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The Continued Fractions of Certain Exponentials[†]

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Abstract

In 1954, Perron constructed simple continued fractions of $e^{1/k}$ and $e^{2/k}$ where k is a positive integer. These are called Hurwitz continued fractions. Using the method given in Perron's book, we determine explicit shapes of simple continued fractions of $ke^{1/k}$, $\frac{1}{k}e^{1/k}$ and $2e$.

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Introduction

A simple continued fraction is an expression of form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} := [a_0, a_1, a_2, \ldots], \tag{1}$$

where $a_0 \in \mathbb{Z},\ a_i \in \mathbb{N}\,(i \geq 1)$. The a_i's are called the *partial quotients*, the value $[a_0, a_1, \ldots, a_n] := p_n/q_n$ is called the nth *convergent*, and the tail $[a_n, a_{n+1}, \ldots]$ is called the nth *complete quotient* of the continued fraction (1). Let

$$\begin{array}{l} \varphi_0(0), \varphi_0(1), \varphi_0(2), \ldots \\ \varphi_1(0), \varphi_1(1), \varphi_1(2), \ldots \\ \quad\vdots \\ \varphi_{k-1}(0), \varphi_{k-1}(1), \varphi_{k-1}(2), \ldots \end{array} \tag{2}$$

be k arithmetic sequences. The continued fraction

$$[a_0, \ldots, a_{k-1}, \varphi_0(0), \varphi_1(0), \ldots, \varphi_{k-1}(0), \varphi_0(1), \varphi_1(1), \ldots, \varphi_{k-1}(1), \varphi_0(2), \varphi_1(2), \ldots, \varphi_{k-1}(2), \ldots] \tag{3}$$

is referred to as a *Hurwitz continued fraction*. We denote the continued fraction (3) for short by the symbol

$$\Big[a_0, \ldots, a_{k-1}, \overline{\varphi_0(\lambda), \varphi_1(\lambda), \ldots, \varphi_{k-1}(\lambda)}\Big]_{\lambda=0}^{\infty}. \tag{4}$$

There have already appeared several papers dealing with continued fraction expansions of $e, e^{1/k}$ and $e^{2/k}$ for positive odd integer k, e.g. [1–3]. Here we determine the explicit forms of the continued fractions of $2e, ke^{1/k}$ and $\frac{1}{k}e^{1/k}$, which to our knowledge have never appeared before.

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Preliminaries

We shall make use of the following known facts about simple continued fractions whose proofs can be found in [4, Sections 28-29].

Lemma 1. *Let ξ_0, η_0 be two irrational numbers such that*

$$\eta_0 = \frac{a\xi_0 + b}{c\xi_0 + d} \qquad (c\xi_0 + d > 0, \quad ad - bc = n > 0) \tag{5}$$

where $a, b, c, d \in \mathbb{Z}$. Let A_ν, B_ν be the numerator and denominator of the νth convergent of

$$\xi_0 = [a_0, a_1, a_2, ...]. \tag{6}$$

For a suitable fixed index ν_0, if

$$B_{\nu_0 - 1}(c\xi_0 + d) \geq 1 \ \text{ and } \ a_{\nu_0} \geq 2n + |c|, \tag{7}$$

then the fraction $\frac{aA_{v_0-1}+bB_{v_0-1}}{cA_{v_0-1}+dB_{v_0-1}}$ has a positive denominator and its value is equal to a convergent of η_0.

Lemma 2. *Let ξ_0, η_0 be two irrational numbers satisfying*

$$\eta_0 = \frac{a\xi_0 + b}{c\xi_0 + d} \qquad (c\xi_0 + d > 0, \quad ad - bc = n > 0), \tag{8}$$

where $a, b, c, d \in \mathbb{Z}$. Let the simple continued fraction of ξ_0 be

$$\xi_0 = [a_0, a_1, a_2, \ldots]. \tag{9}$$

If there are increasing indices $\nu_0, \nu_1, \nu_2, \ldots$ such that

$$B_{\nu_0 - 1}(c\xi_0 + d) \geq 1, \qquad a_{\nu_0} \geq 2n + |c|, \quad \text{and} \quad a_{\nu_i} \geq 2n \qquad (i = 1, 2, 3, \ldots), \tag{10}$$

then the simple continued fractions for ξ_0 and η_0 correspond in sections as

$$\xi_0 = [a_0, a_1, \ldots, a_{\nu_0 - 1} | a_{\nu_0}, a_{\nu_0+1}, \ldots, a_{\nu_1 - 1} | a_{\nu_1}, a_{\nu_1+1}, \ldots, a_{\nu_2 - 1} | \ldots], \tag{11}$$
$$\eta_0 = [d_0, d_1, \ldots, d_{\mu_0 - 1} | d_{\mu_0}, d_{\mu_0+1}, \ldots, d_{\mu_1 - 1} | d_{\mu_1}, d_{\mu_1+1}, \ldots, d_{\mu_2 - 1} | \ldots], \tag{12}$$

