WALAILAK JOURNAL
http://wjst.wu.ac.th Applied Mathematics

The Riccati Equation Mapping Method for Solving Nonlinear Partial Differential Equations in Mathematical Physics

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Received: 18 January 2013, Revised: 29 April 2013, Accepted: 23 January 2014

Abstract

In this article, many new exact solutions of the (2+1)-dimensional nonlinear Boussinesq-Kadomtsev-Petviashvili equation and the (1+1)-dimensional nonlinear heat conduction equation are constructed using the Riccati equation mapping method. By means of this method, many new exact solutions are successfully obtained. This method can be applied to many other nonlinear evolution equations in mathematical physics.

Keywords: The Riccati equation mapping method, the (2+1)-dimensional nonlinear Boussinesq-Kadomtsev-Petviashvili equation, the (1+1)-dimensional nonlinear heat conduction equation, exact solutions.

Introduction

Nonlinear evolution equations are often used to describe the motion of isolated waves, localized in a small part of space, in many fields, such as hydrodynamics, plasma physics, and nonlinear optics. The investigation of exact solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. With the development of computerized symbolic computation, much work has been focused on the various extensions and applications of the well known algebraic methods to construct the exact solutions of the nonlinear evolution equations [1-32].

The objective of this article is to apply the Riccati equation mapping method to find many new exact solutions of the following nonlinear (2+1)-dimensional Boussinesq-Kadomtsev-Petviashvili equations;

$$u_t = u_{xxx} + u_{yyy} + 6(uv)_x + 6(u\omega)_y, \tag{1}$$
$$v_y = u_x\,, \tag{2}$$
$$w_x = u_y\,, \tag{3}$$

as well as the following nonlinear (1+1)-dimensional heat conduction equation;

$$u_t - (u^2)_{xx} = pu - qu^2 \tag{4}$$

where p and q are known constants.

Recently, Zheng [32] has discussed about the 2 models (1) - (4) using a different approach, namely, the (G′/G)-expansion method, and found some exact solutions of these 2 models. Comparison between the results in the present paper and these well-known results will be given later.

The rest of this article is organized as follows: In section 2, the Riccati equation mapping method is described. In section 3, this method is applied to solve the2 models (1) - (4). In section 4, some conclusions are given.

Description of the Riccati equation mapping method

Consider a given nonlinear partial differential equation (PDE) with the independent variable $X = (t, x_1, x_2, \ldots\ldots, x_n)$ and the dependent variable u(X) in the following form;

$$H\left(u, u_t, u_{x_i}, u_{x_i x_i}, u_{x_i x_j}, \ldots\right) = 0, \tag{5}$$

where *H* is a general polynomial function of its arguments. In order to solve Eq. (5) by using the proposed method , the following main steps are given;

Step 1 Using the general form of the wave transformation;

$$u = u(\xi)\ , \qquad \xi = \xi(X), \tag{6}$$

where ξ is a real function of the independent variable X.

Step 2 Substituting (6) into (5) yields an ordinary differential equation (ODE) in ξ of the form;

$$Q(u, u'(\xi), u''(\xi), \ldots\ldots\ldots) = 0, \tag{7}$$

where Q is a general polynomial function of its arguments.

Step 3 Assuming that Eq. (7) has the formal solution;

$$u(\xi) = a_0 + \sum_{i=1}^{M} \left\{ a_i \phi^i(\xi) + b_i \phi^{i-1}(\xi)\sqrt{\sigma + \phi^2(\xi)} \right\}, \tag{8}$$

with ϕ satisfying the Riccati equation;

$$\phi'(\xi) = \sigma + \phi^2(\xi), \tag{9}$$

where σ, a_0, a_i, and b_i are constants to be determined later, while M is a positive integer, which is called the balance number.

Step 4 Determining the positive integer M by balancing the highest order derivatives and the nonlinear terms in Eq. (7).

Step 5 Substituting (8) along with Eq. (9) into Eq. (7) and collecting all the coefficients of $\phi^n(\zeta)\left(\sqrt{\sigma + \phi^2(\zeta)}\right)^m$, $(n = 0, \pm 1, \pm 2, \pm 3, \ldots;\ m = 0, \pm 1)$, then setting each coefficient to zero, a set of algebraic equations is obtained for a_0, a_i, b_i and ξ.

Step 6 Solving the system of algebraic equations in Step 5 using Maple or Mathematica software to find the value of a_0, a_i, b_i and ξ.

