

## **A New Efficient Algorithm to Solve Non-Linear Fractional Ito Coupled System and Its Approximate Solution**

**Sunil KUMAR**

*Department of Mathematics, National Institute of Technology, Jamshedpur, Jharkhand 831014, India*

**(Corresponding author's e-mail: [skitbhu28@gmail.com](mailto:skitbhu28@gmail.com), [skumar.math@nitjsr.ac.in](mailto:skumar.math@nitjsr.ac.in))**

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### **Abstract**

The aim of the present article is to obtain the approximate analytical solution of time fractional Ito coupled equations by using the homotopy perturbation method. The fractional derivatives are described in the Caputo sense. The highlight of the paper is error analysis between the exact solutions and approximate solutions which shows that our approximate solutions converge very rapidly to the exact solutions. The method gives analytical solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. The results reveal that the proposed method is very effective and simple.

**Keywords:** Ito coupled system, fractional derivative, Caputo derivatives, approximate solution, fractional Brownian motion

### **Introduction**

Nonlinear partial differential equations have many applications in various fields of science and engineering such as fluid mechanics, thermodynamics, mass and heat transfer, micro electromechanics etc. It is difficult to handle the nonlinear part of these equations. Construction of the exact and explicit solutions of nonlinear partial differential equations is very important in mathematical sciences and it is one of the most stimulating and particularly active areas of the research. It is well known that all nonlinear partial differential equations can be separated into parts: the integrable partial differential equations and non-integrable ones.

Fractional differential equations have received more attention in recent years. During the past few years, there has been a growing interest in the field of fractional derivatives. The main reason consists in the fact that the theory of derivatives of fractional (non-integer) order stimulates considerable interest in the areas of mathematics, physics and engineering. Many physical phenomena [1-5] can be modelled by fractional differential equations which have diverse applications in various physical processes such as acoustics, electromagnetism, control theory, robotics, viscoelastic materials, diffusion, edge detection, turbulence, signal processing, anomalous diffusion and fractured media.

In this paper, we will use the homotopy perturbation method (HPM) to study a time fractional Ito coupled system. The homotopy perturbation method was first proposed by the Chinese mathematician He [6-8]. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. The essential idea of this method is to introduce a homotopy parameter, say  $p$ , which takes values from 0 to 1, when  $p = 0$ , the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As  $p$  gradually increases to 1, the system goes through a sequence of deformations, the solution for each of which is close to that of the previous stage of deformation. Eventually at  $p = 1$ , the system takes the original form of the equation and the final stage of deformation gives the desired solution. The method was successfully applied to the

space-time fractional advection-dispersion equation by Yildirim and Kocak [9], fractional Zakharov-Kuznetsov equations by Yildirim and Gulkanat [10], Fokker Planck equation by Yildirim [11], fractional modified Kdv equation by Abdulaziz *et al.* [12], inversion of the Abel integral equation by Kumar and Singh [13], the fractional Riccati equation by Odibat and Momani [14], the fractional Kdv equation by Wang [15], time-fraction generalized Hirota-Satsuma coupled KdV equation by Ganji *et al.* [16], multi-order time fractional differential equations by Golbabai and Sayevand [17], explicit analytical solutions of the generalized Burger and Burger-Fisher equations by Rashidi *et al.* [18], 2 dimensional viscous flow in the extrusion process by Rashidi and Ganji [19], nonlinear coupled equations by Chen and An [20], and a one phase inverse Stefan problem by Slota [21].

Recently, many experts have paid great attention to the construction of solutions of the Ito coupled as symmetry analysis of an integrable Ito coupled system by Zedan [22], numerical solutions for Ito coupled system by Kawala [23], numerical solutions for a generalized Ito system by Zedan and Aidrous [24], double periodic solutions for the generalized Ito system by Zhao *et al.* [25], Hamiltonian structure for Ito system by Liu [26], invariant for a Ito system by Samoilenko *et al.* [27], the proposed relation between oHS and Ito systems by Karasu [28], Ito type coupled Kdv equations by Xu and Shu [29], the integrability of a generalized Ito system by Ayse *et al.* [30], the Hirota-Satsuma coupled KdV equation and a coupled Ito system by Tam *et al.* [31].

