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On the Diophantine Equation $3^x + p5^y = z^2$ †

Kittipong LAIPAPORN , Saeree WANANIYAKUL and Prathomjit KHACHORNCHAROENKUL*

School of Science, Walailak University, Nakhon Si Thammarat 80160 Thailand

(*Corresponding author's e-mail: prathomjit@gmail.com)

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Abstract

In this paper, we present new series of solutions of the Diophantine equation $3^x + p5^y = z^2$ where p is a prime number and x, y and z are nonnegative integers using elementary techniques. Moreover, the equation has no solution if p is equivalent to 5 or 7 modulo 24.

Keywords: Exponential Diophantine equation, Catalan's conjecture

Mathematics Subject Classification: 11D61

Introduction

In 2004, Catalan's conjecture was exhibited by Mihailescu [1]. During the past fifteen years, many researchers have studied the Diophantine equation of the form $a^x + b^y = z^2$ by considering a and b. In 2007, Acu [2] showed that $(3, 0, 3)$ and $(2, 1, 3)$ are the only two solutions (x, y, z) satistying the equation $2^x + 5^y = z^2$. In 2011, Suvarnamani [3] considered in the form $2^x + p^y = z^2$ and found that solutions of this equation according to value of p for example, $(3, 0, 3)$ is a solution for $p > 2$, besides, $(4, 2, 5)$ is another solution to the equation for $p = 3$. If $p = 2$, the solutions consist of three types. In 2012, Sroysang [4] found that $(1, 0, 2)$ is the unique solution to the equation $3^x + 5^y = z^2$. In 2013, Ninrata [5] showed that there is no nonnegative solution to the modified Sroysang's equation as 5^y is added. In the same year, Sroysang [6] extended Suvarnamani's result in case $p = 3$ that $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are the only three solutions satistying the equation $2^x + 3^y = z^2$. In 2014, Bacani and Rabago [7] generalized the Diophantine equation as $3^x + 5^y + 7^z = w^2$ and showed that $(0, 0, 1, 3)$, $(1, 1, 0, 3)$ and $(3, 1, 2, 9)$ are the only three solutions (x, y, z, w) satistying this equation.

In this paper, we consider the Diophantine equation in the particular form of

$$3^x + p5^y = z^2$$

where p is a prime number not equal to 2 or 5 and x, y and z are nonnegative integers.

Main Results

First, we begin this section by providing a lemma that is used through our discussion.

Lemma 1. *[8] If l is an integer relatively prime to 24, then there are infinitely many primes p such that* $p \equiv l \mod 24$.

We consider in case x is zero.

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Proposition 2. *The solution of the equation*

$$1 + p5^y = z^2 \tag{1}$$

is $(y, z, p) \in \{(0, 2, 3), (1, 4, 3), (1, 6, 7), (2, 24, 23)\} \cup A \cup B$
where $A = \{(y, 5^y - 1, 5^y - 2) \mid y$ *is even and a prime* $5^y - 2 \equiv 23 \mod 24$ *for* $y \geq 3\}$
and $B = \{(y, 5^y + 1, 5^y + 2) \mid y$ *is odd and a prime* $5^y + 2 \equiv 7 \mod 24$ *for* $y \geq 3\}$.

Proof. From (1), we have $p5^y = z^2 - 1 = (z - 1)(z + 1)$. If $z - 1 = 1$, then $y = 0$ and $p = 3$. Hence, $(y, z, p) = (0, 2, 3)$. Now, we attend to the case $z - 1 > 1$. Clearly, in the case $y < 3$, the solution of the equation (1) is $(y, z, p) \in \{(1, 4, 3), (1, 6, 7), (2, 24, 23)\}$. We consider in the case $y \geq 3$.
Case 1 $p|z - 1$. If $5|z - 1$, then $5 \nmid z + 1$ and we conclude that $p5^y = z - 1 < z + 1 = 1$ which is impossible. Thus

