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Closed-Form Expansions for Transition Densities of Convenience Yield Processes

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ABSTRACT

In this paper, we derive closed-form expansions for transition densities of convenience yield processes modeled by using extended Cox-Ingersoll-Ross (ECIR) processes. The closed-form expansions can be used for all cases of model having appropriate drift and diffusion coefficients. In terms of the efficiency, the closed-form expansions are close to the exact transition densities when the number of terms in the expansions is large and the time step size is small.

Keywords: Closed-form expansions, transition densities, convenience yield, ECIR processes

INTRODUCTION

In the literature of commodity price modeling based on continuous-time stochastic processes [1-3], the assumption of the models is that the commodity spot price process $(S_t)_{t\in[0,T]}$ and the instantaneous convenience yield process $(\delta_t)_{t\in[0,T]}$, under an equivalent martingale measure $\mathbb{Q}$, follow the stochastic differential equations (SDEs) of the following forms:

$$dS_t = \mu_S(S_t, \delta_t; \theta)\, dt + \sigma_S(S_t, \delta_t; \theta) dW_t^{(1)}, \tag{1}$$

$$d\delta_t = \mu_\delta(t, \delta_t; \theta)\, dt + \sigma_\delta(t, \delta_t; \theta)\, dW_t^{(2)}, \tag{2}$$

where $W \equiv (W_t^{(1)}, W_t^{(2)})_{t\in[0,T]}$, $T > 0$, is a two-dimensional Brownian motion under a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The functions μ_S, μ_δ, σ_S, and σ_δ are the drifts and the volatilities, respectively, of the 2 processes. In order to compute the expectation of the commodity spot price at a final time *T*, i.e. $E_{\mathbb{Q}}[S_T]$, employing Monte Carlo simulations, the sample paths of the 2 processes in the time interval $[0, T]$ must be generated by using traditional numerical schemes such as a *Milstein scheme*.

This means the model parameter vector $\theta \in \mathbb{R}^p$ for some positive integer p has to be known before plugging its value into the recursive formulas of (1) and (2) which are derived from a selected numerical scheme. In general, the true parameter vector θ, however, is unknown and its components depend on the price characteristics of a selected commodity. Hence, in order to obtain consistent estimators of the true parameters, maximum likelihood (ML) estimation is a method of choice.

Suppose that μ_δ and σ_δ depend on all components of θ. In other words, it is sufficient to construct maximum likelihood estimators (MLEs) of the unknown parameters using only a discrete sample data of δ_t, i.e. $\delta_{t_0}, \delta_{t_1} \ldots, \delta_{t_N}$, at observed dates $\{t_n = n\Delta t \mid n = 0,1,\ldots,N\}$ for some positive integer N and $\Delta t > 0$. Therefore, if we know the transition (probability) density of the Markov process δ_t, using Bayes's rule, we can get the log-likelihood function of the simple form:

$$L_N(\theta) := \sum_{n=1}^{N} \ln\left\{p_\delta(\delta_{t_n}, t_n \mid \delta_{t_{n-1}}, t_{n-1}; \theta)\right\}, \tag{3}$$

where $p_\delta(\cdot \mid \cdot;\theta)$ denotes the transition density of δ_t. The vector of MLEs refers to the one that maximizes L_N over a parameter space Θ. When δ_t is assumed to follow a well-known model, the transition density $p_\delta(\cdot \mid \cdot;\theta)$ exists in closed-form expression. For example in the Black-Scholes model (the geometric Brownian motion), the Vasiček model (the Ornstein-Unlenbeck process), and the Cox-Ingersoll-Ross model (the CIR process), the transition densities of these processes exist in closed form expressions [4]. However, in the case that δ_t is assumed to follow an extended CIR (ECIR) process which can be written in a general form as follows:

$$d\delta_t = \kappa(t;\theta)(\alpha(t;\theta) - \delta_t)dt + \sigma(t;\theta)\sqrt{\delta_t}\, dW_t^{(2)}, \tag{4}$$

for some given time-dependent functions $\kappa(\cdot;\theta)$, $\alpha(\cdot;\theta)$, $\sigma(\cdot;\theta)$, the true transition densities and their closed-form expansions are mostly unknown. The model (4) in the case that $\kappa(t;\theta) \equiv \kappa$, $\sigma(\cdot;\theta) \equiv \sigma$ are assumed to be constants and $\alpha(t;\theta)$ is a parameterized linear combination of trigonometric functions, is used in Rujivan [3] in order to describe seasonal variation in commodity prices. Thus, the derivation of the closed-form expansions of the transition densities of the convenience yield processes is necessary for the estimation of the model parameters based on ML estimation.

