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New Exact Solutions for Isothermal Magnetostatic Atmosphere Equations

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Abstract

Here, an extended, (G'/G)-expansion method with a computerized symbolic computation is used for constructing the exact travelling wave solutions for isothermal magnetostatic atmospheres equations. These equations depend on arbitrary functions that must be specified with choices of the different choice of the different arbitrary functions. The proposed method has been successfully used to obtain some exact travelling wave solutions for the Liouville and sinh-Poisson equations. The obtained travelling wave solutions are expressed by hyperbolic, triangular and exponential function. The solutions obtained via the propose method have many potential applications in physics.

Keywords: Isothermal magnetostatic atmospheres equations, extended (G'/G)-expansion method, Liouville and sinh-Poisson equations, Liouville equation, travelling wave solutions

Introduction

Nonlinear evolution equations in mathematical physics play a major role in various fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics, and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction, and convection are very important in nonlinear wave equations.

The investigation of the exact solutions of nonlinear evolution equations play an important role in the study of nonlinear physical phenomena, and has gradually become a most important and significant area. In the past several decades, many effective methods for obtaining the exact solutions of NLEEs have been presented [1-26].

The equations of magnetostatic equilibrium have been used extensively to model the solar magnetic structure [27-31]. An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry was carried out. The force balance consisted of the JB force B as the magnetic field induction, and J as the electric current density, the gravitational force, and gas pressure gradient force. However, in many models, the temperature distribution is specified *a priori*, and direct reference to the energy equations is eliminated. In solar physics, the equations of magnetostatic have been used to model diverse phenomena, such as the slow evolution stage of solar flares, or the magnetostatic support of prominences [32,33]. The nonlinear equilibrium problem has been solved in several cases [34-36].

The rest of this paper is arranged as follows. In **section 2**, we describe the extended (G'/G)-expansion method. In **section 3**, to illustrate the method, we consider the equations of magnetohydrostatic equilibrium for plasma in a gravitational field and obtain abundant exact solutions, which include the hyperbolic, triangular, and exponential function. Finally, a conclusion and discussion are given in **section 4**.

Methodology

Here, we give a brief description of the extended $\frac{G'(\xi)}{G(\xi)}$ - expansion method [8-12]. For a given nonlinear equation, say in 2 independent variables x and t,

$$\phi(u, u_t, u_x, u_{tt}, u_{xx}, \ldots\ldots\ldots\ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, ϕ is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear term are involved, combining the independent variables x and t into one variable $\xi = x - ct$, suppose that;

$$u(x,t) = u(\xi), \xi = x - ct, \tag{2}$$

Eq. (1) becomes;

$$\psi(u, -cu', u', c^2u'', u'', \ldots\ldots) = 0 \tag{3}$$

The solution of Eq. (3) can be expressed by a polynomial in $(\frac{G'(\xi)}{G(\xi)})$;

$$u(\xi) = \sum_{j=-M}^{M} a_j \left(\frac{G'(\xi)}{G(\xi)}\right)^j, \tag{4}$$

where $G = G(\xi)$ satisfies;

$$G(\xi)G''(\xi) = AG^2(\xi) + BG(\xi)G'(\xi) + C(G'(\xi))^2, \tag{5}$$

which can be rewritten as;

$$\frac{d}{d\xi}\left(\frac{G'(\xi)}{G(\xi)}\right) = -(1-C)\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + B\left(\frac{G'(\xi)}{G(\xi)}\right) + A, \tag{6}$$

where *A*, *B*, *C* are real parameters to be determined later, $G' = \frac{dG(\xi)}{d\xi}, G'' = \frac{d^2G(\xi)}{d\xi^2}$, $a_i \neq 0$; the unwritten part in (4) is also a polynomial in $(\frac{G'}{G})$, but where the degree of which is generally equal to or less than $M - 1$, the positive integer M can be determined by balancing the highest order derivative terms with the nonlinear term appearing in Eq. (3).