$$z = 5^y - 1 \text{ and } p = 5^y - 2.$$

Since $5^y \equiv (-1)^y \mod 3$, we obtain that $5^y - 2$ is divisible by 3 if and only if y is odd. Thus, $y = 2k$ for some $k \in \mathbb{Z}$. As a result, $p = 5^y - 2 = 5^{2k} - 2 = 25^k - 2 \equiv -1 \equiv 23 \mod 24$. Hence, this case may have infinitely many solutions in the form of $(y, z, p) = (y, 5^y - 1, 5^y - 2)$ where y is even and $5^y - 2$ is a prime number for $y \geq 3$ by Lemma 1.
Case 2 $p|z + 1$. By the same manner as in Case 1, we obtain that

$$z = 5^y + 1 \text{ and } p = 5^y + 2.$$

Moreover, $y = 2k + 1$ for some $k \in \mathbb{Z}$, and hence, $p = 5^y + 2 = 5^{2k+1} + 2 = 5(25^k) + 2 \equiv 5 + 2 \equiv 7 \mod 24$. Therefore, this case may have infinitely many solutions in the form of $(y, z, p) = (y, 5^y + 1, 5^y + 2)$ where y is odd and $5^y + 2$ is a prime number for $y \geq 3$ by Lemma 1.

From now on, we consider in case x is greater than zero and y is zero.

Proposition 3. *The solution of the equation*

$$3^x + p = z^2 \tag{2}$$

is $(x, z, p) \in A \cup B \cup C$
where $A = \{(x, 3^{\frac{x}{2}} + 1, 2(3^{\frac{x}{2}}) + 1) \mid x$ *is even and a prime* $2(3^{\frac{x}{2}}) + 1 \equiv 7, 19 \mod 24\}$,
$B = \{(x, 4u + 2, 24v + 1) \mid x$ *is odd and* $u, v \in \mathbb{Z}\}$ *and* $C = \{(x, 4u, 24v + 13) \mid x$ *is odd and* $u, v \in \mathbb{Z}\}$.

Proof. **Case 1** x is even. Then, $x = 2k$ for some $k \in \mathbb{N}$. Moreover, $p = z^2 - 3^{2k} = (z - 3^k)(z + 3^k)$. But $z + 3^k \neq 1$ so that

$$z = 1 + 3^k \text{ and } p = 2(3^k) + 1. \tag{3}$$

Hence, $(x, z, p) = (x, 3^{\frac{x}{2}} + 1, 2(3^{\frac{x}{2}}) + 1)$ where $2(3^{\frac{x}{2}}) + 1$ is a prime number. Moreover, $p \equiv -1 \mod 4$ from (2) and $p \equiv 1 \mod 3$ from (3). By the Chinese remainder theorem, we obtain $p \equiv 7 \mod 24$ or $p \equiv 19 \mod 24$. Consequently, this case may have infinitely many solutions in the form of $(x, z, p) = (x, 3^{\frac{x}{2}} + 1, 2(3^{\frac{x}{2}}) + 1)$ where x is even and $2(3^{\frac{x}{2}}) + 1$ is a prime number by Lemma 1.
Case 2 x is odd. Then, $x = 2k + 1$ for some $k \in \mathbb{N}_0$. Now,

$$3^{2k+1} + p = z^2. \tag{4}$$

Thus, $p = 3$ or $p \equiv 1 \mod 3$.
<u>Subcase 2.1</u> $p = 3$. Obviously, $9|3^{2k+1} + 3$ which is impossible. Hence, there is no solution.

Subcase 2.2 $p \equiv 1 \mod 3$. Then, $p \equiv 1 \mod 12$ because of $p = z^2 - 3^{2k+1} \equiv -(-1)^{2k+1} \equiv 1 \mod 4$. It implies that $p \equiv 1 \mod 24$ or $p \equiv 13 \mod 24$. By Lemma 1, the prime p may be infinitely many terms satisfying the equation (4). If $p \equiv 1 \mod 24$, then $p \equiv 1 \mod 8$ and $z^2 \equiv 4 \mod 8$ by (4). Thus, $4 \nmid z$. Since z is even, we have $z = 4u + 2$ for some $u \in \mathbb{Z}$. Similarly, if $p \equiv 13 \mod 24$, then $p \equiv 5 \mod 8$ and $z^2 \equiv 0 \mod 8$, and hence, $z = 4u$ for some $u \in \mathbb{Z}$. Hence, this case may have infinitely many solutions in the form of $(x, z, p) = (x, 4u + 2, 24v + 1)$ or $(x, z, p) = (x, 4u, 24v + 13)$ for some u and $v \in \mathbb{Z}$ by Lemma 1.