Recently, Aït-Sahalia [5] gave a method to get the approximate transition densities in closed forms by using Hermite expansions in the case of univariate time-homogeneous diffusion models. Aït-Sahalia's work has been extended by Schaumburg [6] to univariate jump diffusions. Egorov and co-workers [7] extended Aït-Sahalia's work to univariate time-inhomogeneous diffusion models. Closed-form likelihood expansions for multivariate diffusions have also been obtained by Aït-Sahalia [8], which was generalized to multivariate jump diffusions by Yu [9]. In 2005, Choi [10] extended the results to multivariate time-inhomogeneous cases.

The purpose of this research is to derive a closed-form expansion for transition densities of convenience yield processes modeled by an ECIR model. The method we used is called "Choi's method" [10]. The obtained approximate transition densities can be used for all cases of models having appropriate drift and diffusion coefficients included in the convenience yield processes.

The remainder of this paper is organized as follows. In Section 2, we set up the model and assumptions. In Section 3, Choi's method for univariate time-inhomogeneous diffusions is reviewed. The closed-form expansions for transition densities of convenience yield processes obtained by using Choi's method are derived in Section 4 and Mathematica codes for computing the coefficients of the expansions are provided. The efficiency of the approximate transition densities is demonstrated and discussed with examples. Finally, Section 5 concludes the paper.

SETUP AND ASSUMPTIONS

Consider the univariate time-inhomogeneous diffusion

$$dX_t = \mu_X(X_t, t; \theta)dt + \sigma_X(X_t, t; \theta)dW_t, \tag{5}$$

where W_t is the standard one-dimensional Brownian motion, and $\theta \in \mathbb{R}^p$ is a vector of the model parameters that need to be estimated. The drift and diffusion terms, i.e. $\mu_X(\cdot;\theta)$ and $\sigma_X(\cdot;\theta)$, respectively, are functions depending not only on X_t and θ, but also time *t*. Note that by setting $\mu_X \equiv \kappa(t;\theta)(\alpha(t;\theta) - X_t)$ and $\sigma_X \equiv \sigma(t;\theta)\sqrt{X_t}$ in (5), the obtained process is a general form for ECIR processes which are often used to describe the dynamics of a convenience yield process.

Let Θ be a compact subset of $\mathbb{R}^p$ and $D_X := (0,\infty)$ denote the domain of the diffusion process X_t. We modify the regularity conditions of Egorov and co-workers [7] to suit ECIR processes and these conditions guarantee the existence and uniqueness of the strong solution of the SDE (5), and hence, the transition density of X_t is unique.

Assumption 1 (Smoothness of the Drift and Diffusion Coefficients)

Functions $\kappa(t;\theta)$, $\alpha(t;\theta)$, and $\sigma(t;\theta)$ are infinitely differentiable in $t \in [0,\infty)$ and twice continuously differentiable in $\theta \in \Theta$.

Assumption 2 (Non-degeneracy of the Diffusions)

The diffusion coefficient $\sigma_X(x,t;\theta) = \sigma(t;\theta)\sqrt{x}$ is non-degenerate away from $x = 0$: there exists a constant c such that $\sigma_X(x,t;\theta) > c > 0$ for all $(x,t,\theta) \in D_X \times [0,\infty) \times \Theta$. However, $\sigma_X(x,t;\theta)$ can be locally degenerate at $x = 0$: (i) for any positive T and ξ, there exists a constant c_ξ such that $\sigma_X(x,t;\theta) > c_\xi > 0$ for all $(x,t,\theta) \in [\xi,\infty)\times[0,T]\times\Theta$, and (ii) if $\sigma_X(0,t;\theta)=0$ for some positive t, then for any positive T there exist positive constants ξ_0,ω,ρ such that $\sigma_X(x,t;\theta) \geq \omega x^\rho$ for all $(x,t,\theta)\in[\xi_0,\infty)\times[0,T]\times\Theta$.

