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A New (G'/G)-Expansion Method and Its Application to the Burgers Equation

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Abstract

In this article, a new (G'/G)-expansion method is proposed, where $G = G(\xi)$ satisfies a second order nonlinear ordinary differential equation to seek the travelling wave solutions of nonlinear evolution equations. The Burgers equation is chosen to illustrate the validity and advantages of proposed method. Hyperbolic function, trigonometric function and rational function solutions with arbitrary constants are obtained from which some special solutions, including the known solitary wave solution, are derived by setting the appropriate values of constants. It is shown that the new (G'/G)-expansion method is effective, and gives new, more general, travelling wave solutions than the existing methods, such as the basic (G'/G)-expansion method, the extended (G'/G)-expansion method, the improved (G'/G)-expansion method, the generalized and improved (G'/G)-expansion method etc.

Keywords: (G'/G)-expansion method, the Burgers equation, homogeneous balance, traveling wave solutions, solitary wave solutions

Introduction

The investigation of the traveling wave solutions of nonlinear partial differential equations (NPDEs) plays an important role in the study of nonlinear physical phenomena, especially in fluid mechanics, solid-state physics, biophysics, chemical kinematics, geochemistry, electricity, propagation of shallow water waves, plasma physics, high-energy physics, condensed matter physics, quantum mechanics, optical fibers, elastic media, and so on. As a key problem, finding their analytical solutions is of great interest, and is carried out through various methods to construct exact solutions of nonlinear evolution equations (NLEEs). With the invention of symbolic computation software, like Maple or Mathematica, direct methods to search for exact solutions of NLEEs have attracted more attention. As a result, researchers have developed and established many methods, for example, the inverse scattering transform [1], the Darboux transformation method [2], the Backlund transformation method [3], the Hirota bilinear method [4], the tanh method [5], the symmetry method [6], the Painleve expansion method [7], the Exp-function method [8-12], and so on, to construct exact solution of NLEEs.

Wang *et al*. [13] introduced a simple and straightforward method, called the (G'/G)-expansion method, to investigate traveling wave solutions of nonlinear evolution equations. Applications of the (G'/G)-expansion method to NLEEs can be found in the references [14-22] for better conception.

In order to establish the effectiveness and reliability of the (G'/G)-expansion method, and to extend the range of its applicability, further research has been carried out by several researchers, such as Zhang *et al*. [23], who developed a generalized (G'/G)-expansion method to deal with evolution equations with variable coefficients. Zhang *et al.* [24] also presented an improved (G'/G)-expansion method to look into more general traveling wave solutions. Zayed [25] presented an alternative approach of the (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptical equation $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$. Zayed [26] again presented a further alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = A + B G^2(\xi)$. Akbar *et al.* [18] presented a generalized and improved (G'/G)-expansion method which provided further new solutions than the improved (G'/G)-expansion method [24].

In this article, a new (G'/G)-expansion method is offered to look for solutions of NLEEs in mathematical physics. This approach is new and has not been used previously. To show the originality, reliability and advantages of the projected method, the Burgers equation has been solved and further new families of exact solutions are found.

Materials and methods

Let us consider the NLPDE in 2 independent variables x and t of the form;

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0, \tag{1}$$

where $u(x,t)$ is an unknown function and F is a polynomial of its arguments. In order to solve the NLPDE (1), we have to execute the following fundamental steps;

Step 1 At this step, we introduce the traveling wave ansatz;

$$u(x,t) = u(\xi), \quad \xi = x \pm V t, \tag{2}$$

where V is the velocity of the wave. Substituting (2) into Eq. (1) yields a nonlinear ordinary differential equation (ODE) for $u(\xi)$;

$$H(u, u', u'', u''', \cdots) = 0, \tag{3}$$

where prime indicates the ordinary derivatives with respect to ξ.

Step 2 For the suggested method, we assume that the solution of Eq. (3) can be presented in the following form;

$$u(\xi) = \sum_{j=-N}^{N} \alpha_j \left(d + G'(\xi)/G(\xi)\right)^j \tag{4}$$

wherein d and α_j $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$ are constants to be determined, such that α_{-N} or α_N may be zero, but together they cannot be zero and the unknown function $G(\xi)$ satisfies the second order nonlinear ODE;

$$G G'' = \lambda G G' + \mu G^2 + \nu (G')^2 \tag{5}$$

where prime indicates the derivative with respect ξ, λ, μ and ν are arbitrary constants. Eq. (5) has not been used by anybody previously as an auxiliary equation.

The Cole-Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\eta$ transforms the Eq. (5) into the generalized Riccati type equation in terms of $\Phi(\xi)$;

$$\Phi'(\xi) = \mu + \lambda\, \Phi(\xi) + (\nu - 1)\, \Phi^2(\xi) \tag{6}$$

where $\Phi(\xi) = (G'(\xi)/G(\xi))$. The generalized Riccati equation has 25 distinct solutions (see **Appendix** for details).

Step 3 The positive integer N can be determined by considering the homogeneous balance between the highest order derivative $u^{(n)}(\xi)$ and the nonlinear terms of the highest order $u^r(\xi)\, u^s(\xi)$ appearing in Eq. (3).