Step 7 As Eq. (9) possesses the solutions;

$$\phi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\,\xi\right), & \sigma<0,\\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\xi\right), & \sigma<0,\\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\,\xi\right), & \sigma>0,\\ \sqrt{\sigma}\cot\left(\sqrt{\sigma}\,\xi\right), & \sigma>0,\\ -\dfrac{1}{\xi}, & \sigma=0,\end{cases} \tag{10}$$

substituting a_0, a_i, b_i ξ and (10) into (8) to obtain the exact solutions of Eq. (5).

Applications

In this section, many exact solutions of the nonlinear (2 +1)-dimensional nonlinear Boussinesq-Kadomtsev-Petviashvili Eqs. (1) - (3) and the nonlinear (1 + 1)-dimensional heat conduction Eq. (4) are determined using the Riccati equation mapping method described in section 2.

Example 1 The exact solutions of the nonlinear (2 + 1)-dimensional Boussinesq-Kadomtsev-Petviashvili equations (1) - (3)

In order to construct the exact solutions of Eqs. (1) - (3), suitable wave transformations are chosen;

$$u=u(\xi),\quad v=v(\xi),\quad w=w(\xi),\quad \xi=fx+gy-ht, \tag{11}$$

where *f, g* and *h* are constants to be determined later. The transformation (11) is a special case of the general form (6). Substituting (11) into Eqs. (1) - (3) and integrating once, the ODEs;

$$(f^3+g^3)\,u''(\xi)+\frac{6}{fg}(f^3+g^3)u^2(\xi)+\frac{1}{fg}[6(c_2f^2+c_3g^2)+fgh]\,u(\xi)=c_1, \tag{12}$$

$$u(\xi)=\frac{f\,u(\xi)+c_2}{g}, \tag{13}$$

$$w(\xi)=\frac{gu(\xi)+c_3}{f}, \tag{14}$$

are obtained, where $c_i\ (i=1-3)$ are constants of integration and $'=\frac{d}{d\xi}$.

By balancing u"(ξ) with $u^2(\xi)$ in Eq. (12) , we get M = 2. On using (8) the formal solutions;

$$u(\xi)=a_0+a_1\phi(\xi)+a_2\phi^2(\xi)+b_1\sqrt{\sigma+\phi^2(\xi)}+b_2\phi(\xi)\sqrt{\sigma+\phi^2(\xi)}, \tag{15}$$

are obtained, where $a_0,\ a_1,\ a_2,\ b_1\ and\ b_2$ are constants to be determined. Substituting (15) along with (9) into Eq. (12) and collecting all the coefficients of $\phi^n(\xi)\left(\sqrt{\sigma+\phi^2(\xi)}\right)^m$, then setting each coefficient to zero, the following system of algebraic equations;

$\phi^5(\xi): 12f^3a_2b_2+6fg^4b_2+6f^4g\,b_2+12g^3a_2b_2=0,$

$\phi^4(\xi): 12f^3a_2b_1+12g^3a_1b_2+12g^3a_2b_1+2fg^4b_1+12f^3a_1b_2+2f^4g\,b_1=0,$

$\phi^3(\xi): 12f^3a_1b_1+hfgb_2+12g^3a_0b_2+11f^4g\sigma b_2+12\sigma g^3a_2b_2+6c_2f^2b_2+12f^3a_0b_2 + 12\sigma f^3a_2b_2+12g^3a_1b_1+11fg^4\sigma b_2+6c_3g^2b_2=0,$

$\phi^2(\xi): 6c_2f^2b_1+12\sigma f^3a_1b_2+6c_3g^2b_1+3f^4g\sigma b_1+hfgb_1+12\sigma g^3a_1b_2+12\sigma g^3a_2b_1 + 12\sigma f^3a_2b_1+3fg^4\sigma b_1+12g^3a_0b_1+12f^3a_0b_1=0,$