MATERIALS AND METHODS

In this section we review Choi's method which will be used to derive closed-form likelihood expansions. We need to transform the original diffusion process X_t in order to make the transition density of the transformed diffusion process close to the Normal. Using the Jacobian formula, the transition density of X_t can be retrieved.

We transform the univariate diffusion process (5) to a unit diffusion process (7) by using the following transformation from X to Y,

$$y := \gamma(x,t;\theta) = \int^x \frac{1}{\sigma_X(\omega,t;\theta)}d\omega. \tag{6}$$

Using the Itô formula,

$$dY_t = \mu_Y(Y_t,t;\theta)dt + dW_t, \tag{7}$$

where

$$\mu_Y(y,t;\theta) = \frac{\mu_X(\gamma^{-1}(y,t;\theta),t;\theta)}{\sigma_X(\gamma^{-1}(y,t;\theta),t;\theta)} + \frac{\partial\gamma}{\partial t}(\gamma^{-1}(y,t;\theta),t;\theta) - \frac{1}{2}\frac{\partial\sigma_X}{\partial x}(\gamma^{-1}(y,t;\theta),t;\theta). \tag{8}$$

Let p_X and p_Y be transition densities of X_t and Y_t, respectively. These 2 transition densities satisfy the Fokker Planck equations:

$$\frac{\partial p_X}{\partial t} + \frac{\partial}{\partial x}(\mu_X p_X) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma_X^2 p_X) = 0 \text{ and } \frac{\partial p_Y}{\partial t} + \frac{\partial}{\partial y}(\mu_Y p_Y) - \frac{1}{2}\frac{\partial^2 p_Y}{\partial y^2} = 0.$$

Suppose that Assumptions 1 and 2 are fulfilled. The following theorem proved in [10] shows that $l_X \equiv \ln p_X$ and $l_Y \equiv \ln p_Y$ have closed-form expansions.

Theorem 1 (Applications of Theorem 3.2 in [10])

Let $\triangle t = t - t_0$. The closed-form approximate logarithm of the transition density for time-inhomogeneous diffusion process Y_t is given by

$$l_Y(y,t \mid y_0,t_0;\theta) = -\frac{1}{2}\ln(2\pi\Delta t) + \frac{C_Y^{(-1)}(y,t \mid y_0,t_0;\theta)}{\Delta t} + \sum_{k=0}^{\infty} C_Y^{(k)}(y,t \mid y_0,t_0;\theta)\frac{\Delta t^k}{k!}, \quad (9)$$

where the coefficient functions are

$$C_Y^{(-1)}(y,t \mid y_0,t_0;\theta) = -\frac{(y-y_0)^2}{2}, \quad (10)$$

$$C_Y^{(0)}(y,t \mid y_0,t_0;\theta) = (y-y_0)\int_0^1 \mu_Y(y_0+u(y-y_0),t;\theta)\,du. \quad (11)$$

Given $C_Y^{(-1)}, C_Y^{(0)},\ldots,C_Y^{(k-1)}$, *the other coefficients* $C_Y^{(k)}, k \geq 1$ *can be computed recursively by*

$$C_Y^{(k)}(y,t \mid y_0,t_0;\theta) = k\int_0^1 G_Y^{(k)}(y_0+u(y-y_0),t \mid y_0,t_0;\theta)u^{k-1}du, \quad (12)$$

where

$$G_Y^{(1)}(y,t \mid y_0,t_0;\theta) = -\frac{1}{2}\left[\mu_Y^2(y,t;\theta) + \frac{\partial\mu_Y(y,t;\theta)}{\partial y}\right] - \int_{y_0}^{y}\frac{\partial\mu_Y(\omega,t;\theta)}{\partial t}d\omega, \quad (13)$$

and for $k \geq 2$

$$\begin{aligned} G_Y^{(k)}(y,t \mid y_0,t_0;\theta) = {} & \frac{1}{2}\frac{\partial^2 C_Y^{(k-1)}(y,t \mid y_0,t_0;\theta)}{\partial y^2} - \frac{\partial C_Y^{(k-1)}(y,t \mid y_0,t_0;\theta)}{\partial t} \\ & + \frac{1}{2}\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial C_Y^{(h)}(y,t \mid y_0,t_0;\theta)}{\partial y}\frac{\partial C_Y^{(k-1-h)}(y,t \mid y_0,t_0;\theta)}{\partial y}. \end{aligned} \quad (14)$$

