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Generalized Fractional Integration of the $\overline{H}$-Function Involving General Class of Polynomials

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Abstract

In the present paper, we consider 2 integral transforms involving the Appell function F_3 in the kernels. They generalize the fractional integral operators given by Saigo (1978). Formulas for compositions of such generalized fractional integrals with the product of the $\overline{H}$-function and a general class of polynomials are proved. The results are established in terms of $\overline{H}$-function due to Inayat-Hussain (1987(a), 1987(b)). The obtained results of this paper provide an extension of the results given by the literature.

Keywords: Generalized fractional calculus operators, $\overline{H}$-function, Generalized Wright hypergeometric function, Generalized Wright-Bessel function, the poly-logarithm function, Generalized Riemann Zeta function and Whittaker function

Introduction

A significantly large number of works on the subject of fractional calculus give interesting accounts of the theory and applications of fractional calculus operators in many different areas of mathematical analysis. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical systems, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, non-linear biological systems and astrophysics. In the last 5 decades, a number of workers including Love [1], Srivastava and Saxena [2], Saxena *et al*. [3-6], Saigo [7], Kilbas [8], Samko, Kilbas and Marichev [9], Miller and Ross [10], Baleanu, Mustafa and Agarwal [11], Baleanu *et al*. [12,13], Agarwal [14-17], Srivastava and Agarwal [18] and Ram and Kumar [19,20], etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration.

Several definitions of the operators of the classical and generalized fractional calculus are already well known and widely used in the applications to mathematically model fractional order. Motivated by these avenues of applications, a remarkably large number of fractional integral formulas involving a variety of special functions have been developed by many authors (see, for example [21-25]).

For our purpose, we begin by recalling some known functions and earlier works.

Let $\gamma > 0$ and $\alpha, \alpha', \beta, \beta', \gamma \in C$, then the fractional integral operators $I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}$ and $I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}$ of a function $f(x)$ is defined by Saigo and Maeda [26], in the following form;

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\right) f(t)\,dt, \quad \mathrm{Re}(\gamma) > 0, \tag{1}$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x\right) f(t)\,dt, \quad \mathrm{Re}(\gamma) > 0 \cdot \tag{2}$$

These operators reduce to the fractional integral operators introduced by Saigo [7], due to the following relations;

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x), \qquad (\gamma \in C), \tag{3}$$

and

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f(x), \qquad (\gamma \in C). \tag{4}$$

Recently, Ram and Kumar [19] have obtained the images of the product of 2 H-functions in Saigo-Maeda operators; Saxena, Ram and Kumar [3] have obtained the generalized fractional integral formulae of the product of Bessel functions of the first kind involving Saigo-Maeda fractional integral operators. Similarly, generalized fractional calculus formulae of the Aleph-function associated with the Appell function F_3 is given by Saxena *et al.* [5,6], and Ram and Kumar [20].

A lot of research work has been recently come up on the study and development of a function that is more general than Fox's H-function, known as the $\overline{H}$-function. The $\overline{H}$-function was introduced by Inayat-Hussain [27,28] and is defined and represented in the following manner (see for example [29,30]).

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \overline{\theta}(\xi)\, z^{\xi}\, d\xi, \quad (z \neq 0), \tag{5}$$

where $i = \sqrt{-1}$ and;

$$\overline{\theta}(s) = \frac{\prod_{j=1}^{M} \Gamma\left(b_j - \beta_j \xi\right) \prod_{j=1}^{N} \left\{\Gamma\left(1-a_j+\alpha_j\xi\right)\right\}^{A_j}}{\prod_{j=M+1}^{Q} \left\{\Gamma\left(1-b_j+\beta_j\xi\right)\right\}^{B_j} \prod_{j=N+1}^{P} \Gamma\left(a_j-\alpha_j\xi\right)}. \tag{6}$$

