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Sequences Generated by Polynomials over Integral Domains†

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Abstract

Let D be an integral domain. For sequences $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $I = (i_1, i_2, \ldots, i_n)$ in D^n with distinct i_j, call $\bar{a}$ a (D^n, I)-polynomial sequence if there exists $f(x) \in D[x]$ such that $f(i_j) = a_j$ $(j = 1, \ldots, n)$. Criteria for a sequence to be a (D^n, I)-polynomial sequence are established and explicit structures of $D^n/P_{n,I}$ where $P_{n,I}$ is the set of all (D^n, I)-polynomial sequences are determined.

Keywords: Polynomial sequences, sequence over integral domain, interpolation polynomials

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Introduction

For a fixed $n \in \mathbb{N}$, by a polynomial sequence (of length n), we mean a sequence $\bar{a} := (a_1, a_2, ..., a_n)$ in $\mathbb{Z}^n$ for which there exists $f(x) \in \mathbb{Z}[x]$ such that $f(i) = a_i$ for all $i = 1, 2, \ldots, n$; we refer to $f(x)$ as a polynomial which generates the sequence $\bar{a}$. Denote by P_n the set of all polynomial sequences. Cornelius, Jr. and Schultz in [1] characterized P_n using Lagrange and (implicitly) Newton interpolation polynomials and determined the structure of $\mathbb{Z}^n/P_n$.

The main objectives of this work are first to extend the characterization of Cornelius-Schultz from $\mathbb{Z}$ to an integral domain D and second, to determine their corresponding structure.

Throughout, let $I = (i_1, i_2, \ldots, i_n) \in D^n$ with distinct i_j and let

$$P_{n,I} = \{\bar{a} = (a_1, \ldots, a_n) \in D^n \mid \text{there exists } f(x) \in D[x] \text{ such that } f(i_j) = a_j \text{ for all } 1 \leq j \leq n\} \quad (1)$$

be the set of all (D^n, I)-polynomial sequences. It is easy to see that the set $P_{n,I}$ is a group under addition and if $\bar{a} \in P_{n,I}$ then $c\bar{a} \in P_{n,I}$ for any $c \in D$.

Characterization

For a fixed sequence I as above and a sequence $\bar{a} := (a_1, \ldots, a_n) \in D^n$, the Lagrange interpolation polynomial, [2, page 33], which interpolates the points (i_j, a_j) $(1 \leq j \leq n)$, is defined by

$$L_{a,I}(x) := \sum_{j=1}^{n} a_j \prod_{m=1, m\neq j}^{n} \frac{x - i_m}{i_j - i_m} \in D_Q[x] \quad (D_Q \text{ the quotient field of } D) \quad (2)$$

and satisfies

$$L_{a,I}(i_j) = a_j \ \ (1 \leq j \leq n). \quad (3)$$

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Theorem 1. *Let* $I = (i_1, i_2, \ldots, i_n) \in D^n$ *with distinct* i_j. *Then* $\bar{a} = (a_1, \ldots, a_n) \in D^n$ *is a* (D^n, I)-*polynomial sequence if and only if* $L_{a,I}(x) \in D[x]_n$, *the set of all polynomials in* $D[x]$ *of degree* $< n$. *Furthermore,* $L_{a,I}(x)$ *is the unique polynomial of degree* $< n$ *in* $D_Q[x]$ *that generates* $\bar{a}$.

Proof. If $\bar{a} \in P_{n,I}$, then there is $f(x) \in D[x]$ such that $f(i_j) = a_j$ $(1 \le j \le n)$. We next let a polynomial $p(x) := (x - i_1)\cdots(x - i_n) \in D[x]$, $\deg p(x) = n$. Since $p(x)$ is monic, by the division algorithm, $f(x) = q(x)p(x) + r(x)$, where $q, r \in D[x]$ with $\deg r < n$. Evaluating at the points i_j $(1 \le j \le n)$, we see that $r(x)$ generates the sequence $\bar{a}$ which shows that both $r(x)$ and $L_{a,I}(x)$ are polynomials in $D_Q[x]$ of degree $< n$ which agree at n distinct points and so both must be identical. The remaining assertions are trivial.