in such a way that $\mu_i \equiv \nu_i \pmod 2$ and

$$\frac{a[a_0, a_1, \ldots, a_{\nu_0 - 1}] + b}{c[a_0, a_1, \ldots, a_{\nu_0 - 1}] + d} = [d_0, d_1, \ldots, d_{\mu_0 - 1}], \tag{13}$$

$$\frac{r_i[a_{\nu_i}, a_{\nu_i+1}, \ldots, a_{\nu_{i+1}-1}] - t_i}{s_i} = [d_{\mu_i}, d_{\mu_i+1}, \ldots, d_{\mu_{i+1}-1}], \tag{14}$$

where $r_i, s_i, t_i \in \mathbb{Z}$ are defined recursively by

$$r_0 = \gcd(aA_{v_0-1} + bB_{v_0-1}, cA_{v_0-1} + dB_{v_0-1}), \tag{15}$$

$$s_0 = \frac{n}{r_0}, \quad t_0 = s_0 \frac{D_{\mu_0-2}}{D_{\mu_0-1}} - r_0 \frac{cA_{\nu_0-2} + dB_{\nu_0-2}}{cA_{\nu_0-1} + dB_{\nu_0-1}}; \tag{16}$$

in general, $r_{i+1} = \gcd(r_i A_{\nu_{i+1}-\nu_i-1,\nu_i} - t_i B_{\nu_{i+1}-\nu_i-1,\nu_i},\ s_i B_{\nu_{i+1}-\nu_i-1,\nu_i})$,

$$s_{i+1} = \frac{n}{r_{i+1}}, \quad t_{i+1} = s_{i+1} \frac{D_{\mu_{i+1}-\mu_i-2,\mu_i}}{D_{\mu_{i+1}-\mu_i-1,\mu_i}} - r_{i+1} \frac{B_{\nu_{i+1}-\nu_i-2,\nu_i}}{B_{\nu_{i+1}-\nu_i-1,\nu_i}}, \tag{17}$$

where A_ν/B_ν, C_ν/D_ν are the νth convergents of $\xi_0 := [a_0, a_1, \ldots]$, $\eta_0 := [d_0, d_1, \ldots]$, respectively, and $A_{\nu,\nu_i}/B_{\nu,\nu_i}$, $C_{\nu,\mu_i}/D_{\nu,\mu_i}$ denote the νth convergents of $\xi_{\nu_i} := [a_{\nu_i}, a_{\nu_i+1}, \ldots]$, $\eta_{\mu_i} := [d_{\mu_i}, d_{\mu_i+1}, \ldots]$, respectively.

Lemma 3. *Under the hypothesis of Lemma 2, both sections*

$$|a_{\nu_i}, a_{\nu_i+1}, \ldots, a_{\nu_{i+1}-1}| \quad and \quad |a_{\nu_j}, a_{\nu_j+1}, \ldots, a_{\nu_{j+1}-1}| \tag{18}$$

differ only in the starting element, which are congruent modulo n. If $\quad r_i = r_j, \quad s_i = s_j, \quad t_i = t_j$, then the two corresponding sections

$$|d_{\mu_i}, d_{\mu_i+1}, \ldots, d_{\mu_{i+1}-1}| \quad and \quad |d_{\mu_j}, d_{\mu_j+1}, \ldots, d_{\mu_{j+1}-1}| \tag{19}$$

differ only in the starting element, with

$$d_{\mu_i} = d_{\mu_j} + r_i^2 \frac{a_{\nu_i} - a_{\nu_j}}{n}. \tag{20}$$

Moreover, we have

$$r_{i+1} = r_{j+1}, \quad s_{i+1} = s_{j+1}, \quad t_{i+1} = t_{j+1}. \tag{21}$$

Theorem 4. *(Hurwitz) Let ξ_0, η_0 be two irrational numbers such that*

$$\eta_0 = \frac{a\xi_0 + b}{c\xi_0 + d} \qquad (c\xi_0 + d > 0, \quad ad - bc = n > 0) \tag{22}$$

where $a, b, c, d \in \mathbb{Z}$, and if the simple continued fraction for ξ_0 is a Hurwitz continued faction, then the simple continued fraction for η_0 is also a Hurwitz continued faction, and the order of arithmetic sequence for η_0 is equal to that of ξ_0, except the order 0 that appear in a continued fraction many fail in the other.