Here, $a_j\ (j=1,\ldots,P)$ and $b_j\ (j=1,\ldots,Q)$ are complex parameters $\alpha_j = 0\ (j=1,\ldots,P)$, $\beta_j = 0\ (j=1,\ldots,Q)$ (not all zero simultaneously) and the exponents $a_j\ (j=1,\ldots,N)$ and $b_j\ (j=M+1,\ldots,Q)$ can take on integer values, the reader can also refer the papers [27,28] for more detail. The following sufficient conditions for the absolute convergence of the defining integral for the $\overline{H}$-function given by Bushman and Srivastava [29];

$$\Omega = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} \left| A_j \alpha_j \right| - \sum_{j=M+1}^{Q} \left| B_j \beta_j \right| - \sum_{j=N+1}^{P} \alpha_j > 0, \tag{7}$$

and $\left|\arg z\right| < \frac{1}{2} \Omega \pi$.

Also $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [31];

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k, \quad n = 0,1,2,... \tag{8}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}\,(n,k \geq 0)$ are arbitrary constants, real or complex. Suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as special cases [32].

The paper is organized as follows: The power function formula for the operators $\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)$ and $\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)$ are presented in Section 2. The composition formula of the Marichev-Saigo-Maeda fractional integral operators (1) and (2) with the product of the $\overline{H}$-function and a general class of polynomials in Section 3. Special cases giving compositions of fractional integrals with the product of the generalized Wright hypergeometric function, generalized Wright-Bessel function, Poly-logarithm function, generalized Riemann Zeta and Whittaker function are considered in Section 4. In the sequel, conclusions are considered in Section 5.

Preliminary Lemmas

In this section, we presented the power function formula for the operators $\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)$ and $\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)$, are required to establish our main results:

Lemma 1 [26]. Let $\alpha,\alpha',\beta,\beta',\gamma \in C$; if $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\rho) > \max\left[0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')\right]$, then;

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}. \tag{9}$$

In particular, from relation (3), we have;

$$\left(I_{0+}^{\alpha,\beta,\eta}\, t^{\rho-1}\right)(x) = x^{\rho-\beta-1} \frac{\Gamma(\rho)\Gamma(\rho+\eta-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\eta)}, \tag{10}$$

and if we put $\beta = -\alpha$ in (10), then we have;

$$\left(I_{0+}^{\alpha}\, t^{\rho-1}\right)(x) = x^{\rho+\alpha-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}, \qquad \mathrm{Re}(\alpha)>0, \mathrm{Re}(\rho)>0\,. \tag{11}$$

Lemma 2 [26]. Let $\alpha,\alpha',\beta,\beta',\gamma\in C$; if $\mathrm{Re}(\gamma)>0,$

$\mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right]$, then;

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\, x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}. \tag{12}$$

In particular, from relation (4), we have;

$$\left(I_{-}^{\alpha,\beta,\eta}\, t^{\rho-1}\right)(x) = x^{\rho-\beta-1}\frac{\Gamma(\beta-\rho+1)\Gamma(\eta-\rho+1)}{\Gamma(1-\rho)\Gamma(\alpha+\beta+\eta-\rho+1)}, \tag{13}$$

and if we put $\beta=-\alpha$ in (13) then we have;

$$\left(I_{-}^{\alpha}\, t^{\rho-1}\right)(x) = x^{\rho+\alpha-1}\frac{\Gamma(1-\alpha-\rho)}{\Gamma(1-\rho)}, \qquad 0<\mathrm{Re}(\alpha)<1-\mathrm{Re}(\rho)\,. \tag{14}$$

Main results

In this section, we shall establish 4 fractional integral formulae for the Marichev-Saigo-Maeda fractional integral operators (1) and (2). The results are given in the form of the theorems. The theorems are derived and then the remaining 2 results are deduced as their corollaries.