Taking $I = (1, 2, \ldots, n)$ in Theorem 1, we recover [1, Theorem 2.1].

Given a set of n points (i_k, a_k) $(k = 1, \ldots, n)$, with distinct i_k and a_k being in D, the Newton interpolation polynomial corresponding to the points (i_k, a_k) $(k = 1, \ldots, n)$ is defined as

$$N_{a,I}(x) = b_{0,I} + b_{1,I}(x - i_1) + b_{2,I}(x - i_1)(x - i_2) + \cdots + b_{n-1,I}(x - i_1)(x - i_2)\cdots(x - i_{n-1}), \quad (4)$$

where $b_{k,I} = \displaystyle\sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{m=1, m\ne j+1}^{k+1}(i_{j+1} - i_m)}$ $\quad (0 \le k \le n-1)$. Note that the Newton interpolation polynomial can be obtained by solving the system of equations

$$N_{a,I}(i_k) = a_k \qquad (1 \le k \le n) \quad (5)$$

which can be done inductively. The elements

$$1, p_{i_1} := (x - i_1), p_{i_2} := (x - i_1)(x - i_2), \ldots, p_{i_{n-1}} := (x - i_1)(x - i_2)\cdots(x - i_n) \quad (6)$$

are referred to as the corresponding Newton basis polynomials [2, page 39-40].

Theorem 2. *With the above notations, we have*

$$N_{a,I}(x) = L_{a,I}(x). \quad (7)$$

Proof. By Theorem 1, $L_{a,I}(x)$ is the unique polynomial with coefficients in D_Q of degree less than n generating $\bar{a}$. Since $N_{a,I}(i_j) = a_j = L_{a,I}(i_j)$ for $1 \le j \le n$ and $\deg N_{a,I} < n$, they are identical.

Corollary 3. *Let* $\bar{a} \in D^n$. *Then* $\bar{a} \in P_{n,I}$ *if and only if*

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{m=1, m\ne j+1}^{k+1}(i_{j+1} - i_m)} \qquad (k = 0, 1, \ldots, n-1) \quad (8)$$

is an element in D.

Proof. The result follows immediately from Theorems 1 and 2.

Taking $I = (1, 2, 3, \ldots, n)$ in Theorems 1 and 2, we get the following corollary.

Corollary 4. *Let* $\bar{a} \in \mathbb{Z}^n$.
A) ([1, Lemma 2.2]) If $N_a(x) = b_0p_0(x) + b_1p_1(x) + \cdots + b_{n-1}p_{n-1}(x)$, $b_k = \sum_{j=0}^{k} \frac{(-1)^{k+j}}{j!(k-j)!} a_{j+1}$ $(k = 0, \ldots, n-1)$, *then*

$$N_a(x) = L_a(x). \tag{9}$$

B) ([1, Corollary 2.4]) A sequence $\bar{a}$ *is a polynomial sequence if and only if each number*

$$b_k = \sum_{j=0}^{k} \frac{(-1)^{k+j} a_{j+1}}{j!(k-j)!} \quad (k = 0, 1, \ldots, n-1) \tag{10}$$

is an integer.

It is of interest to investigate the above results for small values of n. Thus we obtain the following result.