Step 4 By making use of Eq. (4) and with the help of (6), from Eq. (3) we obtain polynomials in $(d + G'(\xi)/G(\xi))^j$ and $(d + G'(\xi)/G(\xi))^{-j}$, $(j = 0,1,2,\cdots,N)$. Collecting the coefficients of the like power of the resulted polynomials to zero, yields an over-determined set of algebraic equations for α_j $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$, d and V.

Step 5 Since the general solutions of Eq. (6) are known to us, then by substituting α_j, d, V and the solutions of (6) into (4), we have more traveling wave solutions of the nonlinear evolution equation (1).

Remark 1 It is important to notice that by the suitable substitution of λ, μ and ν, the projected method coincides with the generalized and improved (G'/G)-expansion method studied by Akbar *et al*. [15], the improved (G'/G)-expansion method presented by Zhang *et al*. [24], the basic (G'/G)-expansion method introduced by Wang *et al*. [13], and with the generalized (G'/G)-expansion method developed by Zhang *et al*. [23] if α_j ($j = 1,2,3,\cdots,N$) are functions of x and t instead of constants. Therefore, the methods presented in the Ref. [13,18,23,24] are only particular cases of the proposed (G'/G)-expansion method.

Application of the method

In this section, the proposed method is used to obtain new and more general exact traveling wave solutions of the celebrated Burgers equation.

Let us consider the Burgers equation;

$$u_t + uu_x - u_{xx} = 0\,. \tag{7}$$

The traveling wave transformation $\xi = x - V\,t$, permits us to transform (7) into the ODE;

$$-Vu' + uu' - u'' = 0\,. \tag{8}$$

Integrating Eq. (8) obtains;

$$C - Vu + \frac{1}{2}u^2 - u' = 0, \tag{9}$$

where C is an integration constant. Substituting Eq. (4) into Eq. (9) and balancing the highest order derivative u' with the nonlinear term of the highest order u^2 obtains $N = 1$.

Therefore, the solution of Eq. (9) takes the form;

$$u(\xi) = \alpha_{-1}\left(d + G'(\xi)/G(\xi)\right)^{-1} + \alpha_0 + \alpha_1\left(d + G'(\xi)/G(\xi)\right). \tag{10}$$

Substituting Eq. (13) into Eq. (12), the left hand side is transformed into polynomials of $\left(d + G'(\xi)/G(\xi)\right)^j$ and $\left(d + G'(\xi)/G(\xi)\right)^{j}$, $(j = 0, 1, 2, \cdots, N)$. Equating the coefficients of like power of these polynomials to zero obtains an over-determine set of algebraic equations (for simplicity, these equations are not displayed) for α_0, α_1, α_{-1}, d, C and V. Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple, obtains;

Set 1 $\alpha_{-1} = 0$, $\alpha_0 = \alpha_0$, $\alpha_1 = 2(\nu - 1)$, $d = d$, $V = -2d - \lambda + 2\nu d + \alpha_0$,

$$C = -2\lambda d\nu + 2\lambda d + 2\nu^2 d^2 - 4\nu d^2 + 2\mu\nu - 2\mu + 2d^2 + \frac{1}{2}\alpha_0{}^2 - 2\alpha_0 d - \alpha_0\lambda + 2\alpha_0\nu d. \tag{11}$$

Set 2 $\alpha_{-1} = 2d^2 - 2\mu + 2\lambda d - 2\nu d^2$, $\alpha_0 = \alpha_0$, $\alpha_1 = 0$, $d = d$, $V = \lambda + 2d - 2\nu d + \alpha_0$,

$$C = 2d^2 - 2\mu + 2\lambda d - 4\nu d^2 + 2\mu\nu - 2\lambda d\nu + 2\nu^2 d^2 + \frac{1}{2}\alpha_0{}^2 + \alpha_0\lambda + 2\alpha_0 d - 2\alpha_0\nu d. \tag{12}$$

Set 3 $\alpha_{-1} = -\frac{1}{2}\left(\frac{4\mu\nu - 4\mu - \lambda^2}{\nu - 1}\right)$, $\alpha_0 = \alpha_0$, $\alpha_1 = 2(\nu - 1)$, $V = \alpha_0$, $d = \frac{1}{2}\left(\frac{\lambda}{\nu - 1}\right)$,

$$C = 8\mu\nu - 8\mu + \frac{1}{2}\alpha_0{}^2 - 2\lambda^2, \tag{13}$$

where α_0, λ, μ and ν are arbitrary constants.

Substituting Eqs. (11) - (13) into Eq. (10) respectively obtains;

$$u_1(x,t) = 2(\nu - 1)\left(d + (G'/G)\right) + \alpha_0, \tag{14}$$

where $\xi = x - \{-2d \quad \lambda + 2\nu d + \alpha_0\}t$, and α_0, d, λ and ν are arbitrary constants.

$$u_2(x,t) = \left(2d^2 - 2\mu + 2\lambda d - 2\nu d^2\right)\left(d + (G'/G)\right)^{-1} + \alpha_0, \tag{15}$$

where $\xi = \mathrm{x} - (\lambda + 2d - 2\nu d + \alpha_0)t$, and α_0, d, λ, μ and ν are arbitrary constants, and

$$u_3(x,t) = 2(\nu - 1)\left(\frac{\lambda}{2(\nu - 1)} + (G'/G)\right) - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu - 1)}\right)\left(\frac{\lambda}{2(\nu - 1)} + (G'/G)\right)^{-1} + \alpha_0. \tag{16}$$

where $\xi = x - \alpha_0 t$, and α_0, λ, μ and ν are arbitrary constants.

Substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (14) and simplifying obtains the following solutions;

when $\Omega = \lambda^2 - 4\mu\nu + 4\mu > 0$ and $\lambda(\nu-1) \neq 0$ (or $\mu(\nu-1) \neq 0$),

$$u_{1_1}(x,t) = 2(\nu-1)\times\left\{d - \frac{1}{2(\nu-1)}\left(\lambda + \sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi)\right)\right\} + \alpha_0. \tag{17}$$

$$u_{1_2}(x,t) = 2(\nu-1)\times\left\{d - \frac{1}{2(\nu-1)}\left(\lambda + \sqrt{\Omega}\coth(\frac{1}{2}\sqrt{\Omega}\,\xi)\right)\right\} + \alpha_0. \tag{18}$$

$$u_{1_3}(x,t) = 2(\nu-1)\times\left[d - \frac{1}{2(\nu-1)}\left\{\lambda + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}\,\xi) \pm i\,\mathrm{sec}\,h(\sqrt{\Omega}\,\xi)\right)\right\}\right] + \alpha_0. \tag{19}$$

$$u_{1_4}(x,t) = 2(\nu-1)\times\left[d - \frac{1}{2(\nu-1)}\left\{\lambda + \sqrt{\Omega}\left(\coth(\sqrt{\Omega}\,\xi) \pm \mathrm{csc}\,h(\sqrt{\Omega}\,\xi)\right)\right\}\right] + \alpha_0. \tag{20}$$

$$u_{1_5}(x,t) = 2(\nu-1)\times\left[d - \frac{1}{4(\nu-1)}\left\{2\lambda + \sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\,\xi) + \coth(\frac{1}{4}\sqrt{\Omega}\,\xi)\right)\right\}\right] + \alpha_0. \tag{21}$$

$$u_{1_6}(x,t) = 2(\nu-1)\times\left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \frac{\pm\sqrt{\Omega(A^2+B^2)} - A\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi)}{A\sinh(\sqrt{\Omega}\,\xi) + B}\right\}\right] + \alpha_0. \tag{22}$$

$$u_{1_7}(x,t) = 2(\nu-1)\times\left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \frac{\pm\sqrt{\Omega(A^2+B^2)} + A\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi)}{A\sinh(\sqrt{\Omega}\,\xi) + B}\right\}\right] + \alpha_0, \tag{23}$$

where A and B are real constants.

$$u_{1_8}(x,t) = 2(\nu-1)\times\left\{d + \frac{2\mu\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \lambda\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)}\right\} + \alpha_0. \tag{24}$$

$$u_{1_9}(x,t) = 2(\nu-1)\times\left\{d + \frac{2\mu\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \lambda\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)}\right\} + \alpha_0. \tag{25}$$

$$u_{1_{10}}(x,t) = 2(\nu-1)\times\left\{d + \frac{2\mu\cosh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\sinh(\sqrt{\Omega}\,\xi) - \lambda\cosh(\sqrt{\Omega}\,\xi) \pm i\sqrt{\Omega}}\right\} + \alpha_0. \tag{25}$$

$$u_{1_{11}}(x,t) = 2(\nu-1)\times\left\{d + \frac{2\mu\sinh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi) - \lambda\sinh(\sqrt{\Omega}\,\xi) \pm \sqrt{\Omega}}\right\} + \alpha_0. \tag{27}$$

When $\Omega = \lambda^2 - 4\mu\nu + 4\mu < 0$ and $\lambda(\nu-1) \neq 0$ (or $\mu(\nu-1) \neq 0$),

$$u_{1_{12}}(x,t) = 2(\nu-1)\times\left\{d + \frac{1}{2(\nu-1)}\left(-\lambda + \sqrt{-\Omega}\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\} + \alpha_0. \tag{28}$$

$$u_{1_{13}}(x,t) = 2(\nu-1)\times\left\{d - \frac{1}{2(\nu-1)}\left(\lambda + \sqrt{-\Omega}\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\} + \alpha_0. \tag{29}$$

$$u_{1_{14}}(x,t) = 2(\nu-1)\times\left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}\,\xi) \pm \sec(\sqrt{-\Omega}\,\xi)\right)\right\}\right] + \alpha_0. \tag{30}$$

$$u_{1_{15}}(x,t) = 2(\nu-1)\times\left[d - \frac{1}{2(\nu-1)}\left\{\lambda + \sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}\,\xi) \pm \csc(\sqrt{-\Omega}\,\xi)\right)\right\}\right] + \alpha_0. \tag{31}$$

$$u_{1_{16}}(x,t) = 2(\nu-1)\times\left[d + \frac{1}{4(\nu-1)}\left\{-2\lambda + \sqrt{-\Omega}\left(\tan(\frac{1}{4}\sqrt{-\Omega}\,\xi) - \cot(\frac{1}{4}\sqrt{-\Omega}\,\xi)\right)\right\}\right] + \alpha_0. \tag{32}$$

$$u_{1_{17}}(x,t) = 2(\nu-1)\times\left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \frac{\pm\sqrt{-\Omega(A^2-B^2)} - A\sqrt{-\Omega}\,\cos(\sqrt{-\Omega}\,\xi)}{A\sin(\sqrt{-\Omega}\,\xi) + B}\right\}\right] + \alpha_0. \tag{33}$$

$$u_{1_{18}}(x,t) = 2(\nu-1)\times\left[d + \frac{1}{2(\nu-1)}\left\{-\lambda + \frac{\pm\sqrt{-\Omega(A^2-B^2)} + A\sqrt{-\Omega}\,\cos(\sqrt{-\Omega}\,\xi)}{A\sin(\sqrt{-\Omega}\,\xi) + B}\right\}\right] + \alpha_0, \tag{34}$$

where A and B are arbitrary constants such that $A^2 - B^2 > 0$.