Theorem 1 Let $\alpha,\alpha',\beta,\beta',\gamma,\mu,z\in C,\mathrm{Re}(\gamma)>0$ and $\lambda_j,\sigma>0\ \left(j\in\{1,2,...,s\}\right)$. Further let the constants $M,N,P,Q\in N_0$, $a_j\ (j=1,...,P), b_j\ (j=1,...,Q)$ be complex, $|\arg z|<\pi\Omega/2,\ \Omega>0$ and the exponents $A_j\ (j=1,...,N), B_j\ (j=M+1,...,Q)\notin N$ be given and satisfy the condition;

$\mathrm{Re}(\mu)+\sigma\min\limits_{1\le j\le M}\mathrm{Re}\left(\dfrac{b_j}{\beta_j}\right)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right]$. Then we have the following relation;

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]\overline{H}_{P,Q}^{M,N}\left[z\,t^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\dots c_s^{k_s}\ (x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$\overline{H}^{M,N+3}_{P+3,Q+3}\left[zx^{\sigma}\left|\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right),\left(1-\mu+\alpha+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right),\\ \left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q},\left(1-\mu+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right),\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P}\\ \left(1-\mu+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right),\left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right)\end{matrix}\right]. \tag{15}$$

Proof:

In order to prove (15), we first express the product of a general class of polynomials occurring on its left-hand side in the series from given by (8), replacing the $\overline{H}$-function in terms of Mellin-Barnes contour integral with the help of Eq. (5), interchange the order of summations, we obtain the following form (say I);

$$I=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{\left(-n_1\right)_{m_1k_1}\ldots\left(-n_s\right)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}$$
$$\left\{\frac{1}{2\pi i}\int_L\overline{\theta}(\xi)\,z^{\xi}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi-1}\right)(x)\,d\xi\right\}$$
$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{\left(-n_1\right)_{m_1k_1}\ldots\left(-n_s\right)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}$$
$$\frac{1}{2\pi i}\int_L x^{\mu-\alpha-\alpha'+\gamma+\sum_{j=1}^{s}\lambda_j k_j-1}\left(zx^{\sigma}\right)^{\xi}\frac{\prod_{j=1}^{M}\Gamma\left(b_j-\beta_j\xi\right)\prod_{j=1}^{N}\left\{\Gamma\left(1-a_j+\alpha_j\xi\right)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b_j+\beta_j\xi\right)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma\left(a_j-\alpha_j\xi\right)}$$
$$\frac{\Gamma\left(\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi\right)\Gamma\left(\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi+\gamma-\alpha-\alpha'-\beta\right)\Gamma\left(\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi+\beta'-\alpha'\right)}{\Gamma\left(\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi+\gamma-\alpha-\alpha'\right)\Gamma\left(\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi+\gamma-\alpha'-\beta\right)\Gamma\left(\mu+\sum_{j=1}^{s}\lambda_j k_j+\sigma\xi+\beta'\right)}d\xi$$

Finally, re-interpreting the Mellin-Barnes counter integral in terms of the $\overline{H}$-function, we obtain the right-hand side of (15). This completes the proof of Theorem 1.

Corollary 1.1 Let $\alpha,\beta,\gamma,\mu,z\in C,\mathrm{Re}\left(\alpha\right)>0$ and $\lambda_j,\sigma>0$ $\left(j\in\{1,2,\ldots,s\}\right)$, and satisfy the condition;

$\mathrm{Re}(\mu)+\sigma\min\limits_{1\le j\le M}\mathrm{Re}\left(\dfrac{b_j}{\beta_j}\right)>\max\left[0,\mathrm{Re}\left(\beta-\gamma\right)\right]$. Then we have the following relation;

$$\left\{ I_{0+}^{\alpha,\beta,\gamma}\left(t^{\mu-1}\prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right] \overline{H}_{P,Q}^{M,N}\left[z\,t^{\sigma} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right]\right)\right\}(x)$$

$$= x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!} A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\, c_1^{k_1}\ldots c_s^{k_s}\;(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[zx^{\sigma} \left| \begin{matrix} \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right),\left(1-\mu+\beta-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right), (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right), \left(1-\mu-\alpha-\gamma-\sum_{j=1}^{s}\lambda_j k_j,\sigma;1\right) \end{matrix} \right. \right], \tag{16}$$

where the conditions of existence of the above corollary follow easily with the help of (15).

