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On Generalized Composite Fractional Derivative

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Abstract

In the present paper, we define a generalized composite fractional derivative and obtain results, which include the image of power function, Laplace transform and composition of Riemann-Liouville fractional integral with the generalized composite fractional derivative. We also obtain the closed form solution of a generalized fractional free electron laser equation with this fractional derivative by using the Adomian decomposition method.

Keywords: Adomian decomposition method, free electron laser equation, generalized composite fractional derivative, Laplace transform, Riemann-Liouville fractional integral

Introduction

Fractional calculus is the theory of integrals and derivatives to an arbitrary order, which generalizes integer-order differentiation and integration. Fractional derivatives have proved to be very efficient and adequate to describe many phenomena with memory and hereditary processes. These phenomena are abundant in science, engineering, viscoelasticity, control, porous media, mechanics, electrical engineering, and electromagnetism. Unlike the classical derivatives, fractional derivatives have the ability to characterize adequately the processes involving a past history. Different from classical (or integer-order) derivatives, there are several definitions for fractional derivatives given in different contexts. A few to mention are those given by Liouville [1], Grunwald [2], Letnikov [3], Riemann [4], Riesz [5], Feller [6], Caputo [7], Osler [8], Miller and Ross [9], Nishimoto [10], Hadamard [11,12], Kolwankar and Gangal [13], Hilfer [14] and Jumarie [15].

In the present paper, we give a new definition of a fractional derivative termed as a generalized composite fractional derivative and obtain some basic results for it. We also obtain the solution of a generalized fractional free electron laser equation with the generalized composite fractional derivative using the Adomian decomposition method.

Preliminaries

The Riemann-Liouville fractional integral of order $\alpha > 0, x > a$ [16] is defined as;

$$ {}_aJ_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0, \tag{1}$$

with ${}_aJ_t^0 f(t) = f(t)$.

The semi group property of the Riemann-Liouville fractional integral (1) is given by [16];

$$\left({}_aJ_t^\alpha\, {}_aJ_t^\beta f\right)(t) = \left({}_aJ_t^{\alpha+\beta} f\right)(t),\ \alpha, \beta > 0. \tag{2}$$

The Riemann-Liouville fractional integral of the power function is given by [16];

$${}_aJ_t^\alpha (t-a)^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\alpha)}(t-a)^{\alpha+\lambda-1},\ \alpha, \lambda > 0. \tag{3}$$

The Laplace transform of the Riemann-Liouville fractional integral (1) is given by [16];

$$L\left[{}_0J_t^\alpha f(t); s\right] = s^{-\alpha} L\left[f(t); s\right],\ \alpha > 0. \tag{4}$$

The Riemann-Liouville fractional derivative of order α, $m-1 < \alpha \le m,\ m \in \mathbb{N}$ [16] with $x > a$ is defined as;

$${}_aD_t^\alpha f(t) = D_t^m\, {}_aJ_t^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} D_t^m \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{5}$$

The Riemann-Liouville fractional derivative (5) acts as left-inverse (but not right-inverse) of the Riemann-Liouville fractional integral (1).

The Caputo fractional derivative of order α, $m-1 < \alpha \le m,\ m \in \mathbb{N}$ is defined as [7];

$${}_a^CD_t^\alpha f(t) = {}_aI_t^{m-\alpha} D_t^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{1}{(t-\tau)^{\alpha-m+1}} D_t^m f(\tau) d\tau. \tag{6}$$

The composition of the Riemann-Liouville fractional integral operator (1) and Riemann-Liouville fractional derivative (5) is given by [16];

$$\left({}_aJ_t^\alpha\, {}_aD_t^\alpha f\right)(t) = f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-m+\alpha}}{\Gamma(k-m+\alpha+1)} D_t^k \left({}_aJ_t^{m-\alpha} f(t)\right)\Big|_{t\to a},\ m-1 < \alpha \le m,\ m \in \mathbb{N}. \tag{7}$$

Adomian decomposition method for linear differential/ integro-differential equations [17]

We consider the linear differential equation written in an operator form as;

$$Lu + Ru = g, \tag{8}$$

where L is the lower order derivative which is assumed to be invertible, R is the other linear differential operator and g is a source term.