This paper is committed to the study of a time fractional Ito system by using the HPM. Using the appropriate initial conditions, the approximate analytical solutions for different fractional Brownian motions and also for standard motion are obtained. The variation on the approximate solutions is depicted graphically while error analysis shows the accuracy of the approximate analytical solutions.

### Basic definitions of fractional calculus

In this section, we first give the definitions of fractional order integration and fractional order differentiation. For the concept the fractional derivative, we will adopt Caputo's definition which is a modification of the Riemann Liouville definition and has the advantage of dealing properly with initial value problem.

**Definition 1** A real function  $f(t)$ ,  $t > 0$  is said to be in the space  $C_\mu$ ,  $\mu \in R$  if there exists a real number  $p > \mu$ , such that  $f(t) = t^p f_1(t)$  where  $f_1(t) \in C(0, \infty)$  and it is said to be in the space  $C_n$  if and only if  $f^{(n)} \in C_\mu$ ,  $n \in N$ .

**Definition 2** The Riemann-Liouville fractional integral  $(J_t^\alpha)$  operator of order  $\alpha$  of a function  $f \in C_\mu$ ,  $\mu \geq -1$  is defined as;

$$\begin{cases} J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, (\alpha > 0, t > 0), \\ J_t^0 f(t) = f(t). \end{cases} \quad (1)$$

where  $\Gamma(\cdot)$  is the well-known Gamma function. Some of the properties of the operator  $(J_t^\alpha)$ , can be found in [1-5], we mention only the following. For  $f \in C_\mu$ ,  $\mu \geq -1$ ,  $\alpha, \beta \geq 0$  and  $\gamma \geq -1$ :

$$(1) \quad J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t), \quad (2)$$

$$(2) \quad (J_t^\alpha J_t^\beta) f(t) = (J_t^\beta J_t^\alpha) f(t), \quad (3)$$

$$(3) \quad J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}. \quad (4)$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator ( $D_t^\alpha$ ) proposed by Caputo in his work in the theory of viscoelasticity [32].

**Definition 3** The fractional derivative  $D_t^\alpha$  of  $f(t)$  in the Caputo sense is defined as;

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad (5)$$

where  $m-1 < \alpha \leq m$ ,  $m \in N$ ,  $t > 0$ ,  $f \in C_{-1}^m$ .

The following are 2 basic properties of Caputo's fractional derivative:

**Lemma 1** If  $m-1 < \alpha \leq m$ ,  $m \in N$  and  $f \in C_\mu^n$ ,  $\mu \geq -1$ , then

$$\begin{cases} (D_t^\alpha J_t^\alpha) f(t) = f(t), \\ (J_t^\alpha D_t^\alpha) f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{t^i}{i!}, \end{cases} \quad (6)$$

### Illustrative examples

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the HPM. The simplicity and accuracy of the proposed method is illustrated through the following numerical examples.

**Example 1** We consider the time fractional nonlinear Ito coupled system as [22];

$$\begin{cases} D_t^\alpha u = D_x v, & 0 < \alpha \leq 1, \\ D_t^\beta v = -2(D_x^3 v + 3uD_x v + 3vD_x u) - 12wD_x w, & 0 < \beta \leq 1, \\ D_t^\gamma w = D_x^3 w + 3uD_x w, & 0 < \gamma \leq 1, \end{cases} \quad (7)$$

with initial conditions  $u(x,0) = \frac{ax}{3c}$ ,  $v(x,0) = -\frac{a^2 x^2}{2c^2}$  and  $w(x,0) = 0$ . The exact solution of the nonlinear Ito coupled system (7) for  $\alpha = \beta = \gamma = 1$  are given as

$$u(x,t) = \frac{ax}{3(3at+c)}, \quad v(x,t) = -\frac{a^2 x^2}{2(3at+c)^2} \text{ and } w(x,t) = 0.$$