Lemma 5. *With the above notations, the following statements hold:*

A) For any $I = (i_1) \in \mathbb{Z}$, *we have* $P_{1,I} = \mathbb{Z}$.
B) For any $\overline{a} = (a_1, a_2),\ I = (i_1, i_2) \in \mathbb{Z}^2$ *where* $i_1 < i_2$, *we have*

$$\overline{a} \in P_{2,I} \quad \textit{if and only if} \quad a_1 \equiv a_2 \mod (i_1 - i_2).$$

In fact, if $I = (1, 2)$, *then* $P_2 = \mathbb{Z}^2$.
C) For any $\overline{a} = (a_1, a_2, a_3),\ I = (i_1, i_2, i_3) \in \mathbb{Z}^3$ *where* $i_1 < i_2 < i_3$, *we have*

$$\overline{a} \in P_{3,I} \quad \textit{if and only if} \quad \frac{(a_3 - a_2) + m(i_2 - i_3)}{(i_1 - i_3)(i_2 - i_3)} \quad \textit{and} \quad m = \frac{a_1 - a_2}{i_1 - i_2} \quad \textit{are integers.}$$

In fact, if $I = (1, 2, 3)$, *then* $P_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 \equiv a_3 \mod 2\}$.

Proof. We prove the above results as follows:

A) For any $a \in \mathbb{Z}$ there exists $f(x) = a$ such that $f(i_1) = a$. Thus $P_{1,I} = \mathbb{Z}$.

B) Let $\overline{a} = (a_1, a_2) \in \mathbb{Z}^2$. By Corollary 3, $\overline{a} \in P_{2,I}$ if and only if $b_{0,I} = a_1$ and $b_{1,I} = \frac{a_1}{i_1 - i_2} + \frac{a_2}{i_2 - i_1} = \frac{a_1 - a_2}{i_1 - i_2}$ are integers. Hence, $\overline{a} \in P_{2,I}$ if and only if $a_1 \equiv a_2 \mod (i_1 - i_2)$. If $I = (1, 2)$, then $i_1 - i_2 = 1$, and so $P_2 = \mathbb{Z}^2$.
C) Let $\overline{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$. Then

$$b_{0,I} = a_1, \tag{11}$$

$$b_{1,I} = \frac{a_1}{i_1 - i_2} + \frac{a_2}{i_2 - i_1} = \frac{a_1 - a_2}{i_1 - i_2}, \tag{12}$$

$$b_{2,I} = \frac{a_1}{(i_1 - i_2)(i_1 - i_3)} + \frac{a_2}{(i_2 - i_1)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)} = \frac{(a_3 - a_2) + m(i_2 - i_3)}{(i_1 - i_3)(i_2 - i_3)}, \tag{13}$$

$$\text{where} \quad m = \frac{a_1 - a_2}{i_1 - i_2}. \tag{14}$$

By Corollary 3, $\bar{a} \in P_{3,I}$ if and only if $m = \frac{a_1 - a_2}{i_1 - i_2} \in \mathbb{Z}$ and $\frac{(a_3 - a_2) + m(i_2 - i_3)}{(i_1 - i_3)(i_2 - i_3)}$ are integers . If $I = (1, 2, 3)$, then $m = \frac{a_1 - a_2}{1 - 2} = a_2 - a_1$ is an integer. Hence,

$$\frac{(a_3 - a_2) + m(2 - 3)}{(1 - 3)(2 - 3)} = \frac{(a_3 - a_2) + (a_2 - a_1)(-1)}{2} = \frac{a_3 - a_1}{2} - a_2 \tag{15}$$

is an integer if and only if $2|a_3 - a_1$. Thus $\overline{a} \in \mathbb{Z}^3$ is a polynomial sequence of length 3 if and only if a_1 and a_3 are of the same parity.

The next result shows how to turn a sequence into a (D^n, I)-polynomial sequence.

Theorem 6. *Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ with distinct i_j, let $\bar{a} = (a_1, a_2, \ldots, a_n) \in D^n$ and let*

$$M = \prod_{j=0}^{n-1} M_j, \quad \text{where} \quad M_j = \prod_{m=1, m\neq j+1}^{n} (i_{j+1} - i_m) \ \ (j = 0, 1, 2, \ldots, n-1). \tag{16}$$

Then $M\bar{a} = (Ma_1, Ma_2, \ldots, Ma_n) \in P_{n,I}$.