$$\phi^{1}(\xi) : 6\sigma c_3 g^2 b_2 + 5fg^4\sigma^2 b_2 + 12\sigma f^3 a_1 b_1 + 6\sigma c_2 f^2 b_2 + \sigma hfg b_2 + 12\sigma g^3 a_1 b_1 + 5f^4 g\sigma^2 b_2 + 12\sigma f^3 a_0 b_2 + 12\sigma g^3 a_0 b_2 = 0,$$
$$\phi^{0}(\xi) : f^4 g\sigma^2 b_1 + 12\sigma g^3 a_0 b_1 + fg^4\sigma^2 b_1 + 6\sigma c_3 g^2 b_1 + 12\sigma f^3 a_0 b_1 + 6\sigma c_2 f^2 b_1 + \sigma hfg b_1 = 0,$$
$$\phi^{4}(\xi)\sqrt{\sigma+\phi^2(\xi)} : 6g^3 b_2^2 + 6f^3 a_2^2 + 6fg^4 a_2 + 6f^4 g a_2 + 6g^3 a_2^2 + 6f^3 b_2^2 = 0,$$
$$\phi^{3}(\xi)\sqrt{\sigma+\phi^2(\xi)}: 12f^3 a_1 a_2 + 12g^3 a_1 a_2 + 12\, g^3 b_1 b_2 + 12f^3 b_1 b_2 + 2fg^4 a_1 + 2f^4 g a_1 = 0,$$
$$\phi^{2}(\xi)\sqrt{\sigma+\phi^2(\xi)}: 6f^3 a_1^2 + 6c_2 f^2 a_2 + 8fg^4\sigma a_2 + 12g^3 a_0 a_2 + hfg a_2 + 6\sigma f^3 b_2^2 + 6f^3 b_1^2 + 12f^3 a_0 a_2 + 6g^3 b_1^2 + 8f^4 g\sigma a_2 + 6c_3 g^2 a_2 + 6\sigma g^3 b_2^2 + 6g^3 a_1^2 = 0,$$
$$\phi^{1}(\xi)\sqrt{\sigma+\phi^2(\xi)}: 6c_2 f^2 a_1 + 2f^4 g\sigma a_1 + 12\sigma g^3 b_1 b_2 + 12g^3 a_0 a_1 + 12f^3 a_0 a_1 + 2fg^4\sigma a_1 + 6c_3 g^2 a_1 + hfg a_1 + 12\sigma f^3 b_1 b_2 = 0,$$
$$\phi^{0}(\xi)\sqrt{\sigma+\phi^2(\xi)}: 6g^3 a_0^2 + 6c_2 f^2 a_0 + 6c_3 g^2 a_0 + hfg a_0 + 6\sigma g^3 b_1^2 + 6f^3 a_0^2 + 2fg^4\sigma^2 a_2 + 2f^4 g\sigma^2 a_2 + c_1 fg + 6\sigma f^3 b_1^2 = 0 \tag{16}$$

are obtained. The following cases are used in solving the algebraic system (16) by use of Maple or Mathematica;

Case 1

$$a_0 = \frac{-\sigma fg}{3}, \quad a_1 = 0, \quad a_2 = -fg, \quad b_1 = 0, \quad b_2 = 0, \quad c_1 = 0, \quad f = f, \quad g = g,$$
$$h = \frac{-2}{fg}[3(c_2 f^2 + c_3 g^2) + 2fg\sigma(f^3 + g^3)]. \tag{17}$$

In this case, the exact solutions of Eqs. (1) - (3) have the forms;

$$u(\xi) = \begin{cases} \frac{-\sigma fg}{3}[1 - 3\tanh(\sqrt{-\sigma}\,\xi)], & \sigma < 0 \\ \frac{-\sigma fg}{3}[1 - 3\ \coth^2(\sqrt{-\sigma}\,\xi)], & \sigma < 0 \\ \frac{-\sigma fg}{3}[1 + 3\tan^2(\sqrt{\sigma}\,\xi)], & \sigma > 0 \\ \frac{-\sigma fg}{3}[1 + 3\cot^2(\sqrt{\sigma}\,\xi)], & \sigma > 0 \\ \frac{-fg}{\xi}, & \sigma = 0, \end{cases} \tag{18}$$

$$v(\xi) = \begin{cases} \frac{-\sigma f^2}{3}[1 - 3\tanh(\sqrt{-\sigma}\,\xi)] + \frac{c_2}{g}, & \sigma < 0 \\ \frac{-\sigma f^2}{3}[1 - 3\coth^2(\sqrt{-\sigma}\,\xi)] + \frac{c_2}{g}, & \sigma < 0 \\ \frac{-\sigma f^2}{3}[1 + 3\tan^2(\sqrt{\sigma}\,\xi)] + \frac{c_2}{g}, & \sigma > 0 \\ \frac{-\sigma f^2}{3}[1 + 3\cot^2(\sqrt{\sigma}\,\xi)] + \frac{c_2}{g}, & \sigma > 0 \\ \frac{-f^2}{\xi^2} + \frac{c_2}{g}, & \sigma = 0, \end{cases} \tag{19}$$