We construct the homotopy which satisfies the;

$$\begin{cases} D_t^\alpha u = \varepsilon(D_x v), \\ D_t^\beta v = \varepsilon(-2(D_x^3 v + 3uD_x v + 3vD_x u) - 12wD_x w), \\ D_t^\gamma w = \varepsilon(D_x^3 w + 3uD_x w), \end{cases} \quad (8)$$

and assume the solution of the time fractional Ito coupled system (7) to be in the following form;

$$u(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i u_i(x,t), \quad v(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i v_i(x,t), \quad w(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i w_i(x,t). \quad (9)$$

where  $u_i(x,t), v_i(x,t), w_i(x,t), i = 0,1,2,3,\dots$ , are the functions to be determined. We use the following iterative scheme to evaluate  $u_i(x,t), v_i(x,t), w_i(x,t)$ . Substituting Eq. (8) into Eq. (9) and equating the coefficients of like powers of  $\varepsilon$  we get a system of differential equations. This system of differential equations can be easily solved by applying the operator  $J_t^\alpha$  to obtain the various components  $u_n(x,t), v_n(x,t)$  and  $w_n(x,t)$ , thus enabling the series solution to be entirely determined. The first few components of the homotopy perturbation solutions for the Eq. (9) are given as follows;

$$\begin{aligned} u_0(x,t) &= u(x,0) = \frac{ax}{3c}, & v_0(x,t) &= v(x,0) = \frac{a^2 x^2}{2c^2}, & w_0(x,t) &= w(x,0) = 0, \\ u_1(x,t) &= -\frac{a^2 x}{c^2} \frac{t^\alpha}{\Gamma(\alpha+1)}, & v_1(x,t) &= \frac{3a^3 x^2}{c^3} \frac{t^\beta}{\Gamma(\beta+1)}, & w_1(x,t) &= 0, \\ u_2(x,t) &= \frac{6a^3 x}{c^3} \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, & v_2(x,t) &= -\frac{9a^4 x^2}{c^4} \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{18a^4 x^2}{c^4} \frac{t^{2\beta}}{\Gamma(2\beta+1)}, & w_2(x,t) &= 0, \\ u_3(x,t) &= -\frac{18a^4 x}{c^4} \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{36a^4 x}{c^4} \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}, \\ v_3(x,t) &= \frac{54a^5 x^2}{c^5} \left( \frac{2t^{3\beta}}{\Gamma(3\beta+1)} + \frac{2t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} \right), & w_3(x,t) &= 0, \end{aligned}$$

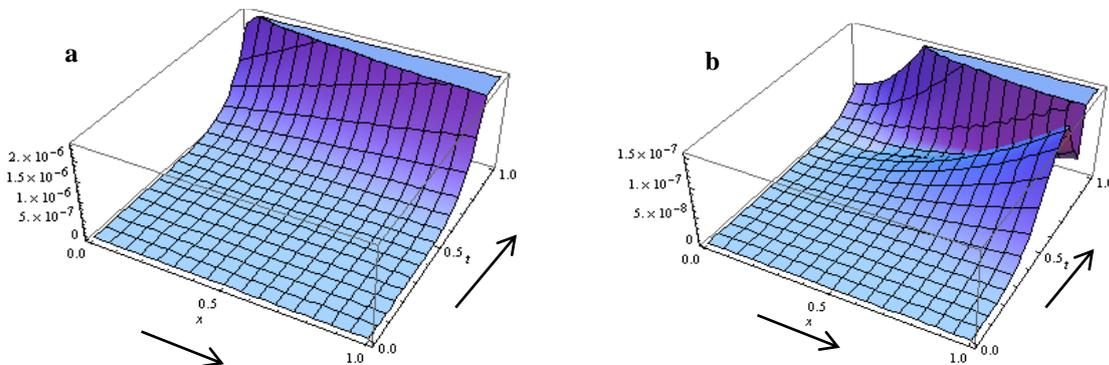
and so on.