Moreover, we have the closed-form approximate logarithm of the transition density for time-inhomogeneous diffusion process X_t,

$$\begin{aligned} l_X(x,t \mid x_0,t_0;\theta) = {} & -D_v(x,t;\theta) - \frac{1}{2}\ln(2\pi\Delta t) + \frac{C_Y^{(-1)}(\gamma(x,t;\theta),t \mid \gamma(x_0,t_0;\theta),t_0;\theta)}{\Delta t} \\ & + \sum_{k=0}^{\infty} C_Y^{(k)}(\gamma(x,t;\theta),t \mid \gamma(x_0,t_0;\theta),t_0;\theta)\frac{\Delta t^k}{k!}, \end{aligned} \quad (15)$$

where the term $D_v(x,t;\theta) = \frac{1}{2}\ln(\sigma_X^2(x,t;\theta)) = \frac{1}{2}\ln(\sigma^2(t;\theta)\,x)$, *arises by the Jacobian formula.*

This theorem says that l_X can be approximated by the right hand side of (15) when $\Delta t \downarrow 0$ and $K \uparrow \infty$ where K is a positive integer denoting the number of terms in the series that we use to approximate the function on the left hand side.

RESULTS AND DISCUSSION

Closed-Form Likelihood Expansions for Convenience Yield Processes

Now we use the results obtained in the previous section to derive closed-form expansions for the transition densities of convenience yield processes. Given a convenience yield process X_t in which its dynamics follows the SDE (4):

$$dX_t = \kappa(t;\theta)(\alpha(t;\theta) - X_t)dt + \sigma(t;\theta)\sqrt{X_t}\,dW_t. \tag{16}$$

From (6)-(8), we have

$$y = \gamma(x,t;\theta) = \int^x \frac{1}{\sigma(t;\theta)\sqrt{\omega}}d\omega = \frac{2\sqrt{x}}{\sigma(t;\theta)}, \tag{17}$$

$$x = \gamma^{-1}(y,t;\theta) = \frac{1}{4}\sigma^2(t;\theta)y^2, \tag{18}$$

$$\mu_Y(y,t;\theta) = \frac{4\alpha(t;\theta)\kappa(t;\theta) - \sigma(t;\theta)(\sigma(t;\theta) + (\alpha(t;\theta)\kappa(t;\theta) + 2\sigma'(t;\theta))y^2)}{2\sigma^2(t;\theta)y}. \tag{19}$$

The leading coefficient $C_Y^{(-1)}(y,t \mid y_0,t_0;\theta)$ is given in (10). Applying Theorem 1, we then obtain the second coefficient:

$$C_Y^{(0)}(y,t \mid y_0,t_0;\theta) = \frac{1}{4\sigma^2(t;\theta)}\left(8\alpha(t;\theta)\kappa(t;\theta)\ln\left(\frac{y}{y_0}\right) - \sigma(t;\theta)\left(\left(2\ln\left(\frac{y}{y_0}\right) + \kappa(t;\theta)(y^2 - y_0^2)\right)\sigma(t;\theta) + 2(y^2 - y_0^2)\sigma'(t;\theta)\right)\right). \tag{20}$$

The remaining coefficients $C_Y^{(1)},\ldots,C_Y^{(K)}$ for some positive integer $K \geq 1$, can be obtained by running the Mathematica codes shown in **Figure 1**. Note that, we omit writing the argument θ in the Mathematica codes to avoid tedious expressions. In **Figure 2**, we show the result obtained by running the Mathematica codes for computing the coefficient $C_Y^{(1)}$ in closed form. Unfortunately, we cannot display all of the coefficients $C_Y^{(k)}, k \geq 2$ within this paper due to limited space. In addition, the closed-form expansion of the logarithm of the transition density of X_t can be obtained using (15).