Theorem 4. *The solutions of this equation*

$$3^x + p5^y = z^2 \tag{5}$$

where p is a prime number not equal to 2 *or* 5 *and* x, y *and* z *are nonnegative integers satisfy the following:*

1. *If* $x = 0$ *and* $y = 0$, *then* $(x, y, z, p) = (0, 0, 2, 3)$ *is a solution.*
2. *If* $x = 0$ *and* $y > 0$, *then* $(x, y, z, p) \in \{(0, 1, 4, 3), (0, 1, 6, 7), (0, 2, 24, 23)\} \cup A \cup B$ *where* $A = \{(0, y, 5^y - 1, 5^y - 2) \mid y$ *is even and a prime* $5^y - 2 \equiv 23 \mod 24$ *for* $y \geq 3\}$ *and* $B = \{(0, y, 5^y + 1, 5^y + 2) \mid y$ *is odd and a prime* $5^y + 2 \equiv 7 \mod 24$ *for* $y \geq 3\}$.
3. *If* $x > 0$ *and* $y = 0$, *then* $(x, y, z, p) \in A \cup B \cup C$ *where* $A = \{(x, 0, 3^{\frac{x}{2}} + 1, 2(3^{\frac{x}{2}}) + 1) \mid x$ *is even and a prime* $2(3^{\frac{x}{2}}) + 1 \equiv 7, 19 \mod 24\}$, $B = \{(x, 0, 4u + 2, 24v + 1) \mid x$ *is odd and* $u, v \in \mathbb{Z}\}$ *and* $C = \{(x, 0, 4u, 24v + 13) \mid x$ *is odd and* $u, v \in \mathbb{Z}\}$.
4. *If* $x > 0$ *and* $y > 0$, *then* x *is even and* $(x, y, z, p) \in A \cup B \cup C \cup D$ *where* $A = \{(x, y, 1 + 3^{\frac{x}{2}}, \frac{2(3^{\frac{x}{2}})+1}{5^y}) \mid y$ *is even and a prime* $\frac{2(3^{\frac{x}{2}})+1}{5^y} \equiv 7, 19 \mod 24\}$, $B = \{(x, y, 1 + 3^{\frac{x}{2}}, \frac{2(3^{\frac{x}{2}})+1}{5^y}) \mid y$ *is odd and a prime* $\frac{2(3^{\frac{x}{2}})+1}{5^y} \equiv 11, 23 \mod 24\}$, $C = \{(x, y, 5^y \pm 3^{\frac{x}{2}}, 5^y \pm 2(3^{\frac{x}{2}})) \mid y$ *is even and a prime* $5^y \pm 2(3^{\frac{x}{2}}) \equiv 7, 19 \mod 24\}$ *and* $D = \{(x, y, 5^y \pm 3^{\frac{x}{2}}, 5^y \pm 2(3^{\frac{x}{2}})) \mid y$ *is odd and a prime* $5^y \pm 2(3^{\frac{x}{2}}) \equiv 11, 23 \mod 24\}$.

Proof. For the other cases, by Proposition 2 and Proposition 3, we obtain the corresponding solution form. It remains to consider in case $x > 0$ and $y > 0$.
Case 1 $p = 3$. If $x = 1$, then we consider $z^2 = 3 + 3(5^y)$. It implies that $3 + 3(5^y) \equiv 3 + 3 \equiv 2 \mod 4$. However, $z^2 \equiv 0 \mod 4$ leads to a contradiction. If $x > 1$, then we consider $z^2 = 3^x + 3(5^y)$ and conclude that $z = 3r$ for some $r \in \mathbb{Z}$. Then, $5^y = 3r^2 - 3^{x-1} \equiv 0 \mod 3$ which is a contradiction.
Case 2 $p > 5$. If x is odd, then $3^x \equiv 3, 7 \mod 10$ and $p5^y \equiv 5 \mod 10$. It implies that $z^2 = 3^x + p5^y \equiv 8, 2 \mod 10$, which contradicts with $z^2 \equiv 0, 4, 6 \mod 10$.
Now, we consider in case x is even. Then, $3^{2k} + p5^y = z^2$ for some $k \in \mathbb{N}$. This implies that