$$w(\xi)=\begin{cases}\frac{-\sigma g^2}{3}\left[1-3\tanh\left(\sqrt{-\sigma}\;\xi\right)\right]+\frac{c_3}{f}, & \sigma<0\\ \frac{-\sigma g^2}{3}\left[1-3\coth^2\left(\sqrt{-\sigma}\;\xi\right)\right]+\frac{c_3}{f}, & \sigma<0\\ \frac{-\sigma g^2}{3}\left[1+3\tan^2\left(\sqrt{\sigma}\;\xi\right)\right]+\frac{c_3}{f}, & \sigma>0\\ \frac{-\sigma g^2}{3}\left[1+3\cot^2\left(\sqrt{\sigma}\;\xi\right)\right]+\frac{c_3}{f}, & \sigma>0\\ \frac{-g^2}{\xi^2}+\frac{c_3}{f}\;, & \sigma=0,\end{cases} \tag{20}$$

where

$$\begin{cases}\xi=fx+gy+\frac{2t}{fg}\left[3(c_2f^2+c_3g^2)+2\sigma fg(f^3+g^3)\right], & \sigma\neq 0\\ \xi=fx+gy+\frac{6t}{fg}(c_2f^2+c_3g^2), & \sigma=0\,.\end{cases} \tag{21}$$

Case 2

$$a_0=-\sigma fg\;,\quad a_1=0\,,\quad a_2=-fg\,,\quad b_1=0\,,\quad b_2=0\,,\quad c_1=0,\quad f=f\,,\quad g=g,$$
$$h=\frac{-2}{fg}\left[3(c_2f^2+c_3g^2)-2fg\sigma(f^3+g^3)\right]. \tag{22}$$

In this case, the exact solutions of Eqs. (1) - (3) have the forms;

$$u(\xi)=\begin{cases}-\sigma fg\,\mathrm{sech}^2\left(\sqrt{-\sigma}\xi\right)\,, & \sigma<0\;,\\ \sigma fgcsch^2\left(\sqrt{-\sigma}\xi\right)\;, & \sigma<0\;,\\ -\sigma fgsec^2\left(\sqrt{\sigma}\xi\right)\;, & \sigma>0\;,\\ -\sigma fg\;csc^2\left(\sqrt{\sigma}\xi\right)\;, & \sigma>0\,,\\ \frac{-fg}{\xi^2}\;, & \sigma=0,\end{cases} \tag{23}$$

$$v(\xi)=\begin{cases}-\sigma f^2\,\mathrm{sech}^2\left(\sqrt{-\sigma}\xi\right)+\frac{c_2}{g}\;, & \sigma<0\;,\\ \sigma f^2csch^2\left(\sqrt{-\sigma}\xi\right)\;+\frac{c_2}{g}\;, & \sigma<0\;,\\ -\sigma f^2sec^2\left(\sqrt{\sigma}\xi\right)\;+\frac{c_2}{g}\;, & \sigma>0,\\ -\sigma f^2csc^2\left(\sqrt{\sigma}\xi\right)\;+\frac{c_2}{g}\;, & \sigma>0\,,\\ \frac{-f^2}{\xi^2}+\frac{c_2}{g}\;, & \sigma=0\;,\end{cases} \tag{24}$$

$$w(\xi)=\begin{cases} -\sigma g^2\,\mathrm{sech}^2\left(\sqrt{-\sigma}\xi\right)+\dfrac{c_3}{f}, & \sigma<0, \\ \sigma g^2 csch^2\left(\sqrt{-\sigma}\xi\right)+\dfrac{c_3}{f}, & \sigma<0, \\ -\sigma g^2 sec^2\left(\sqrt{\sigma}\xi\right)+\dfrac{c_3}{f}, & \sigma>0, \\ -\sigma g^2 csc^2\left(\sqrt{\sigma}\xi\right)+\dfrac{c_3}{f}, & \sigma>0, \\ \dfrac{-g^2}{\xi^2}+\dfrac{c_3}{f}, & \sigma=0, \end{cases} \tag{25}$$

where

$$\begin{cases} \xi=fx+gy+\dfrac{2t}{fg}[3(c_2f^2+c_3g^2)-2\sigma fg(f^3+g^3)], & \sigma\neq 0 \\ \xi=fx+gy+\dfrac{6t}{fg}(c_2f^2+c_3g^2), & \sigma=0. \end{cases} \tag{26}$$