(*Mathematica Codes*)

```
μY[y_, t_] := (4 α[t] κ[t] - σ[t] (σ[t] + y^2 κ[t] σ[t] + 2 y^2 σ'[t])) / (2 y σ[t]^2);
CY[-1][y_, t_, y0_, t0_] := -1/2 (y - y0)^2;
CY[0][y_, t_, y0_, t0_] :=
  1/(4 (σ[t])^2)
   (8 (Log[y] - Log[y0]) α[t] κ[t] +
     σ[t] ((-2 Log[y] + 2 Log[y0] + (-y^2 + y0^2) κ[t]) σ[t] + 2 (-y^2 + y0^2) σ'[t]));
CY[k_][y_, t_, y0_, t0_] :=
  k (Limit[(∫ ((GY[k][y0 + u (y - y0), t, y0, t0])) u^(k - 1) du), u → 1] -
     (Limit[(∫ ((GY[k][y0 + u (y - y0), t, y0, t0])) u^(k - 1) du), u → 0])) /; k ≥ 1;

GY[1][y_, t_, y0_, t0_] :=
  (-∂z μY[z, s] - ∂s CY[0][z, s, z0, s0] - μY[z, s] ∂z CY[0][z, s, z0, s0] +
    1/2 (∂z,z CY[0][z, s, z0, s0] + (∂z CY[0][z, s, z0, s0])^2)) /. {z → y, z0 → y0, s → t, s0 → t0};

GY[k_][y_, t_, y0_, t0_] :=
  ((-mY[z, s] ∂z c[k - 1][z, s, z0, s0] + 1/2 ∂z,z CY[k - 1][z, s, z0, s0] - ∂s CY[k - 1][z, s, z0, s0] +
     1/2 Sum[((k - 1)!/(h! (k - 1 - h)!) ∂z CY[h][z, s, z0, s0] ∂z CY[k - 1 - h][z, s, z0, s0]), {h, 0, k - 1}]) /.
    {z → y, z0 → y0, s → t, s0 → t0}) /; k ≥ 2;
```

Figure 1 Mathematica codes for computing the coefficients $C_Y^{(1)},\ldots,C_Y^{(K)}$.

```
(*Mathematica Codes*)
(* Case y ≠ y0 :*)

CY[1][y_, t_, y0_, t0_] :=
  1/(24 y (y - y0) y0 σ[t]^4)
  (-48 (y - y0) α[t]^2 κ[t]^2 +
    24 α[t] σ[t] (y (y - 2 y0) y0 κ[t]^2 σ[t] + 2 y y0 (y - y0 - y Log[y] + y Log[y0]) σ[t] κ'[t] +
      2 κ[t] ((y - y0) σ[t] + y^2 y0 (-1 + 2 Log[y] - 2 Log[y0]) σ'[t])) +
    σ[t]^2 (σ[t]^2 (-9 y + 9 y0 + 24 y y0^2 κ[t] + (-y^4 y0 + 2 y y0^4) κ[t]^2 + 2 y (y - y0)^2 y0 (y + 2 y0) κ'[t]) +
      4 y y0 (-12 (-y + y0 + y Log[y] - y Log[y0]) κ[t] α'[t] + y (-2 y^2 + 3 y0^2) σ'[t]^2) +
      4 y y0 σ[t] ((12 y0 - (y^3 - 2 y0^3) κ[t]) σ'[t] + (y - y0)^2 (y + 2 y0) σ''[t])));
(*Case y = y0 :*)
CY[1][y_, t_, y0_, t0_] :=
  -1/(8 y0^2 σ[t]^4) (16 α[t]^2 κ[t]^2 - 8 α[t] κ[t] σ[t] ((2 + y0^2 κ[t]) σ[t] + 2 y0^2 σ'[t]) +
    σ[t]^2 ((3 + y0^4 κ[t]^2) σ[t]^2 + 4 y0^4 κ[t] σ[t] σ'[t] + 4 y0^4 σ'[t]^2));
```

Figure 2 The closed-form of $C_Y^{(1)}$ obtained by running the Mathematica codes.

Suppose that $X_{t_0}, X_{t_1}, \ldots, X_{t_N}$ are observed with an equidistant time step size $\triangle t$. The approximate MLE of the true parameter θ_0, denoted by $\hat{\theta}$, is the solution of the optimization problem:

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \sum_{n=1}^{N} l_X^{(K)}(X_{t_n}, t_n \mid X_{t_{n-1}}, t_n; \theta), \tag{21}$$

where $l_X^{(K)}$ is an approximation of l_X obtained from (15) by cutting off the terms containing the coefficients $C_Y^{(k)}, k > K$. Under the regular conditions [7], $\hat{\theta}$ is close to θ_0 when K and N are large and $\triangle t$ is small.