Remark 1 We can also obtain results concerning Riemann-Liouville and Erdélyi-Kober fractional integral operators by putting $\beta=-\alpha$ and $\beta=0$ respectively in Corollary 1.1.

Theorem 2 Let $\alpha,\alpha',\beta,\beta',\gamma,\mu,z\in C,\mathrm{Re}(\gamma)>0$ and $\lambda_j,\sigma>0$ $(j\in\{1,2,\ldots,s\})$. Further let the constants $M,N,P,Q\in N_0, a_j\,(j=1,\ldots,P), b_j\,(j=1,\ldots,Q)$ be complex,
$|\arg z|<\pi\Omega/2,\ \Omega>0$ and the exponents $A_j\,(j=1,\ldots,N),\ B_j\,(j=M+1,\ldots,Q)\notin N$ be given and satisfy the condition;

$$\mathrm{Re}(\mu)-\sigma\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right].$$

Then we have the following relation;

$$\left\{ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}\prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right] \overline{H}_{P,Q}^{M,N}\left[z\,t^{\sigma} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right]\right)\right\}(x)$$

$$= x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!} A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\, c_1^{k_1}\ldots c_s^{k_s}\;(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$\overline{H}_{P+3,Q+3}^{M+3,N}\left[zx^{\sigma} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},\left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma\right), \\ \left(1+\alpha+\alpha'-\gamma-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma\right),\left(1+\alpha+\beta'-\gamma-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma\right), \end{matrix} \right. \right.$$

$$\left.\begin{array}{c}\left(1+\alpha+\alpha'+\beta'-\gamma-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma\right),\left(1+\alpha-\beta-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma\right)\\ \left(1-\beta-\mu-\sum_{j=1}^{s}\lambda_j k_j,\sigma\right),\left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q}\end{array}\right]. \tag{17}$$

Proof:

In order to prove (17), we first express the product of a general class of polynomials occurring on its left-hand side in the series form given by (8), replacing the $\overline{H}$-function in terms of Mellin-Barnes contour integral with the help of Eq. (5), interchange the order of summations, we obtain the following form (say *I*);

$$I=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}$$
$$\left\{\frac{1}{2\pi i}\int_L\overline{\theta}(\xi)\,z^{\xi}\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\mu+\sum_{j=1}^{s}\lambda_jk_j+\sigma\xi-1}\right)(x)\,d\xi\right\}$$
$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}$$
$$\frac{1}{2\pi i}\int_L x^{\mu-\alpha-\alpha'+\gamma+\sum_{j=1}^{s}\lambda_jk_j-1}\left(zx^{\sigma}\right)^{\xi}\frac{\prod_{j=1}^{M}\Gamma\left(b_j-\beta_j\xi\right)\prod_{j=1}^{N}\left\{\Gamma\left(1-a_j+\alpha_j\xi\right)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b_j+\beta_j\xi\right)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma\left(a_j-\alpha_j\xi\right)}$$
$$\frac{\Gamma\left(1+\alpha+\alpha'-\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j-\sigma\xi\right)\Gamma\left(1+\alpha+\beta'-\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j-\sigma\xi\right)\Gamma\left(1-\beta-\mu-\sum_{j=1}^{s}\lambda_jk_j-\sigma\xi\right)}{\Gamma\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j-\sigma\xi\right)\Gamma\left(1+\alpha+\alpha'+\beta'-\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j-\sigma\xi\right)\Gamma\left(1+\alpha-\beta-\mu-\sum_{j=1}^{s}\lambda_jk_j-\sigma\xi\right)}d\xi$$

Finally, re-interpreting the Mellin-Barnes counter integral in terms of the $\overline{H}$-function, we obtain the right-hand side of (17). This completes the proof of Theorem 2.