Case 3

$$a_0=\frac{-\sigma fg}{3}, \quad a_1=0, \quad a_2=\frac{-fg}{2}, \quad b_1=0, \quad b_2=\frac{\pm fg}{2}, \quad c_1=0, \quad f=f, \quad g=g,$$
$$h=\frac{-1}{fg}[6(c_2f^2+c_3g^2)+fg\sigma(f^3+g^3)]. \tag{27}$$

In this case, the exact solutions of Eqs. (1) - (3) have the forms;

$$u(\xi)=\begin{cases} \frac{-\sigma fg}{6}[2-3\tanh^2\left(\sqrt{-\sigma}\xi\right)\pm 3i\, tanh\left(\sqrt{-\sigma}\xi\right)sech\left(\sqrt{-\sigma}\xi\right)], & \sigma<0, \\ \frac{-\sigma fg}{6}[2-3\coth^2\left(\sqrt{-\sigma}\xi\right)\pm 3i\, coth\left(\sqrt{-\sigma}\xi\right)csch\left(\sqrt{-\sigma}\xi\right)], & \sigma<0, \\ \frac{-\sigma fg}{6}[2+3\tan^2\left(\sqrt{\sigma}\xi\right)\pm 3\, tan\left(\sqrt{\sigma}\xi\right)sec\left(\sqrt{\sigma}\xi\right)], & \sigma>0, \\ \frac{-\sigma fg}{6}[2+3\cot^2\left(\sqrt{\sigma}\xi\right)\pm 3\, cot\left(\sqrt{\sigma}\xi\right)csc\left(\sqrt{\sigma}\xi\right)], & \sigma>0, \\ \frac{-fg}{\xi^2}, & \sigma=0, \\ 0, & \sigma=0, \end{cases} \tag{28}$$

$$v(\xi)=\begin{cases}\dfrac{-\sigma f^2}{6}[2-3\tanh^2(\sqrt{-\sigma}\xi)\pm 3i\, tanh(\sqrt{-\sigma}\xi)sech(\sqrt{-\sigma}\xi)]+\dfrac{c_2}{g}, & \sigma<0,\\ \dfrac{-\sigma f^2}{6}[2-3\coth^2(\sqrt{-\sigma}\xi)\pm 3i\, coth(\sqrt{-\sigma}\xi)csch(\sqrt{-\sigma}\xi)]+\dfrac{c_2}{g}, & \sigma<0,\\ \dfrac{-\sigma f^2}{6}[2+3\tan^2(\sqrt{\sigma}\xi)\pm 3\, tan(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)]+\dfrac{c_2}{g}, & \sigma>0,\\ \dfrac{-\sigma f^2}{6}[2+3\cot^2(\sqrt{\sigma}\xi)\pm 3\, cot(\sqrt{\sigma}\xi)csc(\sqrt{\sigma}\xi)]+\dfrac{c_2}{g}, & \sigma>0,\\ \dfrac{-f^2}{\xi^2}+\dfrac{c_2}{g}, & \sigma=0,\\ \dfrac{c_2}{g}, & \sigma=0,\end{cases} \tag{29}$$

$$w(\xi)=\begin{cases}\dfrac{-\sigma g^2}{6}[2-3\tanh^2(\sqrt{-\sigma}\xi)\pm 3i\, tanh(\sqrt{-\sigma}\xi)sech(\sqrt{-\sigma}\xi)]+\dfrac{c_3}{f}, & \sigma<0,\\ \dfrac{-\sigma g^2}{6}[2-3\coth^2(\sqrt{-\sigma}\xi)\pm 3i\, coth(\sqrt{-\sigma}\xi)csch(\sqrt{-\sigma}\xi)]+\dfrac{c_3}{f}, & \sigma<0,\\ \dfrac{-\sigma g^2}{6}[2+3\tan^2(\sqrt{\sigma}\xi)\pm 3\, tan(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)]+\dfrac{c_3}{f}, & \sigma>0,\\ \dfrac{-\sigma g^2}{6}[2+3\cot^2(\sqrt{\sigma}\xi)\pm 3\, cot(\sqrt{\sigma}\xi)csc(\sqrt{\sigma}\xi)]+\dfrac{c_3}{f}, & \sigma>0,\\ \dfrac{-g^2}{\xi^2}+\dfrac{c_3}{f}, & \sigma=0,\\ \dfrac{c_3}{f}, & \sigma=0,\end{cases} \tag{30}$$

where

$$\xi=fx+gy+\frac{t}{fg}[6(c_2f^2+c_3g^2)+\sigma fg(f^3+g^3)],\ \ \sigma\neq 0\ ,$$
$$\xi=fx+gy+\frac{6t}{fg}(c_2f^2+c_3g^2),\qquad \sigma=0. \tag{31}$$