In the same manner the rest of the components can be obtained by the homotopy perturbation solution. Thus the solution  $u(x,t), v(x,t)$  and  $w(x,t)$  of Eq. (7) are obtained by;

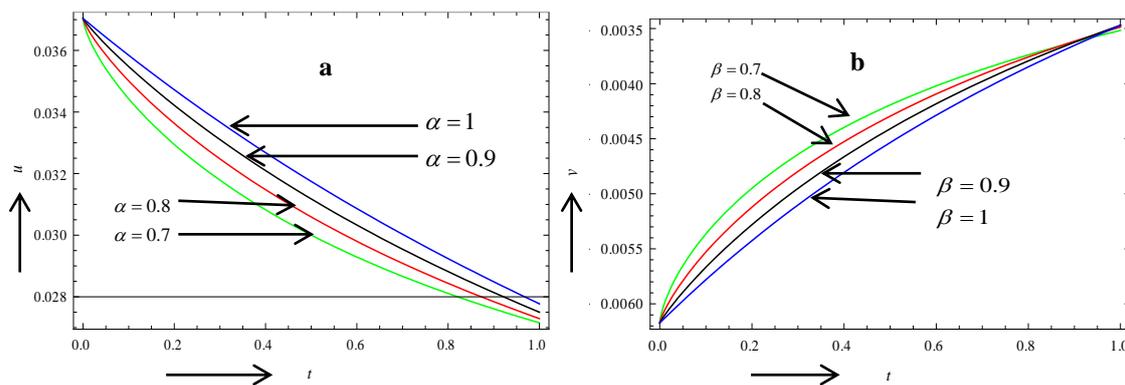
$$\tilde{u}(x,t) = \sum_{n=0}^8 u_n(x,t), \quad \tilde{v}(x,t) = \sum_{n=0}^8 v_n(x,t), \quad \tilde{w}(x,t) = \sum_{n=0}^8 w_n(x,t) \quad (10)$$

The series solution converges very rapidly. The rapid convergence means only a few terms are required to get an analytical function. The simplicity and accuracy of the proposed method is illustrated by computing the absolute errors  $E_{u_8}(x,t) = |u(x,t) - \tilde{u}(x,t)|, E_{v_8}(x,t) = |v(x,t) - \tilde{v}(x,t)|$  where

$a = 1, c = 9$ , and  $u(x, t), v(x, t)$  are exact solutions and  $\tilde{u}(x, t), \tilde{v}(x, t)$  are approximate solutions of (7) obtained by truncating the respective solution series (9) at level  $N = 8$ . **Figure 1** represents the absolute errors between the exact solutions and approximate solutions which shows that our approximate solutions  $\tilde{u}(x, t)$  and  $\tilde{v}(x, t)$  converge to the exact solution very rapidly.



**Figure 1** The absolute errors  $E_{u_8}(x, t)$  and  $E_{v_8}(x, t)$  at  $a = 1$  and  $c = 9$ .



**Figure 2** The approximate solution  $\tilde{u}(x, t)$  and  $\tilde{v}(x, t)$  at  $a = 1, c = 9$ .

The error analysis is depicted in **Figure 1** and we can see that a very satisfactory approximate solution of the system with high accuracy is obtained by truncating the HPM solution series at level  $N = 8$ . **Figure 2** shows the behavior of the approximate solutions for different fractional Brownian motions  $\alpha = \beta = 0.7, 0.8, 0.9$  and standard motions  $\alpha = \beta = 1$  when  $X$  is constant and time  $t$  is varying. The variation of  $\tilde{u}(x, t)$  and  $\tilde{v}(x, t)$  with respect to  $t$  at  $x = 1$  are depicted in **Figure 2**. It is seen from the **Figure 2a** that the approximate solution  $\tilde{u}(x, t)$  decreases with an increase in  $t$  for all values of  $\alpha = 0.70, 0.80, 0.90$  and for the standard case  $\alpha = 1$  at  $a = 1, c = 9$ . Similarly from **Figure 2b**, it can be seen that the approximate solution  $\tilde{v}(x, t)$  increases with an increase in  $t$  for all values of  $\beta = 0.70, 0.80, 0.90$  and for the standard case  $\beta = 1$  at  $a = 1, c = 9$ . In this example,

during the numerical computation only 8<sup>th</sup> order terms of the series solution are considered. The accuracy of the result can be improved by introducing more terms of the approximate solutions.