$$p5^y = z^2 - 3^{2k} = (z - 3^k)(z + 3^k). \tag{6}$$

Subcase 3.1 $p | z - 3^k$. Since $z - 3^k \neq 1$, we get $5 | z + 3^k$ and so $z \equiv -3^k \mod 5$. Beside, $5 \nmid z - 3^k$ because $z - 3^k \equiv -2(3^k) \equiv 3^{k+1} \mod 5$ and $k > 0$. It follows that $p = z - 3^k$ and $5^y = z + 3^k$. Hence, $(x, y, z, p) = (x, y, 5^y - 3^{\frac{x}{2}}, 5^y - 2(3^{\frac{x}{2}}))$ where $5^y - 2(3^{\frac{x}{2}})$ is a prime number.

Subcase 3.2 $p | z + 3^k$ and $z - 3^k = 1$. Then, $(x, y, z, p) = (x, y, 1 + 3^{\frac{x}{2}}, \frac{2(3^{\frac{x}{2}})+1}{5^y})$ where $\frac{2(3^{\frac{x}{2}})+1}{5^y}$ is a prime number. Since $p5^y = 2(3^k) + 1$, we get $p \equiv 2(-1)^k + 1 \equiv 3 \mod 4$ and it is clear that

$$p \equiv \begin{cases} 1 \mod 6 \text{ if } y \text{ is even} \\ 5 \mod 6 \text{ if } y \text{ is odd} \end{cases}.$$

Consequently, $p \equiv 7 \mod 24$ or $p \equiv 19 \mod 24$ where y is even, and $p \equiv 11 \mod 24$ or $p \equiv 23 \mod 24$ where y is odd. By Lemma 1, this case may have infinitely many solutions.
Subcase 3.3 $p|z+3^k$ and $z-3^k > 1$. By (6), we have $5|z-3^k$ and then $z \equiv 3^k \mod 5$. Similarly, since $z+3^k \equiv 2(3^k) \not\equiv 0 \mod 5$, we get $p = z+3^k$ and $5^y = z-3^k$. Hence, $(x,y,z,p) = (x,y,5^y+3^{\frac{x}{2}}, 5^y + 2(3^{\frac{x}{2}}))$ where $5^y + 2(3^{\frac{x}{2}})$ is a prime number.

Furthermore, from the form of solutions in Subcase 3.1 and 3.3, we focus on the pattern of p:

$$p = \begin{cases} 5^y - 2(3^k) \\ 5^y + 2(3^k) \end{cases} \equiv \begin{cases} 1 - 2(-1)^k & \mod 4 \\ 1 + 2(-1)^k & \mod 4 \end{cases} \equiv -1 \mod 4.$$

Besides, $p \equiv (-1)^y \mod 3 \equiv \begin{cases} 1 \mod 3 & \text{if } y \text{ is even} \\ -1 \mod 3 & \text{if } y \text{ is odd} \end{cases}$. Thus, in the case y is even, $p \equiv 7 \mod 24$ or $p \equiv 19 \mod 24$; moreover, in the case y is odd, $p \equiv 11 \mod 24$ or $p \equiv 23 \mod 24$. For these results, we conclude that these cases may have infinitely many solutions by Lemma 1.

More preciously in details of the proof of all propositions and the theorem, we guarantee that $3^x + p5^y = z^2$ has the solution if p must be in the form $24n+l$ where l is a positive remainder and $\gcd(24, l) = 1$ except $l = 5, 17$. This completes the proof of the following corollary.

Corollary 5. 1. *If* $p \equiv 5 \mod 24$ *or* $p \equiv 17 \mod 24$*, then the equation (5) has no nonnegative integer solution.*

2. *For any remainder l modulo* 24 *with a coprime of* 24 *except* 5 *and* 17*, there exists* $p \equiv l \mod 24$ *which the equation (5) has a nonnegative integer solution.*

Example 1. In order to understand deeply Corollary 5, some equations having no nonnegative integer solution are provided: $3^x + 17(5^y) = z^2, 3^x + 29(5^y) = z^2, 3^x + 41(5^y) = z^2$ and $3^x + 53(5^y) = z^2$.