Results and discussion

The simple continued fraction of $2e$

Theorem 5. *We have*

$$2e = [5, 2, 3, \overline{2 + 2\lambda, 3, 1, 2 + 2\lambda, 1, 3}]_{\lambda=0}^{\infty}. \tag{23}$$

Proof. From [4, Section 31], we have

$$\xi_0 = \frac{e-1}{e+1} = [0, 2, 6, 10, 14, 18, \ldots] = [0, 2, 6, \overline{8\lambda + 10, 8\lambda + 14}]_{\lambda=0}^{\infty}, \tag{24}$$

From

$$\eta_0 := 2e = \frac{2\xi_0 + 2}{-\xi_0 + 1}, \tag{25}$$

we have $a = 2, b = 2, c = -1, d = 1$. Thus, $n = ad - bc = 2(1) - 2(-1) = 4 > 0$, and

$$c\xi_0 + d = -\xi_0 + 1 = -\frac{e-1}{e+1} + 1 = \frac{-e-1}{e+1} + \frac{2}{e+1} + 1 = \frac{2}{e+1} > 0. \tag{26}$$

The 0th, 1st and 2nd convergents of $[0, 2, 6, 10, 14, \ldots]$ are, respectively,

$$\frac{A_0}{B_0} = [0] = \frac{0}{1}, \quad \frac{A_1}{B_1} = [0, 2] = \frac{1}{2}, \quad \frac{A_2}{B_2} = [0, 2, 6] = \frac{6}{13}. \tag{27}$$

We subdivide the continued fraction of ξ_0 into sections in the following way

$$\xi_0 = [0, 2, 6|10|14|18|...] = [0, a_1, a_2|a_3|a_4|a_5|...] = [0, a_1, a_2|a_{\nu_0}|a_{\nu_1}|a_{\nu_2}|...]. \tag{28}$$

Thus,

$$B_{\nu_0-1}(c\xi_0 + d) = B_2(c\xi_0 + d) = \frac{26}{e+1} \geq 1,\ a_{\nu_0} = 10 \geq 9 = 2(4) + 1 = 2n + |c| \tag{29}$$

$$a_{\nu_i} \geq 14 \geq 8 = 2(4) = 2n \qquad (i = 1, 2, 3, \ldots). \tag{30}$$

From Lemma 2, we obtain

$$\frac{a[a_0, a_1, a_2] + b}{c[a_0, a_1, a_2] + d} = \frac{2[0, 2, 6] + 2}{-[0, 2, 6] + 1} = [5, 2, 3]. \tag{31}$$

Since it has an odd number of terms, the 1st section of η_0 is $5, 2, 3$ and we find that the 1st and the 2nd convergents of $[5, 2, 3]$, are, respectively,

$$\frac{C_1}{D_1} = [5, 2] = \frac{11}{2}, \quad \frac{C_2}{D_2} = [5, 2, 3] = \frac{38}{7}. \tag{32}$$

Thus,

$$r_0 = \gcd(aA_{\nu_0-1} + bB_{\nu_0-1}, cA_{\nu_0-1} + dB_{\nu_0-1}) = \gcd(38, 7) = 1,\ \ s_0 = \frac{n}{r_0} = \frac{4}{1} = 4. \tag{33}$$

For t_0, we get

$$t_0 = s_0\frac{D_{\mu_0-2}}{D_{\mu_0-1}} - r_0\frac{cA_{\nu_0-2} + dB_{\nu_0-2}}{cA_{\nu_0-1} + dB_{\nu_0-1}} = 4\Big(\frac{2}{7}\Big) - 1\Big(\frac{-1(1) + 1(2)}{-1(6) + 1(13)}\Big) = 1. \tag{34}$$

We proceed to the 2nd section ($[10] = [a_{\nu_0}]$). We have

$$\frac{r_0[a_{\nu_0}] - t_0}{s_0} = \frac{1[10] - 1}{4} = [2, 3, 1] \tag{35}$$

which has an odd number of terms, and the second section of η_0 is $2, 3, 1$. Then we get

$$A_{\nu_1-\nu_0-2,\nu_0} = A_{4-3-2,\nu_0} = A_{-1,\nu_0} = 1,\ B_{\nu_1-\nu_0-2,\nu_0} = B_{4-3-2,\nu_0} = B_{-1,\nu_0} = 0, \tag{36}$$

$$\frac{A_{\nu_1-\nu_0-1,\nu_0}}{B_{\nu_1-\nu_0-1,\nu_0}} = \frac{A_{4-3-1,\nu_0}}{B_{4-3-1,\nu_0}} = \frac{A_{0,\nu_0}}{B_{0,\nu_0}} = [10] = \frac{10}{1} \tag{37}$$

$$\frac{C_{\mu_1-\mu_0-2,\mu_0}}{D_{\mu_1-\mu_0-2,\mu_0}} = \frac{C_{6-3-2,\mu_0}}{D_{6-3-2,\mu_0}} = \frac{C_{1,\mu_0}}{D_{1,\mu_0}} = [2, 3] = \frac{7}{3} \tag{38}$$

$$\frac{C_{\mu_1-\mu_0-1,\mu_0}}{D_{\mu_1-\mu_0-1,\mu_0}} = \frac{C_{6-3-1,\mu_0}}{D_{6-3-1,\mu_0}} = \frac{C_{2,\mu_0}}{D_{2,\mu_0}} = [2, 3, 1] = \frac{9}{4}. \tag{39}$$