Efficiency and Discussions

In this section, we choose 2 models describing the dynamics of convenience yields to test the efficiency of the closed-form expansions.

Model 1 (CIR model):
For $\theta = (\kappa, \alpha, \sigma)$,

$$dX_t = \kappa(\alpha - X_t)dt + \sigma\sqrt{X_t}\,dW_t. \tag{22}$$

Model 2 (ECIR model describing seasonal variation in commodity prices):
For $\theta = (\kappa, \alpha_0, \alpha_1, \alpha_2, \sigma)$,

$$dX_t = \kappa(\alpha_0 + \alpha_1 \sin(2\pi t) + \alpha_2 \cos(2\pi t) - X_t)dt + \sigma\sqrt{X_t}\,dW_t. \tag{23}$$

Running the Mathematica codes, the coefficients $C_Y^{(k)}, k = 0,1,$ of Models 1-2 are obtained and shown, respectively, in **Figures 3** and **4**.

(*Mathematica Codes Model 1*)

$$\text{CY[0][y\_, t\_, y0\_, t0\_] :=} \frac{8\,\alpha\,\kappa\,(\text{Log}[y] - \text{Log}[y0]) + \sigma^2\left(\left(-y^2 + y0^2\right)\kappa - 2\,\text{Log}[y] + 2\,\text{Log}[y0]\right)}{4\,\sigma^2};$$

(*Case $y \neq y_0$:*)

$$\text{CY[1][y\_, t\_, y0\_, t0\_] :=} -\frac{48\,\alpha^2\,\kappa^2 - 24\,\alpha\,\kappa\,(2 + y\,y0\,\kappa)\,\sigma^2 + \left(9 + y^3\,y0\,\kappa^2 + y^2\,y0^2\,\kappa^2 + y\,y0^3\,\kappa^2\right)\sigma^4}{24\,y\,y0\,\sigma^4};$$

(*Case $y = y_0$:*)

$$\text{CY[1][y\_, t\_, y0\_, t0\_] :=} \frac{16\,\alpha^2\,\kappa^2 - 8\,\alpha\,\kappa\,\left(2 + y0^2\,\kappa\right)\sigma^2 + \left(3 + y0^4\,\kappa^2\right)\sigma^4}{8\,y0^2\,\sigma^4};$$

Figure 3 The closed-forms of $C_Y^{(0)}$ and $C_Y^{(1)}$ of Model 1.

(*Mathematica Codes Model 2*)

```
CY[0][y_, t_, y0_, t0_] :=
  σ^2 ((-y^2 + y0^2) κ - 2 Log[y] + 2 Log[y0]) + 8 κ (Log[y] - Log[y0]) (α0 + α2 Cos[2 π t] + α1 Sin[2 π t])
  ---------------------------------------------------------------------------------------------------------- ;
                                                  4 σ^2
```

(*Case $y \neq y_0$:*)

```
CY[1][y_, t_, y0_, t0_] :=
          1
  ---------------------
  24 y (y - y0) y0 σ^4
  (σ^2 ((-9 y + 9 y0 + (-y^4 y0 + y y0^4) κ^2) σ^2 - 96 π y y0 κ (-y + y0 + y Log[y] - y Log[y0])
        (α1 Cos[2 π t] - Sin[2 π t])) + 24 (y - y0) κ (2 + y y0 κ) σ^2 (α0 + α2 + Cos[2 π t] + α1 Sin[2 π t]) -
    48 (y - y0) κ^2 (α0 + α2 + Cos[2 π t] + α1 Sin[2 π t])^2);
```

(*Case $y = y_0$:*)

```
CY[1][y_, t_, y0_, t0_] :=
  -     1
   ---------- ((3 + y0^4 κ^2) σ^4 - 8 κ (2 + y0^2 κ) σ^2 (α0 + α2 + Cos[2 π t] + α1 Sin[2 π t]) +
   8 y0^2 σ^4
    16 κ^2 (α0 + α2 + Cos[2 π t] + α1 Sin[2 π t])^2);
```

Figure 4 The closed-forms of $C_Y^{(0)}$ and $C_Y^{(1)}$ of Model 2.