We next apply the inverse operator L^{-1} to both sides of Eq. (8) and use the given conditions to obtain;

$$u = f - L^{-1}\left(Ru\right), \tag{9}$$

where the function f represents the terms that arise due to application of L^{-1} to the source term g and the given conditions that are assumed to be prescribed. Further we decompose the unknown function u into a sum of an infinite number of components given by the decomposition series;

$$u = \sum_{n=0}^{\infty} u_n, \tag{10}$$

where the components $u_0, u_1, u_2, ...$ are usually recurrently determined. Substituting (10) into both sides of (9) leads to;

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{11}$$

This can be written as;

$$u_0 + u_1 + u_2 + u_3 + ... = f - L^{-1}\left(R\left(u_0 + u_1 + u_2 + ...\right)\right). \tag{12}$$

The Adomian method uses the formal recursive relationship as;

$$\begin{aligned} &u_0 = f, \\ &u_{k+1} = -L^{-1}\left(R\left(u_k\right)\right), k \ge 0. \end{aligned} \tag{13}$$

Generalized composite fractional derivative

In this section, we define a generalized composite fractional derivative for $m-1 < \alpha, \beta \le m,\ 0 \le \nu \le 1,\ m \in \mathbb{N}$, as follows;

$$\left({}_aD_t^{\alpha,\beta;\nu} f\right)(t) = \left({}_aJ_t^{\nu(m-\beta)} D_t^m \left({}_aJ_t^{(1-\nu)(m-\alpha)} f\right)\right)(t). \tag{14}$$

In the case that $\nu = 0$, (14) gives the Riemann-Liouville fractional derivative of order α as;

$$\left({}_aD_t^{\alpha,\beta;0} f\right)(t) = D_t^m \left({}_aJ_t^{(m-\alpha)} f\right)(t) = \left({}_aD_t^{\alpha} f\right)(t), \tag{15}$$

and for $\nu = 1$, it gives the Caputo fractional derivative of order β as;

$$\left({}_aD_t^{\alpha,\beta;1} f\right)(t) = \left({}_aJ_t^{(m-\beta)} D_t^m f\right)(t) = \left({}_a^C D_t^{\beta} f\right)(t). \tag{16}$$

For $0 < \nu < 1$, it interpolates continuously between the Riemann-Liouville fractional derivative of order α and the Caputo fractional derivative of order β.

For $\alpha = \beta$, the generalized composite fractional derivative (14) reduces to the fractional derivative defined by Hilfer [14].

Now, we obtain some results for the generalized composite fractional derivative ${}_aD_t^{\alpha,\beta;\nu}$.

Theorem 1 If $m-1<\alpha,\beta\le m,\ 0\le\nu\le 1,\ m\in\mathbb{N}$ and $t>a,\ \lambda>0$, then;

$$ {}_aD_t^{\alpha,\beta;\nu}(t-a)^{\lambda-1}=\frac{\Gamma(\lambda)}{\Gamma\left(\nu(\alpha-\beta)+\lambda-\alpha\right)}(t-a)^{\nu(\alpha-\beta)+\lambda-\alpha-1}. \tag{17}$$

Proof In view of definition (14) and the result (3), we get;

$$\begin{aligned} {}_aD_t^{\alpha,\beta;\nu}(t-a)^{\lambda-1} &= {}_aJ_t^{\nu(m-\beta)}D_t^m\frac{\Gamma(\lambda)}{\Gamma\left(\lambda+(1-\nu)(m-\alpha)\right)}(t-a)^{(1-\nu)(m-\alpha)+\lambda-1}\\ &= {}_aJ_t^{\nu(m-\beta)}\frac{\Gamma(\lambda)}{\Gamma\left(\lambda+\alpha\nu-\alpha-m\nu\right)}(t-a)^{\alpha\nu-\alpha-m\nu+\lambda-1}. \end{aligned}\tag{18}$$

Using result (3) again, we get (17).