Furthermore,

$$r_1 = \gcd(r_0A_{\nu_1-\nu_0-1,\nu_0} - t_0B_{\nu_1-\nu_0-1,\nu_0}, s_0B_{\nu_1-\nu_0-1,\nu_0}) = 1,\ \ s_1 = \frac{n}{r_1} = \frac{4}{1} = 4. \tag{40}$$

For t_1, we obtain

$$t_1 = s_1\frac{D_{\mu_1-\mu_0-2,\mu_0}}{D_{\mu_1-\mu_0-1,\mu_0}} - r_1\frac{B_{\nu_1-\nu_0-2,\nu_0}}{B_{\nu_1-\nu_0-1,\nu_0}} = 4\Big(\frac{3}{4}\Big) - 1\Big(\frac{0}{1}\Big) = 3. \tag{41}$$

We proceed to the 3rd section of η_0 $([14]) = [a_{\nu_1}])$. We compute

$$\frac{r_1[a_{\nu_1}] - t_1}{s_1} = \frac{1[14] - 3}{4} = [2, 1, 3], \tag{42}$$

which has an odd number of terms and the third section of η_0 is $2, 1, 3$. Thus,

$$A_{\nu_2-\nu_1-2,\nu_1} = A_{5-4-2,\nu_1} = A_{-1,\nu_1} = 1, B_{\nu_2-\nu_1-2,\nu_1} = B_{5-4-2,\nu_1} = B_{-1,\nu_1} = 0, \tag{43}$$

$$\frac{A_{\nu_2-\nu_1-1,\nu_1}}{B_{\nu_2-\nu_1-1,\nu_1}} = \frac{A_{5-4-1,\nu_1}}{B_{5-4-1,\nu_1}} = \frac{A_{0,\nu_1}}{B_{0,\nu_1}} = [14] = \frac{14}{1} \tag{44}$$

$$\frac{C_{\mu_2-\mu_1-2,\mu_1}}{D_{\mu_2-\mu_1-2,\mu_1}} = \frac{C_{9-6-2,\mu_1}}{D_{9-6-2,\mu_1}} = \frac{C_{1,\mu_1}}{D_{1,\mu_1}} = [2, 1] = \frac{3}{1} \tag{45}$$

$$\frac{C_{\mu_2-\mu_1-1,\mu_1}}{D_{\mu_2-\mu_1-1,\mu_1}} = \frac{C_{9-6-1,\mu_1}}{D_{9-6-1,\mu_1}} = \frac{C_{2,\mu_1}}{D_{2,\mu_1}} = [2, 1, 3] = \frac{11}{4} \tag{46}$$

yielding

$$r_2 = \gcd(r_1 A_{\nu_2-\nu_1-1,\nu_1} - t_1 B_{\nu_2-\nu_1-1,\nu_1}, s_1 B_{\nu_2-\nu_1-1,\nu_1}) = \gcd(11, 4) = 1, \quad s_2 = \frac{n}{r_2} = \frac{4}{1} = 4. \tag{47}$$

For t_2, we have

$$t_2 = s_2 \frac{D_{\mu_2-\mu_1-2,\mu_1}}{D_{\mu_2-\mu_1-1,\mu_1}} - r_2 \frac{B_{\nu_2-\nu_1-2,\nu_1}}{B_{\nu_2-\nu_1-1,\nu_1}} = s_2 \frac{D_{1,\mu_1}}{D_{2,\mu_1}} - r_2 \frac{B_{-1,\nu_1}}{B_{0,\nu_1}} = 4\Big(\frac{1}{4}\Big) - 1\Big(\frac{0}{1}\Big) = 1. \tag{48}$$

We proceed to the 4th section of η_0 $([18]) = [a_{\nu_2}])$ by computing

$$\frac{r_2[a_{\nu_2}] - t_2}{s_2} = \frac{1[18] - 1}{4} = 4 + \frac{1}{3 + \frac{1}{1}} = [4, 3, 1] \tag{49}$$

which has an odd number of terms, and the 4th section of η_0 is $4, 3, 1$. Then we get

$$A_{\nu_3-\nu_2-2,\nu_2} = A_{6-5-2,\nu_2} = A_{-1,\nu_2} = 1, \ B_{\nu_3-\nu_2-2,\nu_2} = B_{6-5-2,\nu_2} = B_{-1,\nu_2} = 0, \tag{50}$$