It is worth noting that the solutions of Eq. (5) for $(\frac{G'}{G})$ can be written in the form of hyperbolic, triangular and exponential function solutions as given below.

The first type: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC > 0$, then;

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)} \frac{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} + c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} - c_2 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi}} \tag{7}$$

The second type: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC < 0$, then;

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)} \frac{i\, c_1 \cos(\frac{\sqrt{-\Delta_1}}{2}\xi) - c_2 \sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\, c_1 \sin(\frac{\sqrt{-\Delta_1}}{2}\xi) + c_2 \cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)} \tag{8}$$

The third type: when $B = 0$ and $\Delta_2 = A(C-1) > 0$, then;

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{\sqrt{\Delta_2}}{2(1-C)} \frac{c_1 \cos(\sqrt{\Delta_2}\xi) + c_2 \sin(\sqrt{\Delta_2}\xi)}{c_1 \sin(\sqrt{\Delta_2}\xi) - c_2 \cos(\sqrt{\Delta_2}\xi)} \tag{9}$$

The fourth type: when $B = 0$ and $\Delta_2 = A(C-1) < 0$, then;

$$(\frac{G'(\xi)}{G(\xi)}) = \frac{\sqrt{-\Delta_2}}{2(1-C)} \frac{i\, c_1 \cosh(\sqrt{-\Delta_2}\xi) - c_2 \sinh(\sqrt{-\Delta_2}\xi)}{i\, c_1 \sinh(\sqrt{-\Delta_2}\xi) - c_2 \cosh(\sqrt{-\Delta_2}\xi)} \tag{10}$$

Inserting Eq. (4) into Eq. (3) and using Eq. (5), collecting all terms with the same order $(\frac{G'(\xi)}{G(\xi)})$ together, the left hand side of Eq. (3) is converted into another polynomial in $(\frac{G'(\xi)}{G(\xi)})$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for a_i, λ and μ. With the knowledge of the coefficients a_i and general solution of Eq. (5), we have more travelling wave solutions of the nonlinear evolution Eq. (1).

Formulating the problem

The relevant magnetohydrostatic equations consist of the equilibrium equation [29];

$$J \wedge B - \rho \nabla \phi - \nabla P = 0 \tag{11}$$

which is coupled with Maxwell's equations;

$$J = \frac{\Delta \wedge B}{\mu}, \tag{12}$$

$$\nabla . B = 0, \tag{13}$$

where P, ρ, μ and ϕ are the gas pressure, the mass density, the magnetic permeability, and the gravitational potential, respectively. It is assumed that the temperature is uniform in space, and that the plasma is an ideal gas with an equation of state $p = \rho R_0 T_{0,}$ where R_0 is the gas constant, and T_0 is the temperature. Then, the magnetic field *B* can be written as;

$$B = \nabla u e_x + B_x e_x = (B_x, \frac{\partial u}{\partial z}, \frac{-\partial u}{\partial y}) \tag{14}$$

The form of (14) for B ensures that $\nabla . B = 0$, and there is no mono pole or defect structure. Eq. (11) requires the pressure and density to be of the form;

$$P(y, z) = P(u)\exp(-z / h), \quad \rho(y, z) = (1/gh)P(u)\exp(-z / h) \tag{15}$$

where $h = R_0 T_0 / g$, is the scale height and z measure. With substituting (12) - (15) into Eq. (11), we obtain;

$$\nabla^2 u + f(u) e^{-z/h} = 0 \tag{16}$$

$$f(u) = \mu \frac{dP}{du} \tag{17}$$

Then, Eq. (17) gives;

$$P(u) = P_0 + \frac{1}{\mu}\int f(u) du \tag{18}$$

Inserting Eq. (18) into (15), we have;

$$P(y, z) = (P_0 + \frac{1}{\mu}\int f(u) du)\, e^{-z/h}, \tag{19}$$

$$\rho(y, z) = \frac{1}{gh}(P_0 + \frac{1}{\mu}\int f(u) du)\, e^{-z/h} \tag{20}$$

where P_0 is constant. Making the transformation;

$$x_1 + i x_2 = e^{(-z/l)} e^{(-iy/l)} \tag{21}$$

Then Eq. (16) reduces;

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+l^2 f(u)e^{((2/l)-(l/h))z}=0 \tag{22}$$

These equations have been given in [27,29].