Theorem 2 If $m-1<\alpha,\beta\le m,\ 0\le\nu\le 1,\ m\in\mathbb{N}$, then the composition of the Riemann-Liouville fractional integral (1) with the generalized composite fractional derivative (14) is given by;

$$\left({}_aJ_t^{\alpha+\nu(\beta-\alpha)}\,{}_aD_t^{\alpha,\beta;\nu}f\right)(t)=f(t)-\sum_{k=0}^{m-1}\frac{(t-a)^{k-m+\alpha+\nu(m-\alpha)}}{\Gamma\left(k-m+\alpha+\nu(m-\alpha)+1\right)}D_t^k\left({}_aJ_t^{(1-\nu)(m-\alpha)}f\right)(t)\Big|_{t\to a}. \tag{19}$$

Proof For $m-1<\alpha,\beta\le m,\ 0\le\nu\le 1,\ m\in\mathbb{N}$ the generalized composite fractional derivative (14) can be represented as a composition of the Riemann-Liouville fractional integral (1) and the Riemann-Liouville fractional derivative (5) as follows;

$$\left({}_aD_t^{\alpha,\beta;\nu}f\right)(t)=\left({}_aJ_t^{\nu(m-\beta)}D_t^m\left({}_aJ_t^{(1-\nu)(m-\alpha)}f\right)\right)(t)=\left({}_aJ_t^{\nu(m-\beta)}\,{}_aD_t^{\alpha+\nu(m-\alpha)}f\right)(t). \tag{20}$$

Applying ${}_aJ_t^{\alpha+\nu(\beta-\alpha)}$ on both sides and using the semigroup property (2), we get;

$$\left({}_aJ_t^{\alpha+\nu(\beta-\alpha)}\,{}_aD_t^{\alpha,\beta;\nu}f\right)(t)=\left({}_aJ_t^{\alpha+\nu(m-\alpha)}\,{}_aD_t^{\alpha+\nu(m-\alpha)}f\right)(t). \tag{21}$$

In view of the result (7), we arrive at (19).

Theorem 3 For $m-1<\alpha,\beta\le m,\ 0\le\nu\le 1,\ m\in\mathbb{N}$, the Laplace transform of generalized composite fractional derivative is given by;

$$L\left[{}_0D_t^{\alpha,\beta;\nu}f(t);s\right]=s^{\nu(\beta-\alpha)+\alpha}L\left[f(t);s\right]-\sum_{k=0}^{m-1}s^{m-k-1-\nu(m-\beta)}\left(D^k\left({}_0J_t^{(1-\nu)(1-\alpha)}f\right)(t)\right)\Big|_{t\to 0^+}. \tag{22}$$

Proof For convenience, let us write $g(t)=D_t^m\,{}_0J_t^{(1-\nu)(m-\alpha)}f(t)=D_t^m h(t)$. Now by definition of ${}_0D_t^{\alpha,\beta;\nu}$ and in view of the result (4), we have;

$$L\left[{}_0D_t^{\alpha,\beta;\nu}f(t);s\right]=L\left[{}_0J_t^{\nu(m-\beta)}g(t);s\right]=s^{-\nu(m-\beta)}L\left[g(t);s\right]. \tag{23}$$

Now writing $g(t)$ in terms of $h(t)$ and using the formula of Laplace transform of m^{th} derivative of a function [18], we can write the right side of (23) as;

$$= s^{-\nu(m-\beta)}\left[s^m L\left[h(t);s\right]-\sum_{k=0}^{m-1}s^{m-k-1}\lim_{t\to 0^+}\left(D^k h(t)\right)\right] \tag{24}$$

where $h(t)={}_aJ_t^{(1-\nu)(m-\alpha)}f(t)$. Using the result (4) for the Laplace transform of $h(t)$, we arrive at the result (22).

In all the above Theorems 1 to 3 if we take $\alpha=\beta$, we get corresponding results for the composite fractional derivative defined by Hilfer [19] as given in the works of [14,19,20] respectively.

Solution of the generalized fractional free electron laser equation with a generalized composite fractional derivative

We use the Adomian decomposition method, to solve generalized fractional free electron laser equation.

Theorem 4 Consider the generalized FFEL equation;

$${}_0D_\tau^{\alpha,\beta;\nu}a(\tau)=\lambda\int_0^\tau \xi^\delta a(\tau-\xi)\phi(b,\delta+1;i\eta\xi)\,d\xi+\mu\tau^\gamma\phi(c,\gamma+1;i\eta\tau),\ 0\le\tau\le 1, \tag{25}$$

where $\alpha,\beta>0;\ 0\le\nu\le 1;\ \mu,\lambda\in\mathbb{C};\ \gamma,\delta,b,c,\eta\in\mathbb{R},$ with $\gamma,\delta\neq 0,-1,-2,...$ and ${}_0D_\tau^{\alpha,\beta;\nu}$ is the generalized composite fractional derivative defined by (14) with initial conditions;

$$D_\tau^r\,{}_0J_\tau^{(1-\nu)(m-\alpha)}a(\tau)\Big|_{\tau=0}=b_r;\ r=0,1,2,.....,m-1;\ m-1<\alpha\le m. \tag{26}$$