$$u_{1_{19}}(x,t) = 2(\nu-1)\times\left\{d - \frac{2\mu\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi) + \lambda\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}\right\} + \alpha_0. \tag{35}$$

$$u_{1_{20}}(x,t) = 2(\nu-1)\times\left\{d + \frac{2\mu\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi) - \lambda\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}\right\} + \alpha_0. \tag{36}$$

$$u_{1_{21}}(x,t) = 2(\nu-1)\times\left\{d - \frac{2\mu\cos(\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\,\xi) + \lambda\cos(\sqrt{-\Omega}\,\xi) \pm \sqrt{-\Omega}}\right\} + \alpha_0. \tag{37}$$

$$u_{1_{22}}(x,t) = 2(\nu-1)\times\left\{d + \frac{2\mu\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega}\,\xi) - \lambda\sin(\sqrt{-\Omega}\,\xi) \pm \sqrt{-\Omega}}\right\} + \alpha_0. \tag{38}$$

When $\mu = 0$ and $\lambda(\nu-1) \neq 0$,

$$u_{1_{23}}(x,t) = 2(\nu-1)\times\left\{d - \frac{\lambda k}{(\nu-1)\left\{k + \cosh(\lambda\,\xi) - \sinh(\lambda\,\xi)\right\}}\right\} + \alpha_0. \tag{39}$$

$$u_{1_{24}}(x,t)=2(\nu-1)\times\left\{d-\frac{\lambda\{\cosh(\lambda\,\xi)+\sinh(\lambda\,\xi)\}}{(\nu-1)\{k+\cosh(\lambda\,\xi)+\sinh(\lambda\,\xi)\}}\right\}+\alpha_0, \tag{40}$$

where k is an arbitrary constant.

When $(\nu-1)\neq 0$ and $\lambda=\mu=0$, the solution of Eq. (10) is;

$$u_{1_{25}}(x,t)=2(\nu-1)\times\left\{d-\frac{1}{(\nu-1)\,\xi+c_1}\right\}+\alpha_0, \tag{41}$$

where c_1 is an arbitrary constant.

Substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (15) and simplifying obtains the following solutions;
when $\Omega=\lambda^2-4\mu\nu+4\mu>0$ and $\lambda(\nu-1)\neq 0$ (or $\mu\,(\nu-1)\neq 0$),

$$u_{2_1}(x,t)=\left(2d^2-2\mu+2\lambda d-2\,\nu d^2\right)\times\left\{d-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{\Omega}\,\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi)\right)\right\}^{-1}+\alpha_0. \tag{42}$$

$$u_{2_2}(x,t)=\left(2d^2-2\mu+2\lambda d-2\nu d^2\right)\times\left\{d-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{\Omega}\,\coth(\frac{1}{2}\sqrt{\Omega}\,\xi)\right)\right\}^{-1}+\alpha_0. \tag{43}$$

$$u_{2_3}(x,t)=\left(2d^2-2\mu+2\lambda d-2\nu d^2\right)\times\left[d-\frac{1}{2(\nu-1)}\left\{\lambda+\sqrt{\Omega}\left(\tanh(\sqrt{\Omega}\,\xi)\pm i\sec h(\sqrt{\Omega}\,\xi)\right)\right\}\right]^{-1}+\alpha_0. \tag{44}$$

Similarly, the other families of exact solutions of Eq. (7), which are omitted for convenience, can be written down.
When $\Omega=\lambda^2-4\mu\nu+4\mu<0$ and $\lambda(\nu-1)\neq 0$ (or $\mu\,(\nu-1)\neq 0$),

$$u_{2_{12}}(x,t)=\left(2d^2-2\mu+2\lambda d-2\nu d^2\right)\times\left\{d+\frac{1}{2(\nu-1)}\left(-\lambda+\sqrt{-\Omega}\,\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}^{-1}+\alpha_0. \tag{45}$$

$$u_{2_{13}}(x,t)=\left(2d^2-2\mu+2\lambda d-2\nu d^2\right)\times\left\{d-\frac{1}{2(\nu-1)}\left(\lambda+\sqrt{-\Omega}\,\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}^{-1}+\alpha_0. \tag{46}$$

$$u_{2_{14}}(x,t)=\left(2d^2-2\mu+2\lambda d-2\nu d^2\right)\times\left\{d+\frac{1}{2(\nu-1)}\left\{-\lambda+\sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}\,\xi)\pm\sec(\sqrt{-\Omega}\,\xi)\right)\right\}\right\}^{-1}+\alpha_0. \tag{47}$$

When $(\nu-1)\neq 0$ and $\lambda=\mu=0$, the solution of Eq. (7) is;

$$u_{2_{25}}(x,t)=\left(2d^2-2\mu+2\lambda d-2\nu d^2\right)\times\left\{d-\frac{1}{(\nu-1)\,\xi+c_1}\right\}^{-1}+\alpha_0. \tag{48}$$

where c_1 is an arbitrary constant.