Applications of the extended $\frac{G'(\xi)}{G(\xi)}$-expansion method

We employ the extended $\frac{G'(\xi)}{G(\xi)}$-expansion method for solving specific forms of the function $f(u)$ [29].

Liouville equation

Let us first consider the Liouville equation, which is a special case of Eq. (22);

$$u_{xx}+u_{tt}-\alpha^2 l^2 e^{-2\phi}=0 \tag{23}$$

In order to apply the proposed method, we use the wave transformation $u(\xi),\xi=x-kt$ transform Eq. (23) as;

$$(1+k^2)\phi''=\alpha^2 l^2 e^{-2\phi} \tag{24}$$

Making the transformation;

$$v=e^{-2\phi} \tag{25}$$

Then, Eq. (24) reduces;

$$(1+k^2)vv''-(1+k^2)v'^2+2\alpha^2 l^2 v^3=0 \tag{26}$$

In view of the technique of solution, we introduce the anstaz;

$$v(\xi)=\sum_{i=-N}^{N} a_i\left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{27}$$

where a_i are constants to be determined later. Our main goal is to solve Eq. (26) by the means illustrated above. Considering the homogeneous balance between $v(\xi)v''(\xi)$ and $v^3(\xi)$ in Eq. (26), we have N = 2, and we suppose that the solution of Eq. (26) can be expressed by;

$$v(\xi)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right)+a_{-1}\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1}a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2+a_{-2}\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2}, \tag{28}$$

where a_0, a_1, a_{-1}, a_2 and a_{-2} are constants to be determined later. Substituting Eq. (28) with Eq. (5) into Eq. (26), and collecting the coefficients of $\frac{G'(\xi)}{G(\xi)}$, we obtain a set of algebraic equations for $a_0, a_1, a_{-1}, a_2, a_{-2}, k, \lambda$, and μ. Solving this system with the aid of Maple Package, we obtain the two sets of solutions as;

Case 1:

$$a_{-2} = a_{-1} = 0, l = l, k = k, \alpha = \alpha, a_1 = \frac{B(-k^2 + C - 1 + k^2 C)}{l^2 \alpha^2}, a_0 = -\frac{A(-k^2 + C - 1 + k^2 C)}{l^2 \alpha^2},$$
$$a_2 = -\frac{-2k^2 C - 2C + C^2 + k^2 C + 1 + k^2)}{l^2 \alpha^2}, \tag{29}$$

where A, B, C, l *and* k are arbitrary constants.

Case 2:

$$a_2 = a_1 = 0, l = l, k = k, \alpha = \alpha, a_{-1} = -\frac{BA(1 + k^2)}{l^2 \alpha^2}, a_0 = -\frac{A(-k^2 + C - 1 + k^2 C)}{l^2 \alpha^2},$$
$$a_{-2} = \frac{-A(1 + k^2)}{l^2 \alpha^2}, \tag{30}$$

where A, B, C, l *and* k are arbitrary constants.