$$\frac{A_{\nu_3-\nu_2-1,\nu_2}}{B_{\nu_3-\nu_2-1,\nu_2}} = \frac{A_{6-5-1,\nu_2}}{B_{6-5-1,\nu_2}} = \frac{A_{0,\nu_2}}{B_{0,\nu_2}} = [18] = \frac{18}{1} \tag{51}$$

$$\frac{C_{\mu_3-\mu_2-2,\mu_2}}{D_{\mu_3-\mu_2-2,\mu_2}} = \frac{C_{12-9-2,\mu_2}}{D_{12-9-2,\mu_2}} = \frac{C_{1,\mu_2}}{D_{1,\mu_2}} = [4, 3] = \frac{13}{3} \tag{52}$$

$$\frac{C_{\mu_3-\mu_2-1,\mu_2}}{D_{\mu_3-\mu_2-1,\mu_2}} = \frac{C_{12-9-1,\mu_2}}{D_{12-9-1,\mu_2}} = \frac{C_{2,\mu_2}}{D_{2,\mu_2}} = [4, 3, 1] = \frac{17}{4} \tag{53}$$

yielding

$$r_3 = \gcd(r_2 A_{\nu_3-\nu_2-1,\nu_2} - t_2 B_{\nu_3-\nu_2-1,\nu_2}, s_2 B_{\nu_3-\nu_2-1,\nu_2}) = \gcd(17, 4) = 1, \quad s_3 = \frac{n}{r_3} = \frac{4}{1} = 4. \tag{54}$$

For t_3, we have

$$t_3 = s_3 \frac{D_{\mu_3-\mu_2-2,\mu_2}}{D_{\mu_3-\mu_2-1,\mu_2}} - r_3 \frac{B_{\nu_3-\nu_2-2,\nu_2}}{B_{\nu_3-\nu_2-1,\nu_2}} = 4\Big(\frac{3}{4}\Big) - 1\Big(\frac{0}{1}\Big) = 3, \tag{55}$$

and so

$$\frac{r_3[a_{\nu_3}] - t_3}{s_3} = \frac{1[22] - 3}{4} = [4, 1, 3]. \tag{56}$$

Since $t_0 = t_2,\ s_0 = s_2,\ r_0 = r_2$ and $t_1 = t_3,\ s_1 = s_3,\ r_1 = r_3$, by Lemma 3, we get $t_i = t_j,\ s_i = s_j,\ r_i = r_j$ for $j = i+2$. Therefore,

$$\eta_0 = [5,2,3,2,3,1,2,1,3,4,3,1,4,1,3,\ldots] = [5,2,3,\overline{\chi_0(\lambda),3,1,\chi_1(\lambda),1,3}]_{\lambda=0}^{\infty}, \tag{57}$$

i.e., from

$$\xi_0 = [0,2,6,10,14,18,...] = [0,2,6,\overline{8\lambda+10,8\lambda+14}]_{\lambda=0}^{\infty} = [0,2,6,\overline{\psi_0(\lambda),\psi_1(\lambda)}]_{\lambda=0}^{\infty}\ , \tag{58}$$

we have found that

$$2e = [5,2,3,\overline{\chi_0(\lambda),3,1,\chi_1(\lambda),1,3}]_{\lambda=0}^{\infty}\ , \tag{59}$$

where

$$\chi_0(\lambda) = d_{\mu_0} + r_0^2\frac{\psi_0(\lambda)-\psi_0(0)}{n} = 2 + \frac{8\lambda+10-10}{4} = 2+2\lambda \tag{60}$$

$$\chi_1(\lambda) = d_{\mu_1} + r_1^2\frac{\psi_1(\lambda)-\psi_1(0)}{n} = 2 + \frac{8\lambda+14-14}{4} = 2+2\lambda. \tag{61}$$

The simple continued fraction of $ke^{1/k}$

Theorem 6. *For $k \in \mathbb{N}$, we have*

$$ke^{1/k} = [k+1, 2k-1, \overline{2+2\lambda, 1, 2k-1}]_{\lambda=0}^{\infty}. \tag{62}$$

Proof. From [4, Section 31], we have

$$\xi_0 = \frac{e^{1/k}-1}{e^{1/k}+1} = [0,2k,6k,10k,14k,\ldots] = [0,2k,\overline{(4\lambda+6)k}]_{\lambda=0}^{\infty}. \tag{63}$$

Putting

$$\eta_0 = ke^{1/k} = \frac{k\xi_0+k}{-\xi_0+1}, \tag{64}$$

we get $a = k, b = k, c = -1, d = 1$. Thus, $n = ad - bc = k - (-k) = 2k$, and

$$c\xi_0 + d = -\xi_0 + 1 = \frac{-e^{1/k}+1}{e^{1/k}+1} + 1 = \frac{-e^{1/k}-1}{e^{1/k}+1} + \frac{2}{e^{1/k}+1} + 1 = \frac{2}{e^{1/k}+1} > 0. \tag{65}$$