The other families of exact solutions of Eq. (7), which are omitted for convenience, can be written down.

Finally, substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (16) and simplifying obtains the following solutions;

when $\Omega = \lambda^2 - 4\mu\nu + 4\mu > 0$ and $\lambda(\nu-1) \neq 0$ (or $\mu(\nu-1) \neq 0$),

$$u_{3_1}(x,t) = 2(\nu-1) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi) \right) \right\} - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu-1)} \right) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi) \right) \right\}^{-1} + \alpha_0. \tag{49}$$

$$u_{3_2}(x,t) = 2(\nu-1) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{\Omega}\cot h(\frac{1}{2}\sqrt{\Omega}\,\xi) \right) \right\} - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu-1)} \right) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{\Omega}\cot h(\frac{1}{2}\sqrt{\Omega}\,\xi) \right) \right\}^{-1} + \alpha_0. \tag{50}$$

$$u_{3_3}(x,t) = 2(\nu-1) \times \left[\frac{1}{2(\nu-1)} \left\{ \sqrt{\Omega} \left(\tanh(\sqrt{\Omega}\,\xi) \pm i \sec h(\sqrt{\Omega}\,\xi) \right) \right\} \right] - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu-1)} \right) \times \left[\frac{1}{2(\nu-1)} \left\{ \sqrt{\Omega} \left(\tanh(\sqrt{\Omega}\,\xi) \pm i \sec h(\sqrt{\Omega}\,\xi) \right) \right\} \right]^{-1} + \alpha_0. \tag{51}$$

For simplicity, other families of exact solutions are omitted.

When $\Omega = \lambda^2 - 4\mu\nu + 4\mu < 0$ and $\lambda(\nu-1) \neq 0$ (or $\mu(\nu-1) \neq 0$);

$$u_{3_{12}}(x,t) = 2(\nu-1) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{-\Omega}\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi) \right) \right\} - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu-1)} \right) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{-\Omega}\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi) \right) \right\}^{-1} + \alpha_0. \tag{52}$$

$$u_{3_{13}}(x,t) = 2(\nu-1) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{-\Omega}\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi) \right) \right\} - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu-1)} \right) \times \left\{ \frac{1}{2(\nu-1)} \left(\sqrt{-\Omega}\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi) \right) \right\}^{-1} + \alpha_0. \tag{53}$$

$$u_{3_{14}}(x,t) = 2(\nu-1) \times \left\{ \frac{1}{2(\nu-1)} \left\{ \sqrt{-\Omega} \left(\tan(\sqrt{-\Omega}\,\xi) \pm \sec(\sqrt{-\Omega}\,\xi) \right) \right\} \right\} - \left(\frac{4\mu\nu - 4\mu - \lambda^2}{2(\nu-1)} \right) \times \left\{ \frac{1}{2(\nu-1)} \left\{ \sqrt{-\Omega} \left(\tan(\sqrt{-\Omega}\,\xi) \pm \sec(\sqrt{-\Omega}\,\xi) \right) \right\} \right\}^{-1} + \alpha_0. \tag{54}$$

When $(\nu-1) \neq 0$ and $\lambda = \mu = 0$, the solution of the Burgers equation is;

$$u_{3_{25}}(x,t)=2(\nu-1)\times\left\{\frac{\lambda}{2(\nu-1)}-\frac{1}{(\nu-1)\ \xi+c_1}\right\}$$
$$-\left(\frac{4\mu\nu-4\mu-\lambda^2}{2(\nu-1)}\right)\times\left\{\frac{\lambda}{2(\nu-1)}-\frac{1}{(\nu-1)\ \xi+c_1}\right\}^{-1}+\alpha_0. \tag{55}$$

where c_1 is an arbitrary constant.

The other families of exact solutions of Eq. (17) are omitted for convenience.

Physical explanation

Solutions $u_{1_1}(x,t)$, $u_{1_3}(x,t)$, $u_{1_4}(x,t)$, $u_{1_6}(x,t)$, $u_{1_7}(x,t)$, $u_{1_9}(x,t)$, $u_{1_{10}}(x,t)$, $u_{1_{11}}(x,t)$, $u_{1_{23}}(x,t)$, $u_{1_{24}}(x,t)$, $u_{2_1}(x,t)$, and $u_{2_3}(x,t)$, represent kink. Kink waves are traveling waves which arise from one asymptotic state to another. The kink solutions approach to a constant at infinity. **Figure 1** below shows the shape of the exact kink-type solution $u_{1_1}(x,t)$ of the Burgers Eq. (7). Other figures are omitted for convenience.

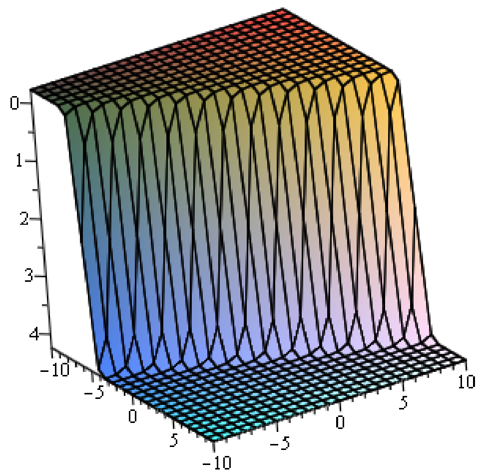

Figure 1 Graph of the solution $u_1(x,t)$ for $\lambda=1$, $\mu=-1$, $\nu=2$, $d=1$, $\alpha=1$ with $-10\le x,t\le 10$.