Moreover, if D is a unique factorization domain, then $M'\bar{a} = (M'a_1, M'a_2, \ldots, M'a_n) \in P_{n,I}$ where $M' = \text{lcm}\{M_j\}_{j=0}^{n-1}$ and M' is the minimal element in D for which this is true for every sequence of length n. The element M' is the minimal in the sense that if $L\bar{a} \in P_{n,I}$ for all n then $M' \mid L$.

Proof. Using the above notation, since

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{m\neq j+1, m=1}^{k+1} (i_{j+1} - i_m)} = \sum_{j=0}^{k} \frac{a_{j+1}}{M_j / \prod_{m=k+2, m\neq j+1}^{n} (i_{j+1} - i_m)} \ \ (0 \le k \le n-1), \tag{17}$$

we see that $Mb_{k,I} \in \mathbb{Z}$ and so $M\bar{a}$ is a (D^n, I)-polynomial sequence.

If D is a unique factorization domain, then letting $M' = \text{lcm}\{M_j\}_{j=0}^{n-1}$, it is easy to see that $M'b_{k,I}$ is in D.

To see that M' is the minimal element with the stated property, consider the following sequences in **Table 1**.

Table 1 Sequences and its corresponding coefficients in the Newton interpolation polynomial

Sequence $\overline{a}$	$b_{0,I}$	$b_{1,I}$	$b_{2,I}$	$\cdots$	$b_{n-1,I}$
$\overline{a}_1 = (1, 0, 0, \ldots, 0)$	1	$\frac{1}{i_1 - i_2}$	$\frac{1}{(i_1-i_2)(i_1-i_3)}$	$\cdots$	$\frac{1}{(i_1-i_2)(i_1-i_3)\cdots(i_1-i_n)}$
$\overline{a}_2 = (0, 1, 0, \ldots, 0)$	0	$\frac{1}{i_2 - i_1}$	$\frac{1}{(i_2-i_1)(i_2-i_3)}$	$\cdots$	$\frac{1}{(i_2-i_1)(i_2-i_3)\cdots(i_2-i_n)}$
$\overline{a}_3 = (0, 0, 1, \ldots, 0)$	0	0	$\frac{1}{(i_3-i_1)(i_3-i_2)}$	$\cdots$	$\frac{1}{(i_3-i_1)(i_3-i_2)(i_3-i_4)\cdots(i_3-i_n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{a}_n = (0, 0, 0, \ldots, 1)$	0	0	0	$\cdots$	$\frac{1}{(i_n-i_1)(i_n-i_2)\cdots(i_n-i_{n-1})}$

For each $\overline{a}_i$ $(1 \le i \le n)$, we see that $M_{i-1}\overline{a}_i \in P_{n,I}$ and for any element $L \in D$ such that $L\overline{a}_i \in P_{n,I}$, we have $M_{i-1}|L$ $(1 \le i \le n)$. Therefore, by the definition of M', we have $M'|L$, showing that M' is the minimal element such that $M'\overline{a} \in P_{n,I}$.

Before proceeding, let us work out two examples.

Example 1. a) Let $D = \mathbb{Z}$, $\overline{a} = (2, 8, 12)$ and $I = (5, 6, 8)$. We see that

$$N_{a,I}(x) = -\frac{4}{3}x^2 + \frac{62}{3}x - 68 \notin \mathbb{Z}[x]. \tag{18}$$

So $\overline{a} \notin P_{3,I}$ over $\mathbb{Z}$. Since $M_0 = 3, M_1 = 2$ and $M_2 = 6$, $M' = 6$. We deduce that $M'\overline{a} = (12, 48, 72)$ is a polynomial sequence generated by $-8x^2 + 24x - 408$ with respect to $I = (5, 6, 8)$ in $\mathbb{Z}$.