The 0th and 1st convergents of $[0,2k,6k,10k,14k,\ldots]$ are, respectively,

$$\frac{A_0}{B_0} = [0] = \frac{0}{1} \quad , \quad \frac{A_1}{B_1} = [0,2k] = \frac{1}{2k}. \tag{66}$$

We subdivide the continued fraction of ξ_0 into sections in the following way

$$\xi_0 = [0,2k|6k|10k|14k|...] = [a_0,a_1|a_2|a_3|a_4|...] = [a_0,a_1|a_{\nu_0}|a_{\nu_1}|a_{\nu_2}|...], \tag{67}$$

to get

$$B_{\nu_0-1}(c\xi_0+d) = B_1(c\xi_0+d) = \frac{4k}{e^{1/k}+1} \geq 1,\ a_{\nu_0} = 6k \geq 4k+1 = 2(2k)+1 = 2n+|c| \tag{68}$$

$$a_{\nu_i} \geq 10k \geq 4k = 2(2k) = 2n, \qquad (i = 1,2,3,\ldots). \tag{69}$$

From Lemma 2, we obtain

$$\frac{a[a_0,a_1]+b}{c[a_0,a_1]+d}=\frac{k[0,2k]+k}{-1[0,2k]+1}=k+1+\frac{1}{2k-1}=[k+1,2k-1]. \tag{70}$$

Since it has an even number of elements, the 1st section of η_0 is $k+1, 2k-1$, and we obtain

$$\frac{C_0}{D_0}=[k+1]=\frac{k+1}{1},\quad \frac{C_1}{D_1}=[k+1,2k-1]=\frac{2k^2+k}{2k-1}. \tag{71}$$

Thus,

$$r_0=\gcd(aA_1+bB_1,cA_1+dB_1)=\gcd(2k^2+k,2k-1)=1,\ s_0=\frac{n}{r_0}=\frac{2k}{1}=2k. \tag{72}$$

For t_0, we have

$$t_0=s_0\frac{D_{\mu_0-2}}{D_{\mu_0-1}}-r_0\frac{cA_{\nu_0-2}+dB_{\nu_0-2}}{cA_{\nu_0-1}+dB_{\nu_0-1}}=\frac{2k-1}{-1+2k}=1. \tag{73}$$

We proceed to the 2nd section ($[6k]$), and find

$$\frac{r_0[a_{\nu_0}]-t_0}{s_0}=\frac{[6k]-1}{2k}=2+\frac{1}{1+\frac{1}{2k-1}}=[2,1,2k-1] \tag{74}$$

which has an odd number of terms, and the second section of η_0 is $2, 1, 2k-1$. Proceeding further, we have

$$A_{\nu_1-\nu_0-2,\nu_0}=A_{3-2-2,\nu_0}=A_{-1,\nu_0}=1,\ B_{\nu_1-\nu_0-2,\nu_0}=B_{3-2-2,\nu_0}=B_{-1,\nu_0}=0, \tag{75}$$

$$\frac{A_{\nu_1-\nu_0-1,\nu_0}}{B_{\nu_1-\nu_0-1,\nu_0}}=\frac{A_{3-2-1,\nu_0}}{B_{3-2-1,\nu_0}}=\frac{A_{0,\nu_0}}{B_{0,\nu_0}}=[6k]=\frac{6k}{1} \tag{76}$$

$$\frac{C_{\mu_1-\mu_0-2,\mu_0}}{D_{\mu_1-\mu_0-2,\mu_0}}=\frac{C_{5-2-2,\mu_0}}{D_{5-2-2,\mu_0}}=\frac{C_{1,\mu_0}}{D_{1,\mu_0}}=[2,1]=\frac{3}{1} \tag{77}$$

$$\frac{C_{\mu_1-\mu_0-1,\mu_0}}{D_{\mu_1-\mu_0-1,\mu_0}}=\frac{C_{5-2-1,\mu_0}}{D_{5-2-1,\mu_0}}=\frac{C_{2,\mu_0}}{D_{2,\mu_0}}=[2,1,2k-1]=\frac{6k-1}{2k} \tag{78}$$

yielding

$$r_1=\gcd(r_0A_{\nu_1-\nu_0-1,\nu_0}-t_0B_{\nu_1-\nu_0-1,\nu_0},s_0B_{\nu_1-\nu_0-1,\nu_0})=\gcd(6k-1,2k^2)=1, \tag{79}$$

$$s_1=\frac{n}{r_1}=\frac{2k}{1}=2k, \tag{80}$$

and so

$$t_1=s_1\frac{D_{\mu_1-\mu_0-2,\mu_0}}{D_{\mu_1-\mu_0-1,\mu_0}}-r_1\frac{B_{\nu_1-\nu_0-2,\nu_0}}{B_{\nu_1-\nu_0-1,\nu_0}}=2k\Big(\frac{1}{2k}\Big)-1\Big(\frac{0}{1}\Big)=1 \tag{81}$$

$$\frac{r_1[a_{\nu_1}]-t_1}{s_1}=4+\frac{1}{1+\frac{1}{2k-1}}=[4,1,2k-1]. \tag{82}$$