**Example 2** We consider the following time fractional nonlinear generalized Ito coupled system as [24,30];

$$\begin{cases} D_t^\alpha u = D_x v, & 0 < \alpha \leq 1, \\ D_t^\beta v = -2D_x^3 v - 6D_x(uv) - 6D_x(wp), & 0 < \beta \leq 1, \\ D_t^\gamma w = D_x^3 w + 3uD_x w, & 0 < \gamma \leq 1, \\ D_t^\eta p = D_x^3 p + 3uD_x p, & 0 < \eta \leq 1, \end{cases} \quad (11)$$

with initial conditions;

$$u(x,0) = r_1 - 2\mu^2 \tanh^2(\mu x), \quad v(x,0) = r_2 + b_2 \tanh^2(\mu x), \quad (12)$$

$$w(x,0) = r_3 + f_1 \tanh(\mu x), \quad p(x,0) = t_0 + t_1 \tanh(\mu x). \quad (13)$$

The exact solution of the nonlinear generalized Ito coupled system are given as;

$$u(x,t) = r_1 - 2\mu^2 \tanh^2 \left[ \mu \left( x - \frac{b_2 t}{2\mu^2} \right) \right], \quad v(x,t) = r_2 + b_2 \tanh^2 \left[ \mu \left( x - \frac{b_2 t}{2\mu^2} \right) \right], \quad (14)$$

$$w(x,t) = r_3 + f_1 \tanh \left[ \mu \left( x - \frac{b_2 t}{2\mu^2} \right) \right], \quad p(x,t) = t_0 + t_1 \tanh \left[ \mu \left( x - \frac{b_2 t}{2\mu^2} \right) \right]. \quad (15)$$

Here, all the result can be verified through substitution  $r_1 = \frac{-b_2 + 4\mu^4}{6\mu^2}$ ,  $r_2 = \frac{-b_2^2 + 4f_1 t_1 \mu^2 - 8b_2 \mu^4}{8\mu^4}$ ,

$r_3 = -\frac{f_1 t_0}{t_1}$ . Where  $\mu, b_2, t_0, t_1$  and  $f_1$  are arbitrary constants.

We construct the homotopy which satisfies the;

$$\begin{cases} D_t^\alpha u = \varepsilon(D_x v), & 0 < \alpha \leq 1, \\ D_t^\beta v = \varepsilon(-2D_x^3 v - 6D_x(uv) - 6D_x(wp)), & 0 < \beta \leq 1, \\ D_t^\gamma w = \varepsilon(D_x^3 w + 3uD_x w), & 0 < \gamma \leq 1, \\ D_t^\eta p = \varepsilon(D_x^3 p + 3uD_x p), & 0 < \eta \leq 1, \end{cases} \quad (16)$$

We seek the solutions of (11) in the following form;

$$u(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i u_i(x,t), \quad v(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i v_i(x,t), \quad (17)$$

$$w(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i w_i(x,t) \quad p(x,t) = \lim_{N \rightarrow \infty} \sum_{i=0}^N \varepsilon^i p_i(x,t). \quad (18)$$

where  $u_i(x,t), v_i(x,t), w_i(x,t), p_i(x,t), i = 0,1,2,3,\dots$ , are the functions to be determined. We use the following iterative scheme to evaluate  $u_i(x,t), v_i(x,t), w_i(x,t), p_i(x,t)$ . Substituting Eq. (17) - (18) into Eq. (16) and the equating the corresponding power of  $\varepsilon$ ;