b) Let $\overline{c} = (4 - i, 5, 6 + 2i) \in \mathbb{Z}[i]^3$ and $I = (i, 3i, 2 + i) \in \mathbb{Z}[i]^3$. We see that

$$N_{c,I}(x) = \frac{-9 + 13i}{8}x^2 + (7 + 4i)x + \frac{55 - 51i}{8} \notin \mathbb{Z}[i][x]. \tag{19}$$

So $\overline{c} \notin P_{3,I}$ over $\mathbb{Z}[i]$. Since $M_0 = -4i, M_1 = -4(1 + i)$ and $M_2 = 4(1 - i), M' = 8$, we get that $M'\overline{c} = (32-8i, 40, 48+16i)$ is a polynomial sequence generated by $(-3+5i)x^2+(24+8i)x+(37-27i)$ with respect to $I = (i, 3i, 2 + i)$ in $\mathbb{Z}[i]$.

If $D = \mathbb{Z}$ and $I = (1, 2, \ldots, n)$, then we have the following result which is [1, Theorem 2.5].

Corollary 7. *If $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, then*

$$(n - 1)!a = ((n - 1)!a_1, (n - 1)!a_2, \ldots, (n - 1)!a_n) \in P_{n,I}. \tag{20}$$

Moreover, $(n - 1)!$ is the least positive integer for which this is true for every sequence of length n.

Proof. Take $I = (1, 2, 3, \ldots, n)$. Using the same notation as in Theorem 6, we compute

$$M_j = \prod_{m=1, m\neq j+1}^{n} (j + 1 - m) = (-1)^{n-j-1}(j)!(n - j - 1)! \ \ (0 \leq j \leq n - 1). \tag{21}$$

Since $(n - 1)! = (j)!(n - j - 1)!\binom{n-1}{j}$ $(0 \leq j \leq n - 1)$, the integer M_j is a divisor of $(n - 1)!$ for all $0 \leq j \leq n - 1$ and $M_{n-1} = (n - 1)!$. Hence, $M = \text{lcm}(M_1, M_2, \ldots, M_n) = (n - 1)!$.

Structure

In this section, we show that $P_{n,I}$ is a rank n subgroup of the free abelian group D^n. We first show that for any $I \in D^n$, we have $P_{n,I} \cong D[x]_n$ as a group where $D[x]_n$ is the set of polynomial in $D[x]$ of degree less than n.

Theorem 8. *The group $P_{n,I}$ is isomorphic to $D[x]_n$.*

Proof. Define $v : D[x] \longrightarrow D^n$ by $v(f(x)) = (f(i_1), f(i_2), \ldots, f(i_n))$. Let $f_1, f_2 \in D[x]_n$. Then

$$\begin{aligned} v((f_1 + f_2)(x)) &= ((f_1 + f_2)(i_1), (f_1 + f_2)(i_2), \ldots, (f_1 + f_2)(i_n)) && (22)\\ &= (f_1(i_1) + f_2(i_1), f_1(i_2) + f_2(i_2), \ldots, f_1(i_n) + f_2(i_n)) = v(f_1(x)) + v(f_2(x)). && (23)\end{aligned}$$

Thus v is an additive homomorphism. We next show that v restricted to $D[x]_n$ is an isomorphism from $D[x]_n$ to $P_{n,I}$. Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in P_{n,I}$. Then there exists $f(x) \in D[x]$ such that $f(x)$ generates $\bar{a}$. Again as in Theorem 1, $f(x) = q(x)p(x) + r(x)$ where $p(x) = (x - i_1)\cdots(x - i_n), q, r \in D[x]$ with $r = 0$ or $\deg r < n$. Evaluating at the points i_j $(1 \leq j \leq n)$, we see that $r(x)$ generates the sequence $\bar{a}$. So v is onto.