Since $t_0=t_1,\ s_0=s_1,\ r_0=r_1$, by Lemma 3, we get $t_i=t_j,\ s_i=s_j,\ r_i=r_j$ for all i,j, and then

$$\eta_0=[k+1,2k-1,2,1,2k-1,4,1,2k-1,\ldots]=[k+1,2k-1,\overline{\chi_0(\lambda),1,2k-1}]_{\lambda=0}^{\infty}, \tag{83}$$

i.e., from

$$\xi_0=[0,2k,6k,10k,14k,\ldots]=[0,2k,\overline{(4\lambda+6)k}]_{\lambda=0}^{\infty}=[0,2k,\overline{\psi_0(\lambda)}]_{\lambda=0}^{\infty}, \tag{84}$$

we get

$$\chi_0(\lambda)=d_{\mu_0}+r_0^2\frac{\psi_0(\lambda)-\psi_0(0)}{n}=2+\frac{(4\lambda+6)k-6k}{2k}=2+2\lambda. \tag{85}$$

The simple continued fraction of $\frac{1}{k}e^{1/k}$

Theorem 7. *For $k \in \mathbb{N}$, we have*

$$\frac{1}{k}e^{1/k} = [0, k-1, 2k, 1, \overline{2+2\lambda, 2k-1, 1}]_{\lambda=0}^{\infty}. \tag{86}$$

Proof. From [4, Section 31], we have

$$\xi_0 = \frac{e^{1/k}-1}{e^{1/k}+1} = [0, 2k, 6k, 10k, 14k, \ldots] = [0, 2k, \overline{(4\lambda+6)k}]_{\lambda=0}^{\infty}, \tag{87}$$

we get $e^{1/k} = \frac{\xi_0+1}{-\xi_0+1}$. Putting

$$\eta_0 = \frac{1}{k}e^{1/k} = \frac{\xi_0+1}{-k\xi_0+k}, \tag{88}$$

we have

$$a = 1, b = 1, c = -k, d = k, \quad n = ad - bc = k - (-k) = 2k > 0, \tag{89}$$

and

$$c\xi_0 + d = -k\xi_0 + k = -k\frac{e^{1/k}-1}{e^{1/k}+1} + k = k\frac{-e^{1/k}-1}{e^{1/k}+1} + \frac{2k}{e^{1/k}+1} + k = \frac{2k}{e^{1/k}+1} > 0. \tag{90}$$

The 0th and the 1st convergents of $[0, 2k, 6k, 10k, 14k, \ldots]$, are, respectively,

$$\frac{A_0}{B_0} = [0] = \frac{0}{1}, \quad \frac{A_1}{B_1} = [0, 2k] = \frac{1}{2k}. \tag{91}$$

We subdivide the continued fraction of ξ_0 into sections in the following way

$$\xi_0 = [0, 2k|6k|10k|14k|...] = [a_0, a_1|a_2|a_3|a_4|\ldots] = [a_0, a_1|a_{\nu_0}|a_{\nu_1}|a_{\nu_2}|\ldots] \tag{92}$$

to get

$$B_{\nu_0-1}(c\xi_0+d) = B_1(c\xi_0+d) = \frac{4k^2}{e^{1/k}+1} \geq 1, \quad a_{\nu_0} = 6k \geq 5k = 2(2k) + k = 2n + |c| \tag{93}$$

$$a_{\nu_i} \geq 10k \geq 4k = 2(2k) = 2n \qquad (i = 1, 2, 3, \ldots). \tag{94}$$

From Lemma 2, we obtain

$$\frac{a[a_0, a_1] + b}{c[a_0, a_1] + d} = \frac{[0, 2k] + 1}{-k[0, 2k] + k} = \frac{1}{k - 1 + \frac{1}{2k + \frac{1}{1}}} = [0, k-1, 2k, 1]. \tag{95}$$

Since it has an even number of terms, the 1st section of η_0 is $0, k-1, 2k, 1$, and we find the 2nd and the 3rd convergents as

$$\frac{C_2}{D_2} = [0, k-1, 2k] = \frac{2k}{2k^2 - 2k + 1}, \quad \frac{C_3}{D_3} = [0, k-1, 2k, 1] = \frac{2k+1}{2k^2 - k}. \tag{96}$$