Corollary 2.1 Let $\alpha,\beta,\gamma,\mu,z\in C,\mathrm{Re}(\alpha)>0$ and $\lambda_j,\sigma>0\ \left(j\in\{1,2,\ldots,s\}\right)$, and satisfy the condition;

$\mathrm{Re}(\mu)-\sigma\min\limits_{1\le j\le M}\mathrm{Re}\left(\dfrac{b_j}{\beta_j}\right)<1+\min\left[\mathrm{Re}(\beta),\mathrm{Re}(\gamma)\right]$. Then we have the following relation;

$$\left\{I_-^{\alpha,\beta,\gamma}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_jt^{\lambda_j}\right]\overline{H}_{P,Q}^{M,N}\left[zt^{\sigma}\left|\begin{array}{c}\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P}\\ \left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q}\end{array}\right.\right]\right)\right\}(x)$$
$$=x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}\,(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{P+2,Q+2}^{M+2,N}\left[zx^{\sigma}\left|\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P},\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma\right),\left(1+\alpha+\beta+\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma\right)\\\left(1+\beta-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma\right),\left(1+\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma\right),\left(b_j,\beta_j\right)_{1,M},\left(b_j,\beta_j;B_j\right)_{M+1,Q}\end{matrix}\right.\right] \tag{18}$$

where the conditions of existence of the above corollary follow easily from Theorem 2.

Remark 2 We can also obtain results concerning Riemann-Liouville and Erdélyi-Kober fractional integral operators by putting $\beta=-\alpha$ and $\beta=0$ respectively in Corollary 2.1.

Special cases and applications

On account of the most general nature of $\overline{H}$-functions and the general class of polynomials occurring in our main results given by (15) and (17) a large number of fractional integral formulas involving simpler functions of one variable can easily be obtained as their special cases. However, we give here only 6 special cases by way of illustration:

(i) If we take $M=1, N=P$ and $Q=Q+1$ in Theorem 1, the $\overline{H}$-function occurring therein breaks up into the generalized Wright hypergeometric function ${}_P\overline{\psi}_Q(.)$ given by [29]. Then, Theorem 1 takes the following form after a little simplification;

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_jt^{\lambda_j}\right]\,{}_P\overline{\psi}_Q\left[-z\,t^{\sigma}\left|\begin{matrix}\left(1-a_j,\alpha_j;A_j\right)_{1,P}\\\left(1-b_j,\beta_j;B_j\right)_{1,Q}\end{matrix}\right.\right]\right)\right\}(x)$$

$$= x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}\,(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{P+3,Q+4}^{1,P+3}\left[zx^{\sigma}\left|\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\\\left(1-b_j,\beta_j;B_j\right)_{1,Q},\left(1-\mu+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\\\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-a_j,\alpha_j;A_j\right)_{1,P}\\\left(1-\mu+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right)\end{matrix}\right.\right]. \tag{19}$$

The conditions of validity of the above result easily follow from (15).

Remark 3 If we set $S_{n_j}^{m_j}=1$ and $A_j,B_j=1$, and use the relation (3) and make suitable adjustment in the parameters in (19), we arrive at the known result given by Kilbas [8].

(ii) If we take $M=1, N=P=0$ and $Q=2$ in (15), and the $\overline{H}$-function occurring therein breaks up into the generalized Wright-Bessel function $\overline{J}_b^{\beta,B}(z)$, then Theorem 1 takes the following form;

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]\overline{J}_b^{\beta,B}\left(-zt^{\sigma}\right)\right)\right\}(x)$$
$$= x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\cdots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}\,(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\overline{H}_{3,5}^{1,3}\left[zx^{\sigma}\left|\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\\ (0,1),(-b,\beta;B),\left(1-\mu+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\\ \left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right)\\ \left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right)\end{matrix}\right.\right]. \tag{20}$$

The conditions of validity of the above result can easily be followed directly from (15).