$$u_0(x,t) = u(x,0) = r_1 - 2\mu^2 \tanh^2(\mu x), \quad v_0(x,t) = v(x,0) = r_2 + b_2 \tanh^2(\mu x), \quad (19)$$

$$w_0(x,t) = w(x,0) = r_3 + f_1 \tanh(\mu x), \quad p_0(x,t) = p(x,0) = t_0 + t_1 \tanh(\mu x). \quad (20)$$

$$u_1(x,t) = 2\mu b_2 \operatorname{sech}^2(\mu x) \tanh(\mu x) \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (21)$$

$$v_1(x,t) = \left[ -2\mu \operatorname{sech}^3(\mu x) \{3(t_0 f_1 + t_1 r_3) \cosh(\mu x) + 2(3r_1 b_2 + 3t_1 f_1 - 2(3r_2 + 4b_2)\mu^2)\} \right] \frac{t^\beta}{\Gamma(\beta+1)}, \quad (22)$$

$$w_1(x,t) = \mu f_1 (3r_1 - 2\mu^2) \operatorname{sech}^2(\mu x) \frac{t^\gamma}{\Gamma(\gamma+1)}, \quad (23)$$

$$p_1(x,t) = \mu t_1 (3r_1 - 2\mu^2) \operatorname{sech}^2(\mu x) \frac{t^\eta}{\Gamma(\eta+1)}, \quad (24)$$

and so on.

Proceeding in the same manner, the rest of the components  $u_n(x,t), v_n(x,t), w_n(x,t)$  and  $p_n(x,t)$  can be obtained and the series solutions are thus entirely determined.

Finally, we approximate the analytical solutions  $u(x,t), v(x,t), w(x,t)$  and  $p(x,t)$  by truncating the series at level  $n = 3$  as;

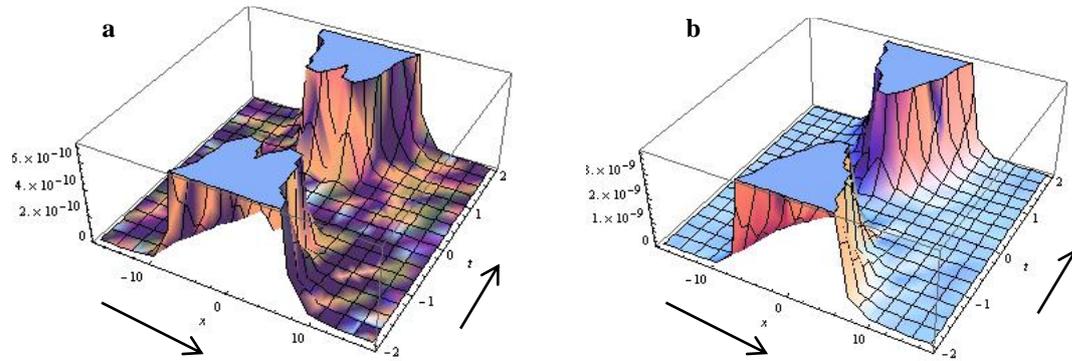
$$u(x,t) = \sum_{i=0}^3 u_i(x,t), \quad v(x,t) = \sum_{i=0}^3 v_i(x,t), \quad (25)$$

$$w(x,t) = \sum_{i=0}^3 w_i(x,t) \quad p(x,t) = \sum_{i=0}^3 p_i(x,t). \quad (26)$$