Let $f, g \in D[x]_n$. Suppose $v(f(x)) = v(g(x))$. Then $f(i_k) = g(i_k)$ for all $1 \leq k \leq n$. Since both $\deg(f)$ and $\deg(g)$ are $< n$ and the polynomials f, g agree at n distinct points, they are identical, i.e., v is one-to-one. Therefore v is an isomorphism from $D[x]_n$ onto $P_{n,I}$.

We next consider the structure of $\mathbb{Z}^n/P_{n,I}$. For $I = (1, 2, \ldots, n) \in \mathbb{Z}^n$, it was shown in [1, Theorem 3.2] that

$$\mathbb{Z}^n/P_n \cong \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}. \tag{24}$$

We use the technique similar to that in [1] to generalize the above result to $D^n/P_{n,I}$.

Theorem 9. *For $n \geq 2$, let $I = (i_1, i_2, \ldots, i_n) \in D^n$. If*

$$\prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) \in D \quad (1 < k < j \leq n), \tag{25}$$

then

$$D^n/P_{n,I} \cong D/(i_2 - i_1)D \oplus D/(i_3 - i_1)(i_3 - i_2)D \oplus \cdots \oplus D/(i_n - i_1)(i_n - i_2)\cdots(i_n - i_{n-1})D. \tag{26}$$

Proof. For $j, k \in \{1, 2, \ldots, n\}$, let

$$a_{jk} = \begin{cases} \prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) & \text{if } j \geq k > 1 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k, \end{cases} \tag{27}$$

so that

$$A_n = (a_{jk}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & \frac{i_3-i_1}{i_2-i_1} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{i_n-i_1}{i_2-i_1} & \frac{(i_n-i_1)(i_n-i_2)}{(i_3-i_1)(i_3-i_2)} & \frac{(i_n-i_1)(i_n-i_2)(i_n-i_3)}{(i_4-i_1)(i_4-i_2)(i_4-i_3)} & \ldots & 1 \end{bmatrix}. \tag{28}$$

Let $e_I(j-1)$ be the j^{th} column of A_n $(j = 1, 2, \ldots, n)$. Since $\det A_n = 1$ and

$$a_{jk} = \prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) \in D \qquad (1 < k < j), \tag{29}$$

the matrix A_n is a unimodular [3, Lemma 1.15]. In this case, we see that $\{e_I(j-1), j = 1, 2, \ldots, n\}$ forms a D-basis for D^n. Now let

$$C_n = (c_{jk}) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & i_2 - i_1 & 0 & \ldots & 0 \\ 1 & i_3 - i_1 & (i_3 - i_1)(i_3 - i_2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & i_n - i_1 & (i_n - i_1)(i_n - i_2) & \ldots & (i_n - i_1)\ldots(i_n - i_{n-1}) \end{bmatrix}, \tag{30}$$

$$c_{jk} = \begin{cases} (i_j - i_1)(i_j - i_2)\cdots(i_j - i_{k-1}) & \text{if } 1 < k \leq j \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k, \end{cases} \tag{31}$$

and let D_n be the diagonal matrix whose j^{th} diagonal entries are

$$d_{j,I} = (i_j - i_1)(i_j - i_2)\cdots(i_j - i_{j-1}) \quad (j = 1, 2, \ldots, n). \tag{32}$$

It is easy to see that $C_n = A_n D_n$. Since $\{1, p_{i_1}(x), \dots, p_{i_{n-1}}(x)\}$ forms a D-basis for $D[x]_n$, by Theorem 8, the map $v : D[x]_n \longrightarrow P_{n,I}$ is an isomorphism. So the image