Case 4

$$a_0=\frac{-\sigma fg}{2}\ ,a_1=0\,,a_2=\frac{-fg}{2}\,,b_1=0\,,b_2=\frac{\pm fg}{2}\,,c_1=0,$$
$$f=f\,,g=g,\ \ h=\frac{-1}{fg}[6(c_2f^2+c_3g^2)-fg\sigma(f^3+g^3)], \tag{32}$$

In this case, the exact solutions of Eqs. (1) - (3) have the forms;

$$u(\xi)=\begin{cases}\frac{-\sigma fg}{2}[\mathrm{sech}^2(\sqrt{-\sigma}\xi)\pm i\,tanh(\sqrt{-\sigma}\xi)sech(\sqrt{-\sigma}\xi)], & \sigma<0,\\ \frac{-\sigma fg}{2}[\mathrm{csch}^2(\sqrt{-\sigma}\xi)\pm coth(\sqrt{-\sigma}\xi)csch(\sqrt{-\sigma}\xi)], & \sigma<0,\\ \frac{-\sigma fg}{2}[\sec^2(\sqrt{\sigma}\xi)\pm\ tan(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)], & \sigma<0,\\ \frac{-\sigma fg}{2}[\sec^2(\sqrt{\sigma}\xi)\pm cot(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)], & \sigma<0\,,\\ \frac{-fg}{\xi^2}, & \sigma=0,\\ 0, & \sigma=0,\end{cases} \tag{33}$$

$$v(\xi)=\begin{cases}\frac{-\sigma f^2}{2}[\mathrm{sech}^2(\sqrt{-\sigma}\xi)\pm i\,tanh(\sqrt{-\sigma}\xi)sech(\sqrt{-\sigma}\xi)]+\frac{c_2}{g}, \sigma<0,\\ \frac{-\sigma f^2}{2}[\mathrm{csch}^2(\sqrt{-\sigma}\xi)\pm coth(\sqrt{-\sigma}\xi)csch(\sqrt{-\sigma}\xi)]+\frac{c_2}{g}, \sigma<0\,,\\ \frac{-\sigma f^2}{2}[\sec^2(\sqrt{\sigma}\xi)\pm\ tan(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)]+\frac{c_2}{g}, & \sigma<0,\\ \frac{-\sigma f^2}{2}[\sec^2(\sqrt{\sigma}\xi)\pm cot(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)]+\frac{c_2}{g}, & \sigma<0,\\ \frac{-f^2}{\xi^2}+\frac{c_2}{g}, & \sigma=0,\\ \frac{c_2}{g}, & \sigma=0,\end{cases} \tag{34}$$

$$w(\xi)=\begin{cases}\frac{-\sigma g^2}{2}[\mathrm{sech}^2(\sqrt{-\sigma}\xi)\pm i\,tanh(\sqrt{-\sigma}\xi)sech(\sqrt{-\sigma}\xi)]+\frac{c_3}{f}, \sigma<0\\ \frac{-\sigma g^2}{2}[\mathrm{csch}^2(\sqrt{-\sigma}\xi)\pm coth(\sqrt{-\sigma}\xi)csch(\sqrt{-\sigma}\xi)]+\frac{c_3}{f}, \sigma<0,\\ \frac{-\sigma g^2}{2}[\sec^2(\sqrt{\sigma}\xi)\pm\ tan(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)]+\frac{c_3}{f}, & \sigma<0,\\ \frac{-\sigma g^2}{2}[\sec^2(\sqrt{\sigma}\xi)\pm cot(\sqrt{\sigma}\xi)sec(\sqrt{\sigma}\xi)]+\frac{c_3}{f}, & \sigma<0,\\ \frac{-g^2}{\xi^2}+\frac{c_3}{f}, & \sigma=0,\\ \frac{c_3}{f}, & \sigma=0,\end{cases} \tag{35}$$

where

$$\xi=fx+gy+\frac{t}{fg}[6(c_2f^2+c_3g^2)-\sigma fg(f^3+g^3)],\quad \sigma\neq 0,$$
$$\xi=fx+gy+\frac{6t}{fg}(c_2f^2+c_3g^2),\qquad \sigma=0\,. \tag{36}$$