Substituting those cases in Eq. (28) provides the following solutions of Eq. (23). These solutions are;

$$v_1(\xi) = [-\frac{A(-k^2 + C - 1 + k^2 C)}{l^2 \alpha^2}] + [\frac{B(-k^2 + C - 1 + k^2 C)}{l^2 \alpha^2}](\frac{G'(\xi)}{G(\xi)})$$
$$-[\frac{-2k^2 C - 2C + C^2 + k^2 C + 1 + k^2)}{l^2 \alpha^2}](\frac{G'(\xi)}{G(\xi)})^2, \tag{31}$$

$$v_2(\xi) = [-\frac{A(-k^2 + C - 1 + k^2 C)}{l^2 \alpha^2}] + [-\frac{BA(1 + k^2)}{l^2 \alpha^2}](\frac{G'(\xi)}{G(\xi)})^{-1} + [\frac{-A(1 + k^2)}{l^2 \alpha^2}](\frac{G'(\xi)}{G(\xi)})^{-2}, \tag{32}$$

$$\phi(\xi) = -\frac{1}{2} v(\xi), \tag{33}$$
$$\xi = x - kt$$

With knowing Eqs. (7) - (10) and Eqs. (31), (32), we obtain the following exponential function solutions, hyperbolic function solutions, and triangular function solutions of Eq. (23);

Family -1: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC > 0$, then;

$$\nu_{1a}(\xi) = [-\frac{A(-k^2 + C - 1 + k^2C)}{l^2\alpha^2}] + [\frac{B(-k^2 + C - 1 + k^2C)}{l^2\alpha^2}] [\frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)} \frac{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} + c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} - c_2 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi}}] + [-[\frac{-2k^2C - 2C + C^2 + k^2C + 1 + k^2)}{l^2\alpha^2}]][\frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)} \frac{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} + c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} - c_2 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi}}]^2 \tag{34}$$

Family -2: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC < 0$, then;

$$\nu_{1b}(\xi) = [-\frac{A(-k^2 + C - 1 + k^2C)}{l^2\alpha^2}] + [\frac{B(-k^2 + C - 1 + k^2C)}{l^2\alpha^2}] [\frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)} \frac{i\, c_1 \cos(\frac{\sqrt{-\Delta_1}}{2}\xi) - c_2 \sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\, c_1 \sin(\frac{\sqrt{-\Delta_1}}{2}\xi) + c_2 \cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)}] + [-[\frac{-2k^2C - 2C + C^2 + k^2C + 1 + k^2)}{l^2\alpha^2}]][\frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)} \frac{i\, c_1 \cos(\frac{\sqrt{-\Delta_1}}{2}\xi) - c_2 \sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\, c_1 \sin(\frac{\sqrt{-\Delta_1}}{2}\xi) + c_2 \cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)}]^2 \tag{35}$$

Family -3: when $B = 0$ and $\Delta_2 = A(C-1) > 0$, then;

$$\nu_{1c}(\xi) = [-\frac{A(-k^2 + C - 1 + k^2C)}{l^2\alpha^2}] + +[-[\frac{-2k^2C - 2C + C^2 + k^2C + 1 + k^2)}{l^2\alpha^2}] [\frac{\sqrt{\Delta_2}}{2(1-C)} \frac{c_1 \cos(\sqrt{\Delta_2}\xi) + c_2 \sin(\sqrt{\Delta_2}\xi)}{c_1 \sin(\sqrt{\Delta_2}\xi) - c_2 \cos(\sqrt{\Delta_2}\xi)}]^2 \tag{36}$$

Family -4: when $B = 0$ and $\Delta_2 = A(C-1) < 0$, then;

$$\nu_{1d}(\xi) = [-\frac{A(-k^2+C-1+k^2C)}{l^2\alpha^2}] + +[-[\frac{-2k^2C-2C+C^2+k^2C+1+k^2}{l^2\alpha^2}]$$
$$[\frac{\sqrt{-\Delta_2}}{2(1-C)}\frac{i\,c_1\cosh(\sqrt{-\Delta_2}\xi)-c_2\sinh(\sqrt{-\Delta_2}\xi)}{i\,c_1\sinh(\sqrt{-\Delta_2}\xi)-c_2\cosh(\sqrt{-\Delta_2}\xi)}]^2 \tag{37}$$

In the same manner, case (2) provides the following exponential function solutions, hyperbolic function solutions, and triangular function solutions of Eq. (23);