The closed form solution to this problem is given by;

$$\begin{aligned}
a(\tau)&=\sum_{r=0}^{m-1}\frac{b_r\tau^{r-m+\alpha+\nu(m-\alpha)}}{\Gamma(r-m+\alpha+\nu(m-\alpha)+1)}\\
&+\sum_{k=1}^{\infty}\left\{\lambda\Gamma(\delta+1)\right\}^k\left[\sum_{r=0}^{m-1}b_r\tau^{k(\delta+\alpha+\nu(\beta-\alpha))+k+r}\phi^*\left(kb,k\left(\alpha+\nu(\beta-\alpha)+\delta\right)+k+r+1;i\eta\tau\right)\right]\\
&+\mu\Gamma(\gamma+1)\sum_{k=0}^{\infty}\Big[\tau^{k(\alpha+\nu(\beta-\alpha)+\delta+1)+\gamma+\alpha+\nu(\beta-\alpha)}\\
&\qquad\phi^*\left(c+kb,k\left(\alpha+\nu(\beta-\alpha)+\delta+1\right)+\gamma+\alpha+\nu(\beta-\alpha)+1;i\eta\tau\right)\Big],
\end{aligned} \tag{27}$$

where ϕ^* is the modified confluent hypergeometric function given by;

$$\phi^*(a,c,z)=\frac{1}{\Gamma(c)}\phi(a,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{28}$$

Proof If we take $s=\tau-\xi$ in (25), it transforms to;

$$ {}_0D_\tau^{\alpha,\beta;\nu} a(\tau) = \mu\tau^\gamma \phi(c,\gamma+1;i\eta\tau) + \lambda\int_0^\tau (\tau-s)^\delta a(s)\phi(b,\delta+1;i\eta(\tau-s))ds, \tag{29} $$

Applying ${}_0J_\tau^{\alpha+\nu(\beta-\alpha)}$ on both sides of Eq. (29) and using the result (19), with $a=0$, and initial conditions (26), we get;

$$ \begin{aligned} a(\tau) &= \sum_{r=0}^{m-1} \frac{b_r \tau^{r-m+\alpha+\nu(m-\alpha)}}{\Gamma(r-m+\alpha+\nu(m-\alpha)+1)} \\ &+ {}_0J_\tau^{\alpha+\nu(\beta-\alpha)} \left[\mu\tau^\gamma \phi(c,\gamma+1;i\eta\tau) + \lambda\int_0^\tau (\tau-s)^\delta a(s)\phi(b,\delta+1;i\eta(\tau-s))ds \right]. \end{aligned} \tag{30} $$

We now decompose the unknown function $a(\tau)$ into a sum of an infinite number of components as;

$$ a(\tau) = \sum_{k=0}^{\infty} a_k(\tau). \tag{31} $$

Using the Adomian decomposition method, these components can recursively be obtained by;

$$ a_0(\tau) = \sum_{r=0}^{m-1} \frac{b_r \tau^{r-m+\alpha+\nu(m-\alpha)}}{\Gamma(r-m+\alpha+\nu(m-\alpha)+1)} + {}_0J_\tau^{\alpha+\nu(\beta-\alpha)} \left[\mu\tau^\gamma \phi(\eta,\gamma+1;i\eta\tau) \right]. \tag{32} $$

and

$$ a_{k+1}(\tau) = {}_0J_\tau^{\alpha+\nu(\beta-\alpha)} \left[\lambda\int_0^\tau (\tau-s)^\delta a_k(s)\phi(b,\delta+1;i\eta(\tau-s))ds \right],\ k=0,1,2,3,.... \tag{33} $$