(iii) If we take $M=1,$ and $N=P=Q=2$ in (15), the $\overline{H}$-function reduces into the Poly-logarithm [33] of complex order ν, denoted by $L^{\nu}(z)$. Then, Eq. (15) takes the following form after simplification;

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]L^{\nu}\left(zt^{\sigma}\right)\right)\right\}(x)$$
$$= x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\cdots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}\,(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\overline{H}_{5,5}^{1,5}\left[-zx^{\sigma}\left|\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\\ (0,1),(0,1;\nu-1),\left(1-\mu+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\\ (0,1,1),(1,1;\nu)\\ \left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right)\end{matrix}\right.\right]. \tag{21}$$

The conditions of validity of the above result can easily be followed directly from Theorem 1.

(iv) If we reduce the $\overline{H}$-function occurring in the left-hand side of (15) to a generalized Riemann Zeta function [34] given by;

$$\phi(z,l,\eta)=\sum_{r=0}^{\infty}\frac{z^r}{(\eta+r)^l}=\overline{H}_{2,2}^{1,2}\left[-z\left|\begin{matrix}(0,1,1),(1-\eta,1;l)\\(0,1),(-\eta,1;l)\end{matrix}\right.\right], \tag{22}$$

then, we arrive at the following result;

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]\phi\left(zt^{\sigma},l,\eta\right)\right)\right\}(x)$$

$$= x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\ldots c_s^{k_s}\,(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{5,5}^{1,5}\left[-zx^{\sigma}\left|\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(0,1;1\right),\left(1-\eta,1;l\right)\\ \left(0,1\right),\left(-\eta,1;l\right),\left(1-\mu+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right)\end{matrix}\right.\right]. \tag{23}$$

The conditions of validity of the above result can easily be followed directly from (15).

(v) If we set the product of polynomials $S_{n_j}^{m_j}$ to unity, and reduce the $\overline{H}$-function to a Whittaker function [35] by taking $A_j = B_j = 1$, then we obtain the following result;

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}e^{\frac{-zt^{\sigma}}{2}}W_{a,b}\left(zt^{\sigma}\right)\right)\right\}(x)$$

$$x^{\mu-\alpha-\alpha'+\gamma-1}\,\overline{H}_{4,5}^{2,3}\left[zx^{\sigma}\left|\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-a,1\right)\\ \left(\tfrac{1}{2}\pm b,1\right),\left(1-\mu+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu+\alpha'+\beta-\gamma-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right),\left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_jk_j,\sigma;1\right)\end{matrix}\right.\right]. \tag{24}$$

The conditions of validity of the above result can easily be followed directly from Theorem 1.

(vi) If we set product of polynomials $S_{n_j}^{m_j}$ to unity, and reduce the $\overline{H}$-function to Fox's H-function, then we can easily obtain the known results given by Saxena and Saigo [6].

Conclusions

In this section, we briefly consider some consequences of the results derived in the previous sections. The $\overline{H}$-function and a general class of polynomials are important special functions that appear widely in science and engineering. In the present paper, we have given the 2 theorems of generalized fractional integral operators given by Saigo-Maeda. The theorems have been developed in terms of the product of the $\overline{H}$-function and a general class of polynomials in a compact and elegant form with the help of Saigo-Maeda power function formulae. Most of the given results have been put in a compact form, avoiding the occurrence of infinite series and thus making them useful in applications. Further, it is interesting to observe that, if we set $\alpha' = 0$ the results given by Agarwal [15] follow as a special case of our main results.

The $\overline{H}$-function defined by (5), possesses the advantage that a number of special functions are found to be particular cases of this function. Therefore, we conclude this paper with the remark that, the results deduced above are significant and can lead to yield numerous other fractional integrals involving various special functions by the suitable specializations of arbitrary parameters in the theorems.

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