Appendices

This section is provided to advocate the existence of infinite primes of the form in Theorem 4. The following tables show some solutions satisfying the above patterns and guaranteeing these p's being prime numbers by using Python codes.

Remark. Solutions in case $x = 0$ and $y > 0$ satisfy the condition of A and B in Theorem 4 (2.). The following y satisfies this pattern and makes its p to be a prime number for $y \leq 20000$, that is, **14**, **26**, **50**, 126, 144, 260, 624, 1424, 10472, 19784 for the set A, and **3**, **17**, **143**, 261, 551, 2285, 18731, 18995, 19751 for another set. The solutions for a specific y are given in **Table 1** and **Table 2**, respectively.

Table 1 Some solutions in case $x = 0$ and $y > 0$ which satisfying the condition of A in Theorem 4 (2.).

x	y	z	p
0	**14**	6 103 515 624	6 103 515 623
0	**26**	1 490 116 119 384 765 624	1 490 116 119 384 765 623
0	**50**	88 817 841 970 012	88 817 841 970 012
		523 233 890 533 447 265 624	523 233 890 533 447 265 623

Table 2 Some solutions in case $x = 0$ and $y > 0$ which satisfying the condition of B in Theorem 4 (2.).

x	y	z	p
0	**3**	126	127
0	**17**	762 939 453 126	762 939 453 127
0	**143**	8 968 310 171 678 829 253 911 869 333 055 463 240 193 676 428 009 700 939 245 237 016 894 662 929 189 507 849 514 484 405 517 578 126	8 968 310 171 678 829 253 911 869 333 055 463 240 193 676 428 009 700 939 245 237 016 894 662 929 189 507 849 514 484 405 517 578 127

Remark. Solutions in case $x > 0$ and $y = 0$ satisfy the condition of A in Theorem 4 (3.). The following x satisfies this pattern and makes its p to be a prime number for $x \leq 20000$, that is, $2, 4, 8,$ **10**$, 12,$ **18**$, 32,$ **34**$, 60, 108, 114, 120, 130, 264, 360, 640, 1392, 1564, 1644, 1794, 2504, 2908, 8434, 10960, 12450, 15684$. The solutions for a specific x are given in **Table 3**.

Solutions in case $x > 0$ and $y = 0$ satisfy the condition of B in Theorem 4 (3.). The following x and z satisfy this pattern and make their p to be a prime number for $x \leq 13$ and $z \leq 2000$, for example, if $x =$ **1**, then z can be $14,$ **110**$, 586, 626, 878, 934, 998, 1450, 1786,$ **1978** etc. If $x =$ **3**, then z can be $10, 22, 34, 50, 206, 346, 578, 614,$ **706**$, 842, 958,$ **1922** etc. If $x =$ **5**, then z can be $22, 38, 46, 58, 422,$ **502**$, 658, 710, 994,$ **1886** etc. If $x =$ **7**, then z can be $50, 62, 70, 86, 98, 106, 242,$ **422**$, 650, 974,$ **1970** etc. The solutions for a specific z are given in **Table 4**.

Solutions in case $x > 0$ and $y = 0$ satisfy the condition of C in Theorem 4 (3.). The following x and z satisfy this pattern and make their p to be a prime number for $x \leq 13$ and $z \leq 2000$, for example, if $x =$ **1**, then z can be **4**$, 8, 80, 92, 112, 296,$ **340**$, 580, 764, 884$ etc. If $x =$ **3**, then z can be $8, 32, 52, 56, 68, 112,$ **388**$, 592, 760, 992,$ **1900** etc. If $x =$ **5**, then z can be $16, 20, 28, 44, 64, 92, 356, 440, 688,$ **988**$,$ **1976** etc. If $x =$ **7**, then z can be $68, 88, 92, 104, 112, 232, 572, 736,$ **988**$, 1000,$ **1940** etc. The solutions for a specific z are given in **Table 5**.