Using (3) and the following formula [21];

$$ \int_0^t \xi^{c-1}\phi^*(a,c;\eta\xi)(t-\xi)^{c'-1}\phi^*(a',c';\eta(t-\xi))d\xi = t^{c+c'-1}\phi^*(a+a',c+c';\eta t), \mathrm{Re}(c)>0, \mathrm{Re}(c')>0, \tag{34} $$

in recursive formulae (32) and (33), we obtain these components as;

$$ \begin{aligned} a_0(\tau) &= \sum_{r=0}^{m-1} \frac{b_r \tau^{r-m+\alpha+\nu(m-\alpha)}}{\Gamma(r-m+\alpha+\nu(m-\alpha)+1)} + \mu\Gamma(\gamma+1)\tau^{\alpha+\nu(\beta-\alpha)+\gamma}\phi^*(c,\alpha+\nu(\beta-\alpha)+\gamma+1;i\eta\tau), \\ a_k(\tau) &= \{\lambda\Gamma(\delta+1)\}^k \left[\sum_{r=0}^{m-1} b_r \tau^{k(\delta+\alpha+\nu(\beta-\alpha))+k+r}\phi^*(kb,k(\alpha+\nu(\beta-\alpha)+\delta)+k+r+1;i\eta\tau) \right. \\ &\left. + \mu\Gamma(\gamma+1)\tau^{k(\alpha+\nu(\beta-\alpha)+\delta+1)+\gamma+\alpha+\nu(\beta-\alpha)}\phi^*(c+kb,k(\alpha+\nu(\beta-\alpha)+\delta+1)+\gamma+\alpha+\nu(\beta-\alpha)+1;i\eta\tau) \right], \\ & k=1,2,3,... \end{aligned} \tag{35} $$

Substituting (35) into (31) we obtain the required result as given by Eq. (27).

Special cases

(1) If we take $\alpha = \beta$ in Theorem 4, we obtain a solution of the following generalized FFEL with a composite fractional derivative defined by Hilfer [14].

Corollary 5 Consider the generalized FFEL equation;

$$ {}_0D_\tau^{\alpha,\nu} a(\tau) = \lambda \int_0^\tau \xi^\delta a(\tau-\xi)\phi(b,\delta+1;i\eta\xi)\,d\xi + \mu\tau^\gamma \phi(c,\gamma+1;i\eta\tau),\ 0 \le \tau \le 1, \tag{36} $$

where $\alpha > 0;\ 0 \le \nu \le 1;\ \mu,\lambda \in \mathbb{C};\ \gamma,\delta,b,c,\eta \in \mathbb{R},$ with $\gamma,\delta \ne 0,-1,-2,...$ and ${}_0D_\tau^{\alpha,\nu}$ is the composite fractional derivative defined by Hilfer [14], with initial conditions;

$$ D_\tau^r\, {}_0J_\tau^{(1-\nu)(m-\alpha)} a(\tau)\Big|_{\tau=0} = b_r,\ r = 0,1,2,.....,m-1,\ m-1 < \alpha \le m. \tag{37} $$

The closed form solution to this problem is given by;

$$ \begin{aligned} a(\tau) = {} & \sum_{r=0}^{m-1} \frac{b_r \tau^{r-m+\alpha+\nu(m-\alpha)}}{\Gamma(r-m+\alpha+\nu(m-\alpha)+1)} \\ & + \sum_{k=1}^{\infty}\left[\left\{\lambda\Gamma(\delta+1)\right\}^k \sum_{r=0}^{m-1} b_r \tau^{k(\delta+\alpha))+k+r} \phi^*\left(kb, k(\alpha+\delta)+k+r+1; i\eta\tau\right)\right] \\ & + \mu\Gamma(\gamma+1)\sum_{k=0}^{\infty}\left[\tau^{k(\alpha+\delta+1)+\gamma+\alpha}\phi^*\left(c+kb, k(\alpha+\delta+1)+\gamma+\alpha+1; i\eta\tau\right)\right], \end{aligned} \tag{38} $$

(2) On taking $\nu = 0$, in Theorem 4, we get the solution of fractional free electron laser equation with Riemann-Liouville fractional derivative, solved earlier by Saxena *et al.* [22] by the method of successive approximations.
(3) On taking $\nu = 1$, in Theorem 4, we get the solution of the generalized fractional free electron laser equation with Caputo fractional derivative studied by Garg and Sharma [23].

Conclusions

In this work, we have defined a generalized composite fractional derivative. We considered generalized fractional free electron laser equation with this fractional derivative. We obtained composition of Riemann-Liouville fractional integral with generalized composite fractional derivative and using this result in Adomian decomposition method, we have solved the fractional free electron laser equation.

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