For Model 1, the transition density of X_t is known in explicit form as follows:

$$p_X(x,t \mid x_0,t_0:\theta) = c_t p_{\chi^2(\upsilon,\lambda_t)}(c_t x), \tag{24}$$

where $c_t = \dfrac{4\kappa}{\sigma^2(1-e^{-\kappa(t-t_0)})}$, $\upsilon = \dfrac{4\kappa\alpha}{\sigma^2}$, $\lambda_t = c_t x_0 e^{-\kappa(t-t_0)}$, and $p_{\chi^2(\upsilon,\lambda_t)}$ is the density of the noncentral chi-squared random variable $\chi^2(\upsilon,\lambda_t)$.

Setting $\kappa = \alpha = \sigma = 0.03$, $x_0 = 0.1$, $t_0 = 0.1$, and $t \in \{0.2, 0.3, 0.4, 0.5\}$, **Figure 5** shows the absolute errors obtained by approximating the exact transition density p_X by the approximate transition density $p_X^{(1)} \equiv e^{l_X^{(1)}}$, i.e., $e_x := \left|p_X - p_X^{(1)}\right|$. The bound of these errors is less than 0.0004. This implies p_X is very close to $p_X^{(1)}$ when $\triangle t$ is small.

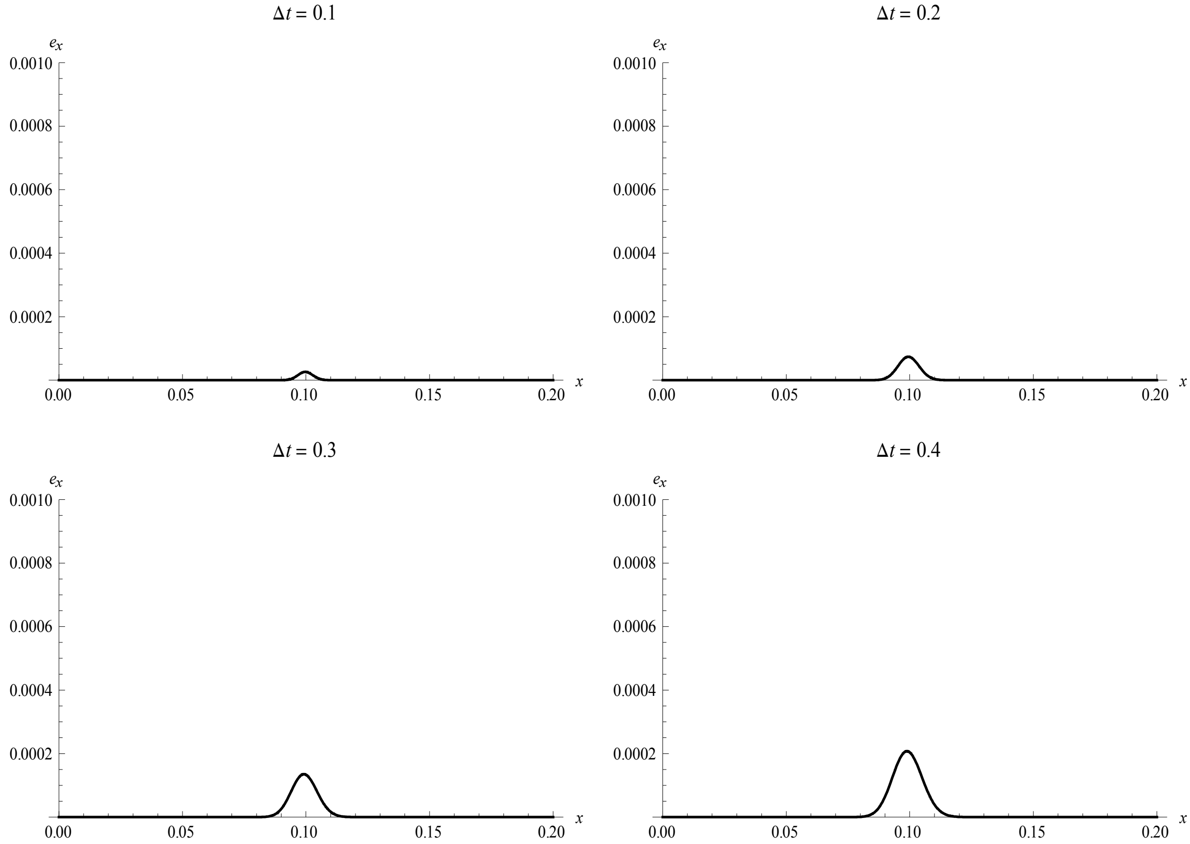

Figure 5 Absolute errors obtained by approximating p_X by $p_X^{(1)}$ in Model 1.