Solutions $u_{1_2}(x,t)$, $u_{1_5}(x,t)$, $u_{1_8}(x,t)$, $u_{1_{25}}(x,t)$ are the singular kink solution. **Figure 2** shows the shape of the exact singular kink-type solution $u_{1_2}(x,t)$ of the Burgers equation.

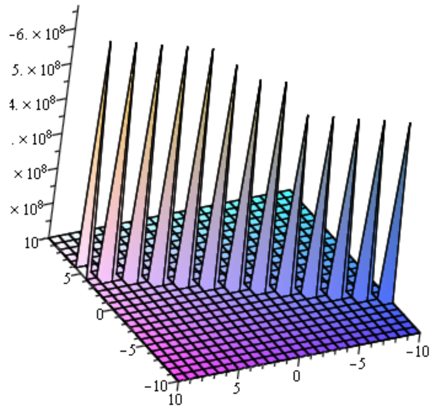

Figure 2 Graph of the solution $u_{1_2}(x,t)$, for $\lambda = 1$, $\mu = -1$, $\nu = 2$, $d = 1$, $\alpha = 1$ with $-10 \le x, t \le 10$.

Solutions $u_{1_{12}}(x,t)$ - $u_{1_{22}}(x,t)$ represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions that are periodic, such as $\cos(x-t)$. **Figure 3** below shows the periodic solution of $u_{1_{12}}(x,t)$. A graph of periodic solution $u_{1_{12}}(x,t)$, for $\lambda = 1$, $\mu = 1$, $\nu = 1$, $d = 1$, $\alpha = 1$ with $-1 \le x, t \le 1$ is shown. For convenience, other figures are omitted.

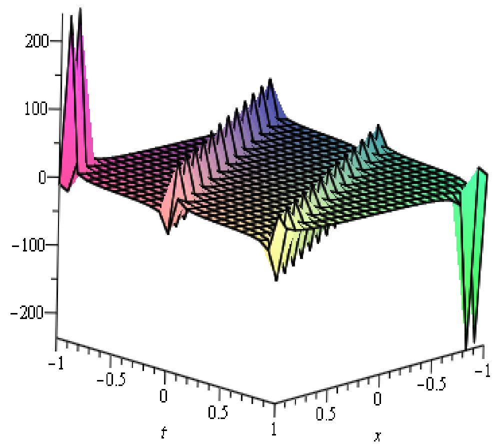

Figure 3 Graph of the solution $u_{1_{12}}(x,t)$ for $\lambda = 1$, $\mu = 1$, $\nu = 1$, $d = 1$, $\alpha = 1$ with $-1 \le x, t \le 1$.

Solutions $u_{2_{12}}(x,t)$ and $u_{2_{14}}(x,t)$ are the exact periodic traveling wave solutions. **Figure 4** shows the outline of $u_{2_{14}}(x,t)$ with $\lambda = 1$, $\mu = -1$, $\nu = 2$, $d = 3$, $\alpha = 1$ with $-10 \le x, t \le 10$.

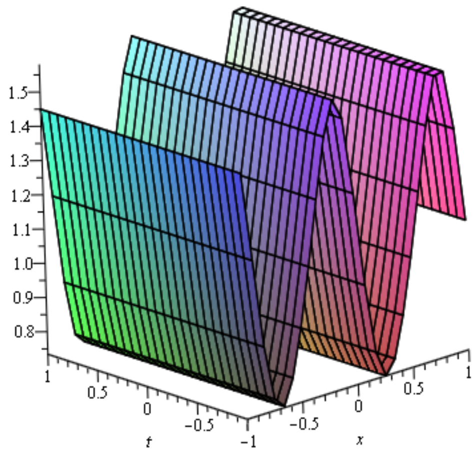

Figure 4 Graph of the solution obtained from $u_{2_{14}}(x,t)$.

Solutions $u_{2_{13}}(x,t)$ and $u_{2_{25}}(x,t)$ are the exact singular periodic traveling wave solutions. **Figure 5** shows the shape of $u_{2_{13}}(x,t)$ with $\lambda = 1$, $\mu = -1$, $\nu = 2$, $d = 3$, $\alpha = 1$ with $-10 \le x,t \le 10$).

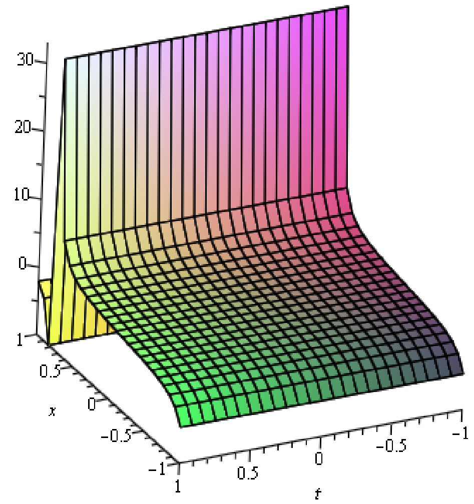

Figure 5 Graph of the solution obtained from $u_{2_{13}}(x,t)$.