Family -1: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC > 0$, then;

$$\nu_{2a}(\xi) = [-\frac{A(-k^2+C-1+k^2C)}{l^2\alpha^2}] + [-\frac{BA(1+k^2)}{l^2\alpha^2}]$$
$$[\frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)}\frac{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}+c_2\exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}-c_2\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}}]^{-1}$$
$$+[\frac{-A(1+k^2)}{l^2\alpha^2}][\frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)}\frac{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}+c_2\exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}-c_2\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}}]^{-2} \tag{38}$$

Family -2: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC < 0$, then;

$$\nu_{2b}(\xi) = [-\frac{A(-k^2+C-1+k^2C)}{l^2\alpha^2}] + [-\frac{BA(1+k^2)}{l^2\alpha^2}]$$
$$[\frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)}\frac{i\,c_1\cos(\frac{\sqrt{-\Delta_1}}{2}\xi)-c_2\sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\,c_1\sin(\frac{\sqrt{-\Delta_1}}{2}\xi)+c_2\cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)}]^{-1}$$
$$+[\frac{-A(1+k^2)}{l^2\alpha^2}][\frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)}\frac{i\,c_1\cos(\frac{\sqrt{-\Delta_1}}{2}\xi)-c_2\sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\,c_1\sin(\frac{\sqrt{-\Delta_1}}{2}\xi)+c_2\cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)}]^{-2} \tag{39}$$

Family -3: when $B = 0$ and $\Delta_2 = A(C-1) > 0,$ then;

$$\nu_{2c}(\xi) = [-\frac{A(-k^2+C-1+k^2C)}{l^2\alpha^2}] + [\frac{-A(1+k^2)}{l^2\alpha^2}][\frac{\sqrt{\Delta_2}}{2(1-C)}\frac{c_1\cos(\sqrt{\Delta_2}\xi)+c_2\sin(\sqrt{\Delta_2}\xi)}{c_1\sin(\sqrt{\Delta_2}\xi)-c_2\cos(\sqrt{\Delta_2}\xi)}]^{-2} \quad (40)$$

Family -4: when $B = 0$ and $\Delta_2 = A(C-1) < 0,$ then;

$$\nu_{2d}(\xi) = [-\frac{A(-k^2+C-1+k^2C)}{l^2\alpha^2}] + [\frac{-A(1+k^2)}{l^2\alpha^2}]$$
$$[\frac{\sqrt{-\Delta_2}}{2(1-C)}\frac{i\,c_1\cosh(\sqrt{-\Delta_2}\xi)-c_2\sinh(\sqrt{-\Delta_2}\xi)}{i\,c_1\sinh(\sqrt{-\Delta_2}\xi)-c_2\cosh(\sqrt{-\Delta_2}\xi)}]^{-2} \quad (41)$$

The sinh-Poisson equation

Secondly, we consider the sinh-Poisson equation, which plays an important role in soliton modeling with BPS Bound. Also, this equation is a special case of Eq. (22);

$$u_{xx} + u_{tt} = \beta^2 \sinh(\phi) \quad (42)$$

To apply the extended, (G'/G)-expansion method, we use the wave transformation $u(\xi), \xi = x - kt$ transform Eq. (42) as;

$$(1+k^2)\phi'' = \beta^2 \sinh(\phi) \quad (43)$$

We next use the transformation;

$$\nu = e^{\phi} \quad (44)$$

Then, Eq. (43) reduces;

$$2(1+k^2)\nu\nu'' - 2(1+k^2)\nu'^2 - \beta^2(\nu^3 - \nu) = 0 \quad (45)$$

We introduce the anstaz;

$$\nu(\xi) = \sum_{i=-N}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \quad (46)$$

where a_i is a constant to be determined later. Considering the homogeneous balance between $\nu(\xi)\nu''(\xi)$ and $\nu^3(\xi)$ in Eq. (45), we have N = 2, and we suppose that the solution of Eq. (45) can be expressed by;