$\{v(1), v(p_{i_1}(x)), \dots, v(p_{i_{n-1}}(x))\}$

forms a D-basis for $P_{n,I}$. From

$$v(p_{i_0}(x)) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, v(p_{i_1}(x)) = \begin{bmatrix} 0 \\ i_2 - i_1 \\ i_3 - i_1 \\ \vdots \\ i_n - i_1 \end{bmatrix}, \dots, v(p_{i_{n-1}}(x)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ (i_n - i_1)(i_n - i_2)\dots(i_n - i_{n-1}) \end{bmatrix},$$

we see that $v(p_{i_{j-1}}(x))$ is the j^{th} column of C_n $(j = 1, 2, \dots, n)$. Since $C_n = A_n D_n$, we have

$$(p_{i_{j-1}}(x)) = (i_j - i_1)(i_j - i_2)\cdots(i_j - i_{j-1})e_I(j-1) = \prod_{m=1}^{j-1}(i_j - i_m)e_I(j-1) \quad (j = 1, 2, \dots, n). \tag{33}$$

Thus,

$$D^n/P_{n,I} = \frac{\langle e_I(0)\rangle \oplus \langle e_I(1)\rangle \oplus \langle e_I(2)\rangle \oplus \cdots \oplus \langle e_I(n-1)\rangle}{\langle e_I(0)\rangle \oplus \prod_{m=1}^{1}(i_2 - i_m)\langle e_I(1)\rangle \oplus \cdots \oplus \prod_{m=1}^{n-1}(i_n - i_m)\langle e_I(n-1)\rangle} \tag{34}$$

$$= \frac{\langle e_I(0)\rangle}{\langle e_I(0)\rangle} \oplus \frac{\langle e_I(1)\rangle}{\prod_{m=1}^{1}(i_2 - i_m)\langle e_I(1)\rangle} \oplus \cdots \oplus \frac{\langle e_I(n-1)\rangle}{\prod_{m=1}^{n-1}(i_n - i_m)\langle e_I(n-1)\rangle} \tag{35}$$

$$\cong D/(i_2 - i_1)D \oplus D/\prod_{m=1}^{2}(i_3 - i_m)D \oplus \cdots \oplus D/\prod_{m=1}^{n-1}(i_n - i_m)D. \tag{36}$$

By Theorem 9, for $1 \le j \le n$, if $a_{jk} = \prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) \in D$ $(1 < k \le j)$, choosing $k = j-1$, we get

$$a_{j,j-1} = \prod_{m=1}^{j-2}(i_j - i_m)/\prod_{m=1}^{j-2}(i_{j-1} - i_m) \in D \quad (j = 0, 1, \dots, n-1). \tag{37}$$

Thus, $d_{j,I} = \prod_{m=1}^{j-1}(i_j - i_m) = a_{j,j-1}\cdot(i_j - i_{j-1})\cdot d_{j-1,I}$, i.e., $d_{j-1,I}$ is the factor of $d_{j,I}$ $(j = 1, 2, \dots, n)$, yielding

Corollary 10. *With the set up above, $D^n/P_{n,I}$ is a finite abelian group of the form*

$D/d_{n-1}D \oplus \cdots \oplus D/d_2D \oplus D/d_1D$

where $d_1 \mid d_2 \mid \cdots \mid d_{n-1}$.

If we take $D = \mathbb{Z}$ and $I = (1, 2, \dots, n)$, we deduce the following result.

Corollary 11. *[1, Corollary 3.3] If $I = (1, 2, \dots, n)$ $(n \ge 3)$, then $\mathbb{Z}^n/P_n$ is a finite abelian group with Smith normal form*

$\mathbb{Z}/(n-1)!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z}$

and Smith invariant $((n-1)!, \dots, 3!, 2!)$. Moreover, $|\mathbb{Z}^n/P_n| = \prod_{i=1}^{n-1} i!$.

We pause to look at one simple example.