Example 2 The exact solutions of the nonlinear (1+1)-dimensional heat conduction equation
In order to find the exact solutions of Eq. (4), suitable wave transformations are chosen;

$$u = u(\xi) \quad , \qquad \xi = kx - ht, \tag{37}$$

where k and h are constants to be determined later. The transformation (37) is also a special case of the general form (6). Substituting (37) into (4), the following ODE is obtained;

$$h\,u'(\xi) + 2k^2\left[\left(u'(\xi)\right)^2 + u(\xi)u''(\xi)\right] + P\,u(\xi) - q\,u^2(\xi) = 0, \tag{38}$$

Balancing $u(\xi)\,u''(\xi)$ with $u'(\xi)$ in Eq. (38), M = −1 is obtained. Using the transformation;

$$u(\xi) = v^{-1}(\xi), \tag{39}$$

to reduce Eq. (38) to the following ODE:

$$h\,v^2(\xi)v'(\xi) + 2k^2\left[3\left(v'(\xi)\right)^2 - v(\xi)v''(\xi)\right] + pv^3(\xi) - q\,v^2(\xi) = 0, \tag{40}$$

by balancing $v(\xi)v''(\xi)$ with $v^2(\xi)\,v'(\xi)$ in Eq. (40), M = 1 is obtained. Consequently, (8) is used to get the following formal solution;

$$v(\xi) = a_0 + a_1\phi(\xi) + b_1\sqrt{\sigma + \phi^2(\xi)}\ , \tag{41}$$

where a_0, a_1 and b_1 are constants to be determined later. Substituting (41) along with (9) into (40) and collecting all the coefficients of $\phi^n(\xi)\left(\sqrt{\sigma+\phi^2(\xi)}\right)^m$, then setting each coefficient to zero, the following system of algebraic equations is obtained;

$$\begin{aligned}
&\phi^5(\xi):\ -h\,b_1^3 + 4k^2a_1b_1 - 3ha_1^2b_1 = 0,\\
&\phi^4(\xi):\ pb_1^3 - 4k^2a_0b_1 + 3pa_1^2b_1 - 4ha_0b_1a_1 = 0,\\
&\phi^3(\xi):\ -ha_0^2b_1 - 5\sigma ha_1^2b_1 - 2\sigma hb_1^3 - 2qa_1b_1 + 6pa_0a_1b_1 + 10k^2\sigma a_1b_1 = 0,\\
&\phi^2(\xi):\ 3pa_0^2b_1 + 3\sigma pa_1^2b_1 - 6\sigma ha_0a_1b_1 - 2qa_0b_1 + 2\sigma pb_1^3 - 6k^2\sigma a_0b_1 = 0,\\
&\phi^1(\xi):\ -\sigma^2hb_1^3 - 2\sigma qa_1b_1 + 6\sigma pa_0a_1b_1 + 6k^2\sigma^2a_1b_1 - \sigma ha_0^2b_1 - 2\sigma^2ha_1^2b_1 = 0,\\
&\phi^0(\xi):\ -2\sigma^2ha_0b_1a_1 + 3\sigma pa_0^2b_1 - 2k^2a_0b_1\sigma^2 + \sigma^2pb_1^3 - 2\sigma qa_0b_1 = 0,\\
&\phi^4(\xi)\sqrt{\sigma+\phi^2(\xi)}:\ 2k^2a_1^2 - ha_1^3 - 3hb_1^2a_1 + 2k^2b_1^2 = 0,\\
&\phi^3(\xi)\sqrt{\sigma+\phi^2(\xi)}:\ -4k^2a_0a_1 - 2ha_0a_1^2 - 2ha_0b_1^2 + pa_1^3 + 3pa_1b_1^2 = 0,\\
&\phi^2(\xi)\sqrt{\sigma+\phi^2(\xi)}:\ -qa_1^2 + 3pa_0b_1^2 - ha_1^3\sigma + 8k^2\sigma a_1^2 - qb_1^2 - 4h\sigma a_1b_1^2 - ha_0^2a_1 + 3pa_0a_1^2 = 0,\\
&\phi^1(\xi)\sqrt{\sigma+\phi^2(\xi)}:\ -2ha_0a_1^2\sigma + 3pa_0^2a_1 + 3\sigma pa_1b_1^2 - 4k^2\sigma a_0a_1 - 2ha_0b_1^2\sigma - 2qa_0a_1 = 0,\\
&\phi^0(\xi)\sqrt{\sigma+\phi^2(\xi)}:\ pa_0^3 - \sigma qb_1^2 - qa_0^2 + 6k^2\sigma^2a_1^2 - ha_0^2\sigma a_1 - 2k^2b_1^2\sigma^2 + 3\sigma pa_0b_1^2\\
&\qquad -\sigma^2hb_1^2a_1 = 0\,.
\end{aligned} \tag{42}$$