Solution $u_{3_3}(x,t)$ shows kink traveling wave solutions. Solutions $u_{3_1}(x,t)$ and $u_{3_2}(x,t)$ are singular Kink solutions. For convenience, other figures are omitted.

Solutions $u_{3_{12}}(x,t)$, $u_{3_{13}}(x,t)$ and $u_{3_{25}}(x,t)$ are the exact singular periodic traveling wave solutions. **Figure 6** below shows the outline of the exact singular periodic traveling wave solution of $u_{3_{12}}(x,t)$, and other figures are omitted for convenience.

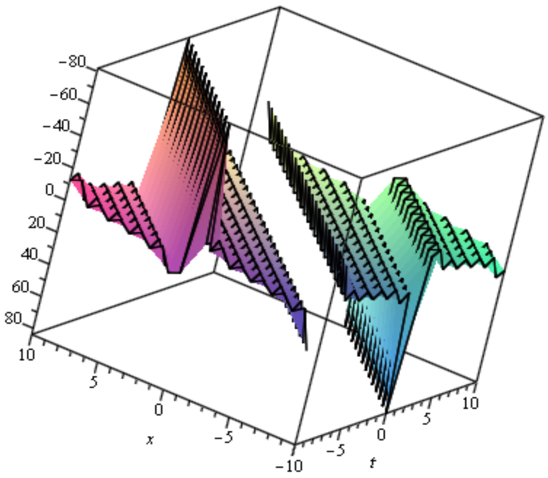

Figure 6 Graph of the solution $u_{3_{12}}(x,t)$ for $\lambda = 1$, $\mu = -1$, $\nu = 2$, $d = 3$, $\alpha = 1$ with $-10 \le x, t \le 10$.

Solution $u_{3_{14}}(x,t)$ is the exact periodic traveling wave solutions. **Figure 7** below shows the outline of the exact periodic traveling wave solution of $u_{3_{14}}(x,t)$.

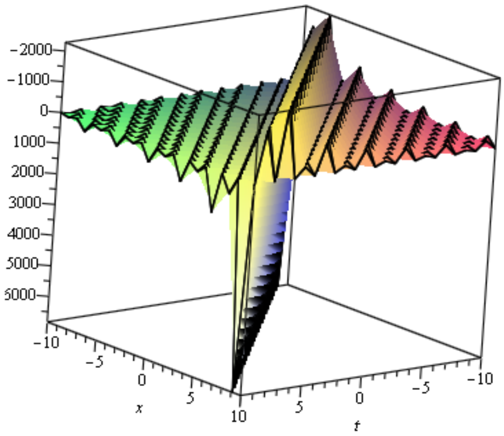

Figure 7 The shape of $u_{3_{14}}(x,t)$ with $\lambda = 2$, $\mu = 3$, $\nu = 2$, $d = 3$, $\alpha = 1$ with $-10 \le x, t \le 10$.

Remark 2 The obtained solutions have been checked by putting them back into the original equation and have been found to be correct.

From the above solutions, it is observed that, if $\nu = 0$, $d = 0$, and $\alpha_0 = \lambda + \sqrt{\lambda^2 - 4\mu}$, and λ and μ are replaced by $-\lambda$ and $-\mu$ respectively in the solutions, then the Kheiri *et al*. [27] solution (13) is identical to the present solution u_{1_1} when $B = 0$, and solution (13) is identical to the present solution u_{1_2} when $A = 0$. Similarly, the Kheiri *et al*. solution (14) is identical to the present solutions $u_{1_{12}}$ and $u_{1_{13}}$

when $B = 0$ and $A = 0$ respectively. On the other hand, the Kheiri *et al*. solution (14) is identical to the present solution $u_{1_{25}}$. Kheiri *et al*. [27] did not find any more solutions, but by using the proposed new (G'/G)-expansion method, for set 1, apart from these solutions, 20 more new solutions are obtained. The solutions obtained in this article for set 2 and for set 3 are not obtained by Kheiri *et al*. It can be shown that solutions obtained by the improved (G'/G)-expansion method [24] and the basic (G'/G)-expansion method [13] are only special cases of the proposed (G'/G)-expansion method.

Conclusions

In this article, a new (G'/G)-expansion method is initiated and applied to the Burgers equation. Abundant exact traveling wave solutions are constructed for this equation by the proposed method. It is noteworthy to observe that our solutions are more general and contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. This study shows that the proposed method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. It is important to note that the basic (G'/G)-expansion method, the improved (G'/G)-expansion and the generalized and improved (G'/G)-expansion method are only special cases of the proposed new (G'/G)-expansion method, and thus the new (G'/G)-expansion method would be a powerful mathematical tool for solving NLEEs.