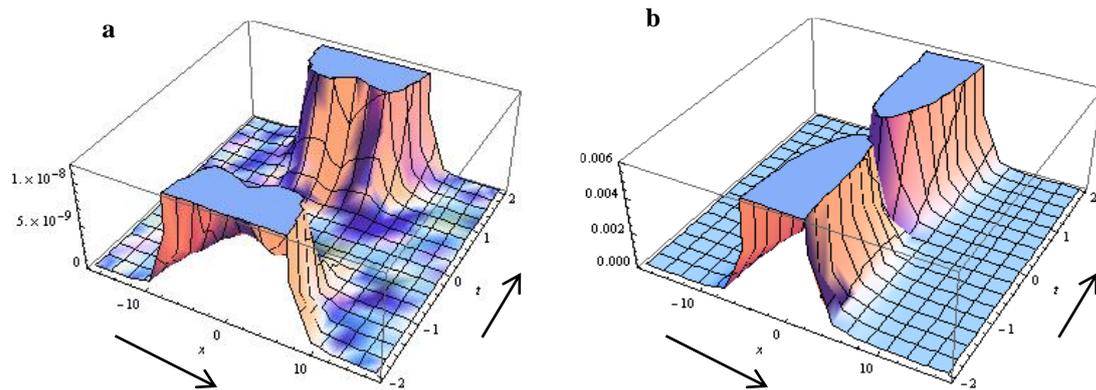
Here, the accuracy of the HPM is illustrated by computing the absolute errors  $E_{u_3}(x,t) = |u(x,t) - \tilde{u}(x,t)|$ ,

$E_{v_3}(x,t) = |v(x,t) - \tilde{v}(x,t)|$ ,  $E_{w_3}(x,t) = |w(x,t) - \tilde{w}(x,t)|$ ,  $E_{p_3}(x,t) = |p(x,t) - \tilde{p}(x,t)|$  for the constants value of  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$  and for the standard generalized Ito system i.e. for  $\alpha = \beta = \gamma = \eta = 1$ , where  $u(x,t), v(x,t), w(x,t), p(x,t)$  are exact solutions and

$\tilde{u}(x,t), \tilde{v}(x,t), \tilde{w}(x,t), \tilde{p}(x,t)$  are approximate solutions of (11) obtained by truncating the respective solutions series (17) - (18). We only consider 3 terms in evaluating the approximate solutions of a generalized Ito system. It achieves a high level of accuracy.

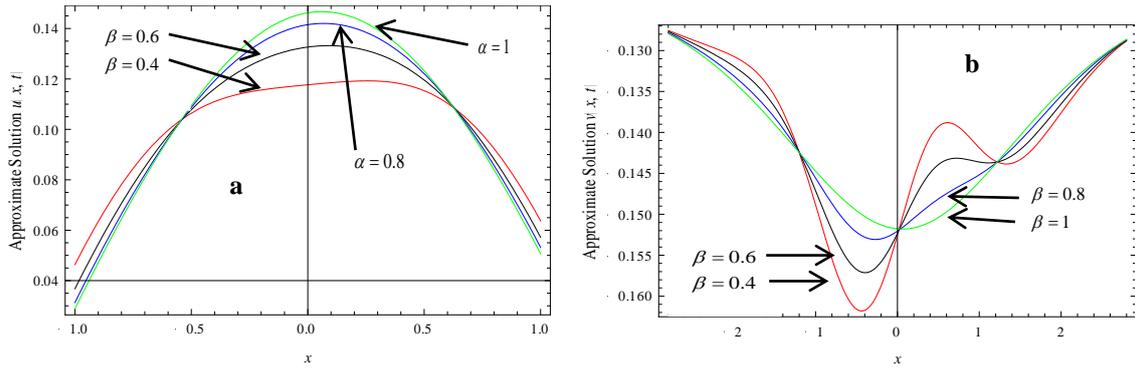


**Figure 3** The absolute errors  $E_{u_3}(x,t)$  and  $E_{v_3}(x,t)$  for a standard generalized Ito system  $\alpha = \beta = \gamma = \eta = 1$  at  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$ .