In contrast to Model 1, the transition density of X_t in Model 2 is not available in explicit form. Hence, to investigate the efficiency of the closed-form approximation for this case, we employ Monte Carlo simulation. Letting a true parameter $\theta_0 = (\kappa, \alpha_0, \alpha_1, \alpha_2, \sigma)$ $= (0.1, 0.05, -0.2, 0.3, 0.04)$, 1,000 sample paths of the process (23) are generated using a *Milstein scheme* with a time interval $[0,1]$ and an initial point $X_0 = 0.1$ and the equidistant time step size $\Delta t = 0.001$, or equivalently $N = 1,000$. For the sample path *i*, we have the artificial sample data $X_{t_1}^{(i)}, X_{t_2}^{(i)}, \ldots, X_{t_{1000}}^{(i)}$. Solving the optimization problem (21) corresponding to these data with $K = 1$ gives us an approximate MLE $\hat{\theta}^{(i)}$. The sample means and the sample standard deviations of all components in the error vectors $\hat{\theta}^{(i)} - \theta_0$, $i = 1, \ldots, 1,000$, are computed and displayed in **Table 1**.

The results shown in **Table 1** suggest that the approximate MLE $\hat{\theta}$ is close to the true parameter θ_0 with small errors. This implies $p_X^{(1)}$ is a good approximation of p_X. To get a better approximation, one can increase the value of K and reduce the time step size $\triangle t$. However, it is very time consuming to solve the optimization problem (21) as K is large. Hence, in practice, we prefer to choose a small $\triangle t$ with $K = 1$.

Table 1 Simulation study for Model 2.

Parameter	$\boldsymbol{\theta_0}$	$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta_0}$	
		Mean	**Std. Dev**
κ	0.10	0.0030	0.0011
α_0	0.05	−0.0300	0.0060
α_1	−0.20	0.0100	0.0037
α_2	0.30	0.0400	0.0034
σ	0.04	0.0002	0.0001

(1,000 samples with $X_0 = 0.1$, $\triangle t = 0.01$ and N=1,000)

CONCLUSION

In this research, closed-form expansions for transition densities of convenience yield processes modeled by ECIR processes are derived. The obtained closed-form expansions can be used for all cases having appropriate drift and diffusion coefficients. In terms of efficiency, the results show that the obtained approximate transition densities are close to the exact transition densities when the number of terms in the expansions is large and the time step size is small. In terms of future work, we can extend this method to derive closed-form expansions for transition densities of the transformed convenience yield processes $Y_t := c_1(\theta)\delta_t + c_0(\theta)$, where $c_i, i = 1,2,$ are functions which depend on the model parameters. This will be a future work of great interest for researchers in the field of commodity price modeling.

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บทคัดย่อ

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การกระจายรูปแบบปิดของฟังก์ชันการเลื่อนความหนาแน่นของกระบวนการผลตอบแทนความสะดวก

งานวิจัยนี้ได้นำเสนอวิธีการประมาณค่าฟังก์ชันการเลื่อนความหนาแน่นในรูปการกระจายแบบปิดของกระบวนการผลตอบแทนความสะดวกซึ่งถูกโมเดลโดยกระบวนการ ECIR (extended Cox-Ingersoll-Ross) ฟังก์ชันการประมาณค่าที่ได้นี้สามารถใช้ได้ในทุกกรณีที่สัมประสิทธิ์การลอยและการแพร่มีความเหมาะสม การประมาณค่าฟังก์ชันการเลื่อนความหนาแน่นด้วยการกระจายแบบปิดนี้มีประสิทธิภาพสูงเมื่อจำนวนพจน์ในอนุกรมมีค่ามากและช่วงห่างของเวลามีค่าน้อย

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