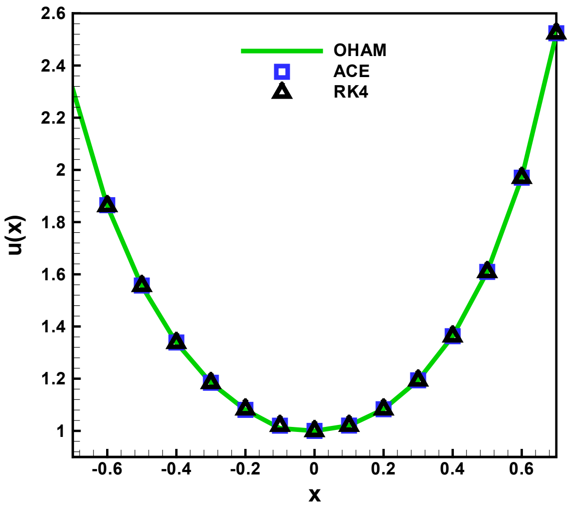

Figure 1 Comparison of solutions using OHAM, ACE [6] and RK4 [6] for $\mu = 2$.

Conclusions

The OHAM is employed to propose a new approximate analytical solution for Painlevé equation II. Our procedure is valid even if the nonlinear differential equation does not contain any small or large parameter. OHAM provides us with a simply and rigorous way to control and adjust the convergence of the solution through the auxiliary function $H(r)$ involving several parameters $C_1, C_2, ...$ which are optimally determined. Actually, the strong point of OHAM is its fast convergence after two or three iteration which proves our procedure is effective in practice. The solution obtained using OHAM is in good agreement with other existing numerical solutions, and thus indicates OHAM's feasibility.

References

[1] M Dehghan and F Shakeri. The numerical solution of the second Painlevé equation. *Numer. Meth. Part. Diff. Equat.* 2008; **25**, 1238-59.
[2] R Ellahi, S Abbasbandy, T Hayat and A Zeeshan. On comparison for series and numerical solutions for second Painlevé equation II. *Numer. Meth. Part. Diff. Equat*. 2009; **26**, 1070-8.
[3] A Saadatmandi. Numerical study of second Painlevé equation. *Comm. Numer. Anal.* 2012; **2012**, cna-00157.
[4] SS Behzadi. Convergence of iterative methods for solving Painlevé equation. *Appl. Math. Sci*. 2010; **4**, 1489-507.
[5] H Esmail and A Peyrovi. The use of variational iteration method and homotopy perturbation method for Painlevé equation I. *Appl. Math. Sci.* 2009; **3**, 1861-71.
[6] H Esmail and A Peyrovi. Homotopy perturbation method for second Painlevé equation and comparison with analytic continuation method and Chebyshev series method. *Int. Math. Forum*. 2010; **5**, 629-37.
[7] AP Bassom, PA Clarkson and AC Hicks. Numerical studies of the fourth Painlevé equation. *J. Appl. Math*. 1993; **50**, 167-93.
[8] CW Clenshaw and HJ Norton. The solutions of non-linear ordinary differential equations in Chebyshev Series. *Comput. J.* 1963; **6**. 88-92.
[9] DG Dastidar and SK Majumdar. The solution of Painlevé equations in Chebyshev series. *F.C. Auluk, F.N.* 1972; **4**, 155-60.

[10] M Mazzocco and MY Mo. The Hamiltonian structure of the second Painlevé hierarchy. *Nonlinearity* 2007; **20**, 2845-82.
[11] V Marinca, N Herisanu and I Nemes. Optimal homotopy asymptotic method with application to thin film flow. *Cent. Eur. J. Phys*. 2008; **6**, 648-53.
[12] F Mabood, AIM Ismail and I Hashim. The application of optimal homotopy asymptotic method for the approximate solution of Riccati equation. *Sains Malays.* 2013; **42**, 863-7.
[13] MS Hashmi, N Khan and S Iqbal. Optimal Homotopy asymptotic method for solving nonlinear Fredholm integral equations of second kind. *Appl. Math. Comput*. 2012; **218**, 10982-9.