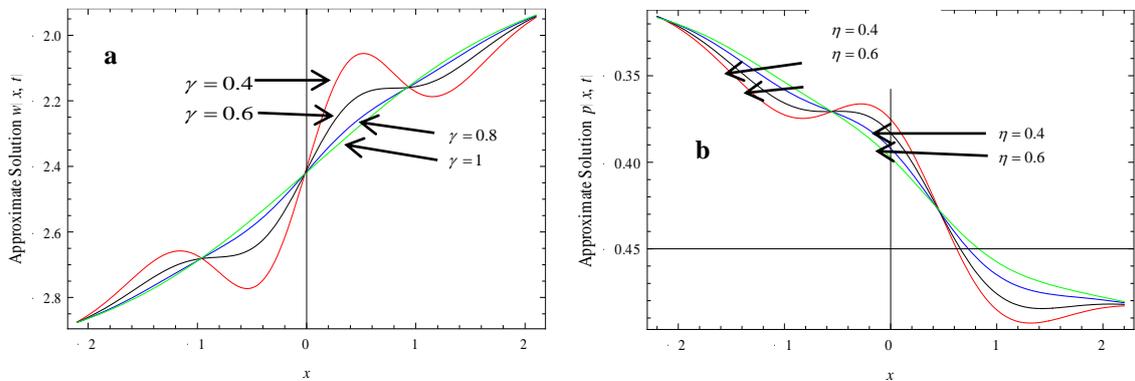


**Figure 4** The absolute errors  $E_{w_3}(x,t)$  and  $E_{p_3}(x,t)$  for a standard generalized Ito system  $\alpha = \beta = \gamma = \eta = 1$  at  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$ .

**Figures 3 - 4** show the absolute errors between the exact and approximate solutions obtained by the proposed method for values of  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$  and for the standard generalized Ito system i.e. for  $\alpha = \beta = \gamma = \eta = 1$ . From **Figures 3 - 4**, it is seen that our approximate solutions obtained by the HPM converge very rapidly to the exact solutions in only the 3<sup>rd</sup> order approximations i.e. approximate solutions are very near to the exact solutions. It achieves a high level of accuracy. In this example, only the 3<sup>rd</sup> order term of the series solution is considered. The accuracy of the result can be improved by introducing more terms of the approximate solutions.



**Figure 5** The approximate solutions  $\tilde{u}(x,t)$  and  $\tilde{v}(x,t)$  for different values of  $\alpha$  and  $\beta$  respectively when  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$  at  $\gamma = \eta = t = 1$ .



**Figure 6** The approximate solutions  $\tilde{w}(x,t)$  and  $\tilde{p}(x,t)$  for different value of  $\gamma$  and  $\eta$  respectively when  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$  at  $\alpha = \beta = t = 1$ .

The evaluation results for the approximate solutions are depicted in **Figures 5 - 6** for different fractional Brownian motions and standard motions. **Figures 5 - 6** show the variation of the approximate solutions  $\tilde{u}(x,t), \tilde{v}(x,t), \tilde{w}(x,t), \tilde{p}(x,t)$  for different value  $\alpha, \beta, \gamma, \eta$  vs.  $x$  at  $t=1$  and for the constants value of  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$ . It is seen from the **Figure 5a** that the approximate solution  $\tilde{u}(x,t)$  increases for the value of  $x < 0$  and decreases for the value of  $x > 0$  at different value of  $\alpha = 0.4, 0.6, 0.8, 1$  vs.  $x$ . Similarly from **Figure 5b**, it can be seen that the approximate solution  $\tilde{v}(x,t)$  has the opposite behavior i.e.  $\tilde{v}(x,t)$  is decreasing for the value of  $x < 0$  and increasing for the value  $x > 0$  at different values of  $\beta = 0.4, 0.6, 0.8, 1$  with an increases in  $x$ . From **Figure 6**, we can see that the approximate solutions  $\tilde{w}(x,t)$  are strictly increasing and  $\tilde{p}(x,t)$  is strictly decreasing with an increases in  $x$  at  $\mu = 0.5, f_1 = 0.6, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1$ .

## Conclusions

In this paper, the approximate solutions of the fractional order Ito system are obtained by employing the homotopy perturbation method. The beauty of the paper is error analysis between the exact solutions and approximate solutions. We demonstrate that our approximate solutions are very near to the exact solutions despite less computational work for both examples. The results show that the homotopy perturbation method is a powerful and efficient technique in finding exact and approximate solutions for a system of nonlinear partial differential equations of fractional order.

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