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Appendix

The solutions of Eq. (7) are;
when

$$\Omega = \lambda^2 - 4\mu\nu + 4\mu > 0 \text{ and } \lambda(\nu-1) \neq 0 \text{ (or } \mu(\nu-1) \neq 0\text{)},$$

$$\Phi_1 = \frac{-1}{2(\nu-1)}\left\{\lambda + \sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi)\right\},$$

$$\Phi_2 = \frac{-1}{2(\nu-1)}\left\{\lambda + \sqrt{\Omega}\coth(\frac{1}{2}\sqrt{\Omega}\,\xi)\right\},$$

$$\Phi_3 = \frac{-1}{2(\nu-1)}\left[\lambda + \sqrt{\Omega}\left\{\tanh(\sqrt{\Omega}\,\xi) \pm i\,\mathrm{sec}\,h(\sqrt{\Omega}\,\xi)\right\}\right],$$

$$\Phi_4 = \frac{-1}{2(\nu-1)}\left[\lambda + \sqrt{\Omega}\left\{\coth(\sqrt{\Omega}\,\xi) \pm \mathrm{csc}\,h(\sqrt{\Omega}\,\xi)\right\}\right],$$

$$\Phi_5 = \frac{-1}{4(\nu-1)}\left[2\lambda + \sqrt{\Omega}\left\{\tanh(\frac{1}{4}\sqrt{\Omega}\,\xi) + \coth(\frac{1}{4}\sqrt{\Omega}\,\xi)\right\}\right],$$

$$\Phi_6 = \frac{1}{2(\nu-1)}\left[-\lambda + \frac{\pm\sqrt{\Omega(A^2+B^2)} - A\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi)}{A\sinh(\sqrt{\Omega}\,\xi) + B}\right],$$

$$\Phi_7 = \frac{1}{2(\nu-1)}\left[-\lambda - \frac{\pm\sqrt{\Omega(A^2+B^2)} + A\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi)}{A\sinh(\sqrt{\Omega}\,\xi) + B}\right],$$

where A and B are non-zero constants.

$$\Phi_8 = \frac{2\mu\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \lambda\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)},$$

$$\Phi_9 = \frac{2\mu\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \lambda\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)},$$

$$\Phi_{10} = \frac{2\mu\cosh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\sinh(\sqrt{\Omega}\,\xi) - \lambda\cosh(\sqrt{\Omega}\,\xi) \pm i\sqrt{\Omega}},$$

$$\Phi_{11} = \frac{2\mu\sinh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi) - \lambda\sinh(\sqrt{\Omega}\,\xi) \pm \sqrt{\Omega}},$$

when $\Omega = \lambda^2 - 4\mu\nu + 4\mu < 0$ and $\lambda(\nu-1) \neq 0$ (or $\mu(\nu-1) \neq 0$)

$$\Phi_{12} = \frac{1}{2(\nu-1)}\left\{-\lambda + \sqrt{-\Omega}\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right\},$$

$$\Phi_{13} = \frac{-1}{2(\nu-1)}\left\{\lambda + \sqrt{-\Omega}\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right\},$$

$$\Phi_{14} = \frac{1}{2(\nu-1)}\left[-\lambda + \sqrt{-\Omega}\left\{\tan(\sqrt{-\Omega}\,\xi) \pm \sec(\sqrt{-\Omega}\,\xi)\right\}\right],$$

$$\Phi_{15} = \frac{-1}{2(\nu-1)}\left[\lambda + \sqrt{-\Omega}\left\{\cot(\sqrt{-\Omega}\,\xi) \pm \csc(\sqrt{-\Omega}\,\xi)\right\}\right],$$

$$\Phi_{16} = \frac{1}{4(\nu-1)}\left[-2\lambda + \sqrt{-\Omega}\left\{\tan(\frac{1}{4}\sqrt{-\Omega}\,\xi) - \cot(\frac{1}{4}\sqrt{-\Omega}\,\xi)\right\}\right],$$

$$\Phi_{17} = \frac{1}{2(\nu-1)}\left[-\lambda + \frac{\pm\sqrt{-\Omega(A^2-B^2)} - A\sqrt{-\Omega}\,\cos(\sqrt{-\Omega}\,\xi)}{A\,\sin(\sqrt{-\Omega}\,\xi) + B}\right],$$

$$\Phi_{18} = \frac{1}{2(\nu-1)}\left\{-\lambda - \frac{\pm\sqrt{-\Omega(A^2-B^2)} + A\sqrt{-\Omega}\,\cos(\sqrt{-\Omega}\,\xi)}{A\,\sin(\sqrt{-\Omega}\,\xi) + B}\right\},$$

where A and B are non-zero constants and satisfy the condition $A^2 - B^2 > 0$.

$$\Phi_{19} = \frac{-2\mu\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi) + \lambda\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)},$$

$$\Phi_{20} = \frac{2\mu\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi) - \lambda\,\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}$$

$$\Phi_{21} = \frac{-2\mu\cos(\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\,\xi) + \lambda\cos(\sqrt{-\Omega}\,\xi) \pm \sqrt{-\Omega}},$$

$$\Phi_{22} = \frac{2\mu\sin(\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega}\,\xi) - \lambda\,\sin(\sqrt{-\Omega}\,\xi) \pm \sqrt{-\Omega}}.$$

When $\mu = 0$ and $\lambda(\nu-1) \neq 0$,

$$\Phi_{23} = \frac{-\lambda\,k}{(\nu-1)\{k + \cosh(\lambda\,\xi) - \sinh(\lambda\,\xi)\}},$$

$$\Phi_{24} = \frac{-\lambda\,\{\cosh(\lambda\,\xi) + \sinh(\lambda\,\xi)\}}{(\nu-1)\{k + \cosh(\lambda\,\xi) + \sinh(\lambda\,\xi)\}},$$

where k is an arbitrary constant.

When $(\nu-1) \neq 0$ and $\lambda = \mu = 0$, the solution of Eq. (6) is;

$$\Phi_{25} = \frac{-1}{(\nu-1)\,\xi + c_1},$$

where c_1 is an arbitrary constant.