Then we get

$$r_0 = \gcd(aA_{\nu_0-1} + bB_{\nu_0-1}, cA_{\nu_0-1} + dB_{\nu_0-1}) = 1, \quad s_0 = \frac{n}{r_0} = \frac{2k}{1} = 2k. \tag{97}$$

For t_0, we have

$$t_0 = s_0 \frac{D_{\mu_0-2}}{D_{\mu_0-1}} - r_0 \frac{cA_{\nu_0-2} + dB_{\nu_0-2}}{cA_{\nu_0-1} + dB_{\nu_0-1}} = \frac{4k^3 - 4k^2 + k}{2k^2 - k} = 2k - 1. \tag{98}$$

We proceed to the 2nd section of η_0 to get

$$\frac{r_0[a_{\nu_0}] - t_0}{s_0} = \frac{[6k] - 2k + 1}{2k} = 2 + \frac{1}{2k - 1 + \frac{1}{1}} = [2, 2k-1, 1] \tag{99}$$

which has an odd number of terms, and the second section of η_0 is $2, 2k-1, 1$. Proceeding as in the previous theorem, we have

$$A_{\nu_1-\nu_0-2,\nu_0} = A_{3-2-2,2} = A_{-1,\nu_0} = 1, \; B_{\nu_1-\nu_0-2,\nu_0} = B_{3-2-2,2} = B_{-1,\nu_0} = 0, \tag{100}$$

$$\frac{A_{\nu_1-\nu_0-1,\nu_0}}{B_{\nu_1-\nu_0-1,\nu_0}} = \frac{A_{3-2-1,\nu_0}}{B_{3-2-1,\nu_0}} = \frac{A_{0,\nu_0}}{B_{0,\nu_0}} = [6k] = \frac{6k}{1} \tag{101}$$

$$\frac{C_{\mu_1-\mu_0-2,\mu_0}}{D_{\mu_1-\mu_0-2,\mu_0}} = \frac{C_{7-4-2,\mu_0}}{D_{7-4-2,\mu_0}} = \frac{C_{1,\mu_0}}{D_{1,\mu_0}} = [2, 2k-1] = \frac{4k-1}{2k-1} \tag{102}$$

$$\frac{C_{\mu_1-\mu_0-1,\mu_0}}{D_{\mu_1-\mu_0-1,\mu_0}} = \frac{C_{7-4-1,\mu_0}}{D_{7-4-1,\mu_0}} = \frac{C_{2,\mu_0}}{D_{2,\mu_0}} = [2, 2k-1, 1] = \frac{4k+1}{2k}. \tag{103}$$

Furthermore,

$$r_1 = \gcd(r_0 A_{\nu_1-\nu_0-1,\nu_0} - t_0 B_{\nu_1-\nu_0-1,\nu_0}, s_0 B_{\nu_1-\nu_0-1,\nu_0}) = \gcd(4k+1, 2k) = 1, \tag{104}$$

$$s_1 = \frac{n}{r_1} = \frac{2k}{1} = 2k. \tag{105}$$

Hence,

$$t_1 = s_1 \frac{D_{\mu_1-\mu_0-2,\mu_0}}{D_{\mu_1-\mu_0-1,\mu_0}} - r_1 \frac{B_{\nu_1-\nu_0-2,\nu_0}}{B_{\nu_1-\nu_0-1,\nu_0}} = 2k\Big(\frac{2k-1}{2k}\Big) - 1\Big(\frac{0}{1}\Big) = 2k - 1, \tag{106}$$

and

$$\frac{r_1[a_{\nu_1}] - t_1}{s_1} = 4 + \frac{1}{2k - 1 + \frac{1}{1}} = [4, 2k-1, 1]. \tag{107}$$

Since $t_0 = t_1, \; s_0 = s_1, \; r_0 = r_1$, by Lemma 3, we get $t_i = t_j, \; s_i = s_j, \; r_i = r_j$ for all i, j, and so

$$\eta_0 = [0, k-1, 2k, 1, 2, 2k-1, 1, 4, 2k-1, 1, ...] = [0, k-1, 2k, 1, \overline{\chi_0(\lambda), 2k-1, 1}]_{\lambda=0}^{\infty}, \tag{108}$$

i.e., from

$$\xi_0 = [0, 2k, 6k, 10k, 14k, \ldots] = [0, 2k, \overline{(4\lambda+6)k}]_{\lambda=0}^{\infty} = [0, 2k, \overline{\psi_0(\lambda)}]_{\lambda=0}^{\infty}, \tag{109}$$

we obtain

$$\chi_0(\lambda) = d_{\mu_0} + r_0^2 \frac{\psi_0(\lambda) - \psi_0(0)}{n} = 2 + \frac{(4\lambda+6)k - 6k}{2k} = 2 + 2\lambda\ . \tag{110}$$

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