Example 2. Let $D = \mathbb{Z}[i]$ and $I = (2+i, 3+4i, 2+11i)$. Since

$$a_{3,2} = \frac{i_3 - i_1}{i_2 - i_1} = \frac{(2+11i) - (2+i)}{(3+4i) - (2+i)} = 3 + i \in \mathbb{Z}[i], \tag{38}$$

all the elements a_{jk} of the matrix A_3 are in $\mathbb{Z}[i]$. By Theorem 9 we get

$$\mathbb{Z}[i]^3/P_{3,I} \cong \frac{\mathbb{Z}[i]}{(1+3i)\mathbb{Z}[i]} \oplus \frac{\mathbb{Z}[i]}{(10i)(-1+7i)\mathbb{Z}[i]} = \frac{\mathbb{Z}[i]}{(1+3i)\mathbb{Z}[i]} \oplus \frac{\mathbb{Z}[i]}{(-70-10i)\mathbb{Z}[i]}. \tag{39}$$

The quotient condition in Theorem 9 simplifies for some particular sets I as witnessed in the next corollary.

Corollary 12. *The following statments hold:*

A) Let a, q be elements in D and $n \geq 2$. If $i_k = aq^k$ $(1 \leq k \leq n)$, then

$$D^n/P_{n,I} \cong D/aq(q-1)D \oplus D/a^2q^{1+2}(q^2-1)(q-1)D \oplus \cdots \oplus D/a^{n-1}q^{1+2+3+\cdots+(n-1)}\prod_{i=1}^{n-1}(q^i-1)D. \tag{40}$$

B) For $n \geq 2,\ 1 \leq k \leq n-1$, if $i_{k+1} - i_k = c$ for some $c \in D$, then

$$D^n/P_{n,I} \cong D/c \cdot D \oplus D/2!c^2D \oplus D/3!c^3D \oplus \cdots \oplus D/(n-1)!c^{n-1}D. \tag{41}$$

Proof. A) Since $i_k = aq^k$, $i_{k+1} - i_k = aq^k(q-1)$ $(1 \leq k \leq n-1)$, we have $i_j - i_k = aq^j - aq^k = aq^k(q^{j-k} - 1)$ $(j > k)$. By the proof of Theorem 9, we get

$$A_n = (a_{jk}),\ \ a_{jk} = \begin{cases} \dfrac{\prod_{m=1}^{k-1}(i_j - i_m)}{\prod_{m=1}^{k-1}(i_k - i_m)} = \dfrac{\prod_{m=1}^{k-1}(q^{j-m}-1)}{\prod_{m=1}^{k-1}(q^m - 1)} & \text{if } j \geq k > 1 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k. \end{cases} \tag{42}$$

For $1 \leq k \leq j \leq n$, since $\prod_{m=1}^{k-1}(q^{j-m}-1)/\prod_{m=1}^{k-1}(q^m-1)$ is a q-binomial coefficient, it is in D and by Theorem 9 we have

$$D^n/P_{n,I} \cong D/aq(q-1)D \oplus D/a^2q^3(q^2-1)(q-1)D \oplus \cdots \oplus D/a^{n-1}q^{\frac{n(n-1)}{2}}\prod_{i=1}^{n-1}(q^i-1)D. \tag{43}$$

B) Since $i_{k+1} - i_k = c$ $(1 \leq k \leq n-1)$, we have

$$i_j - i_k = (i_j - i_{j-1}) + (i_{j-1} - i_{j-2}) + \cdots + (i_{k+1} - i_k) = (j-k)c\ \ (j > k). \tag{44}$$

By the proof of Theorem 9, we get

$$A_n = (a_{jk}),\ \ a_{jk} = \begin{cases} \prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) = \binom{j-1}{k-1} & \text{if } j \geq k > 1 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k. \end{cases} \tag{45}$$

Thus, $a_{jk} \in D$ and by Theorem 9, it is easy to see that

$$D^n/P_{n,I} \cong D/cD \oplus D\big/2!c^2D \oplus \cdots \oplus D\big/(n-1)!c^{n-1}D. \tag{46}$$

Taking $D = \mathbb{Z},\ I = \{1, 2, \ldots, n\}$ and $c = 1$ in Corollary 12 B), we recover [1, Theorem 3.2].

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