Table 3 Some solutions in case $x > 0$ and $y = 0$ which satisfying the condition of A in Theorem 4 (3.).

x	y	z	p
10	0	244	487
18	0	19 684	39 367
34	0	129 140 164	258 280 327

Table 4 Some solutions in case $x > 0$ and $y = 0$ which satisfying the condition of B in Theorem 4 (3.).

x	y	z	p
1	0	**110**	12 097
1	0	**1978**	3 912 481
3	0	**706**	498 409
3	0	**1922**	3 694 057
5	0	**502**	251 761
5	0	**1886**	3 556 753
7	0	**422**	175 897
7	0	**1970**	3 878 713

Table 5 Some solutions in case $x > 0$ and $y = 0$ which satisfying the condition of C in Theorem 4 (3.).

x	y	z	p
1	0	**4**	13
1	0	**340**	115 597
3	0	**388**	150 517
3	0	**1900**	3 609 973
5	0	**988**	975 901
5	0	**1976**	3 904 333
7	0	**988**	973 957
7	0	**1940**	3 761 413

Remark. Solutions in case $x > 0$ and $y > 0$ satisfy the condition of A and B, respectively in Theorem 4 (4.). The following x satisfies this pattern and makes its p to be a prime number for $x \leq 42000$, that is, $\mathbf{6}, \mathbf{14}, \mathbf{30}, \mathbf{118}, 142, 198, 390, 550, 3478, 4510, 6150$. The solutions for a specific x are given in **Table 6**.

Solutions in case $x > 0$ and $y > 0$ satisfy the condition of C and D, respectively in Theorem 4 (4.). The following x and y satisfy this pattern and make their p to be a prime number in the form $p = 5^y + 2(3^{\frac{x}{2}})$ for $x \leq 50$ and $y \leq 3000$, for example, if $x = \mathbf{2}$, then y can be $1, 2, 3, 4, \mathbf{13}, 88, 177, 297, 310, 562, 892$ etc. If $x = \mathbf{6}$, then y can be $1, 3, 7, 14, \mathbf{18}, 27, 86, 162, 179, 224$ etc. If $x = \mathbf{12}$, then y can be $2, 5, \mathbf{14}, 18, 34, 62, 72, 112, 335, 435$ etc. If $x = \mathbf{20}$, then y can be $6, \mathbf{17}, 26, 43, 72, 97, 162, 295, 775$ etc. The solutions for a specific y are given in **Table 7**.

Moreover, the following x and y satisfy this pattern and make their p to be a prime number in the form $p = 5^y - 2(3^{\frac{x}{2}})$ for $x \leq 50$ and $y \leq 3000$, for example, if $x = \mathbf{4}$, then y can be $2, 3, 4, \mathbf{15}, 31, 75, 127, 203, 358, 599, 939$ etc. If $x = \mathbf{6}$, then y can be $3, 4, \mathbf{12}, 21, 22, 40, 81$ etc. If $x = \mathbf{12}$, then y can be $5, \mathbf{8}, 17, 104, 139, 373, 481, 839, 907$ etc. If $x = \mathbf{20}$, then y can be $9, 10, 14, \mathbf{19}, 53, 59, 75, 110, 161, 164, 429, 501, 703$ etc. The solutions for a specific y are given in **Table 8**.

Table 6 Some solutions in case $x > 0$ and $y > 0$ which satisfying the condition of A and B, respectively in Theorem 4 (4.).

x	y	z	p
14	4	2188	7
6	1	28	11
30	1	14 348 908	5 739 563
118	1	14 130 386 091 738 734 504 764 811 068	5 652 154 436 695 493 801 905 924 427

Table 7 Some solutions in case $x > 0$, $y > 0$ and $p = 5^y + 2(3^{\frac{x}{2}})$ which satisfying the condition of C and D, respectively in Theorem 4 (4.).

x	y	z	$p = 5^y + 2(3^{\frac{x}{2}})$
6	**18**	3 814 697 265 652	3 814 697 265 679
12	**14**	6 103 516 354	6 103 517 083
2	**13**	1 220 703 128	1 220 703 131
20	**17**	762 939 512 174	762 939 571 223

Table 8 Some solutions in case $x > 0$, $y > 0$ and $p = 5^y - 2(3^{\frac{x}{2}})$ which satisfying the condition of C and D, respectively in Theorem 4 (4.).

x	y	z	$p = 5^y - 2(3^{\frac{x}{2}})$
6	**12**	244 140 598	244 140 571
12	**8**	389 896	389 167
4	**15**	30 517 578 116	30 517 578 107
20	**19**	19 073 486 269 076	19 073 486 210 027

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