The following cases are used in solving the algebraic system (42) by using Maple or Mathematica;

Case 1

$$a_0 = \frac{q}{2p}\ , \quad a_1 = \frac{\pm q\sqrt{-\sigma}}{2p\sigma}\ , \quad b_1 = 0\ , \quad k = \frac{\pm\sqrt{-q\sigma}}{4\sigma}\ , \quad h = \frac{\pm\, p\,\sqrt{-\sigma}}{4\sigma} \tag{43}$$

In this case, the exact solutions of Eq. (4) have the forms;

$$u(\xi) = \begin{cases} \dfrac{2p}{q[1 \pm \tanh(\sqrt{-\sigma}\,\xi)]}, & \sigma < 0, \\ \dfrac{2p}{q[1 \pm \coth(\sqrt{-\sigma}\,\xi)]}, & \sigma < 0, \\ \dfrac{2p}{q[1 \pm \mathrm{i}\tan(\sqrt{\sigma}\,\xi)]}, & \sigma > 0, \\ \dfrac{2p}{q[1 \pm \mathrm{i}\cot(\sqrt{\sigma}\,\xi)]}, & \sigma > 0, \\ 0, & \sigma = 0, \end{cases} \tag{44}$$

where

$$\xi = \frac{\pm\sqrt{-\sigma}}{4\sigma}\left(x\sqrt{q} - pt\right),\ \ i = \sqrt{-1}\ ,\ \ \sigma \neq 0. \tag{45}$$

Case 2

$$a_0 = \frac{q}{2p}\ ,\ \ a_1 = \frac{\pm q\sqrt{-\sigma}}{2p\sigma},\ \ b_1 = \frac{\pm q\sqrt{-\sigma}}{2p},\ \ k = \frac{\pm\sqrt{-q\sigma}}{2\sigma}\ , \quad h = \frac{\pm p\sqrt{-\sigma}}{2\sigma}. \tag{46}$$

In this case, the exact solutions of Eq. (4) have the forms;

$$u(\xi) = \begin{cases} \dfrac{2p}{q[1 \pm \tanh(\sqrt{-\sigma}\,\xi) \pm i\,\sigma\, sech(\sqrt{-\sigma}\,\xi)]}\ , & \sigma < 0, \\ \dfrac{2p}{q[1 \pm \coth(\sqrt{-\sigma}\,\xi) \pm i\sigma\, cschh(\sqrt{-\sigma}\,\xi)]}\ , & \sigma < 0, \\ \dfrac{2p}{q[1 \pm \mathrm{i}\tan(\sqrt{\sigma}\,\xi) \pm i\sigma\, sec(\sqrt{\sigma}\,\xi)]}\ , & \sigma > 0, \\ \dfrac{2p}{q[1 \pm \mathrm{i}\cot(\sqrt{\sigma}\,\xi) \pm i\sigma\, csc(\sqrt{\sigma}\,\xi)]}\ , & \sigma > 0, \\ 0, & \sigma = 0, \end{cases} \tag{47}$$

where

$$\xi = \frac{\pm\sqrt{-\sigma}}{4\sigma}\left(x\sqrt{q} - pt\right), \qquad \sigma \neq 0. \tag{48}$$

Remark All solutions of this article have been checked with the aid of Maple by putting them back into the original Eqs. (1) - (4).

Conclusions

The Riccati equation mapping method has been successfully used in this paper to seek many new exact solutions of the nonlinear (2+1)-dimensional Boussinesq-Kadomtsev-Petviashvili Eqs. (1) - (3) and the nonlinear (1+1)-dimensional heat conduction Eq. (4). On comparing the new results obtained in this paper using the Riccati equation mapping method with the well-known results obtained in [32] using the (G'/G)-expansion method, more new exact solutions are given using the first method than the second one. Furthermore, it is shown that the proposed method in this article provides a very effective and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Acknowledgements

The authors wish to thank the referees for their interesting suggestions and comments to improve this article.

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