$$\nu(\xi)=a_0+a_1(\frac{G'(\xi)}{G(\xi)})+a_{-1}(\frac{G'(\xi)}{G(\xi)})^{-1}a_2(\frac{G'(\xi)}{G(\xi)})^2+a_{-2}(\frac{G'(\xi)}{G(\xi)})^{-2}, \tag{47}$$

where a_0, a_1, a_{-1}, a_2 and a_{-2} are constants to be determined later. Substituting Eq. (47) with Eq. (5) into Eq. (45), and collecting the coefficients of $\frac{G'(\xi)}{G(\xi)}$, we obtain a set of algebraic equations for $a_0, a_1, a_{-1}, a_2, a_{-2}, k, \lambda$, and μ. Solving this system with the aid of Maple Package, we obtain the two sets of solutions as;

Case 1:

$$\begin{aligned} &a_1=a_2=0, a_0=\frac{B^2}{4AC-B^2-4A}, a_{-1}=\frac{4AB}{4AC-B^2-4A}, a_{-2}=\frac{4A^2}{4AC-B^2-4A}, \\ &k=\pm\sqrt{\frac{4AC-B^2-4A-\beta^2}{4AC-B^2-4A}} \end{aligned} \tag{48}$$

Case 2:

$$\begin{aligned} &a_{-1}=a_{-2}=0, a_0=\frac{B^2}{4AC-B^2-4A}, a_1=\frac{4B(C-1)}{4AC-B^2-4A}, a_2=\frac{4(C-1)^2}{4AC-B^2-4A}, \\ &k=\pm\sqrt{\frac{4AC-B^2-4A-\beta^2}{4AC-B^2-4A}} \end{aligned} \tag{49}$$

with the aid of above 2 cases in Eq. (47), we have the following solutions of Eq. (42) as follows;

$$\nu_2(\xi)=[\frac{B^2}{4AC-B^2-4A}]+[\frac{4AB}{4AC-B^2-4A}](\frac{G'(\xi)}{G(\xi)})^{-1}+[\frac{4A^2}{4AC-B^2-4A}](\frac{G'(\xi)}{G(\xi)})^{-2}, \tag{50}$$

$$\nu_2(\xi)=[\frac{B^2}{4AC-B^2-4A}]+[\frac{4B(C-1)}{4AC-B^2-4A}](\frac{G'(\xi)}{G(\xi)})+[\frac{4(C-1)^2}{4AC-B^2-4A}](\frac{G'(\xi)}{G(\xi)})^2, \tag{51}$$

$$\phi(\xi)=\ln(\nu(\xi)), \tag{52}$$

$$\xi=x-[\pm\sqrt{\frac{4AC-B^2-4A-\beta^2}{4AC-B^2-4A}}]t \tag{53}$$

With using Eqs. (7) - (10) and Eqs. (50), (51), we obtain the following exponential function solutions, hyperbolic function solutions, and triangular function solutions of Eq. (42);

Family -1: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC > 0$, then;

$$\nu_{1a}(\xi) = [\frac{B^2}{4AC - B^2 - 4A}] + [\frac{4AB}{4AC - B^2 - 4A}] [\frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)} \frac{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} + c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} - c_2 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi}}]^{-1} + [\frac{4A^2}{4AC - B^2 - 4A}][\frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)} \frac{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} + c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} - c_2 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi}}]^{-2} \tag{53}$$

$$\xi = x - [\pm\sqrt{\frac{4AC - B^2 - 4A - \beta^2}{4AC - B^2 - 4A}}]t \tag{54}$$

Family -2: when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC < 0$, then;

$$\nu_{2b}(\xi) = [\frac{B^2}{4AC - B^2 - 4A}] + [\frac{4AB}{4AC - B^2 - 4A}] [\frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)} \frac{i\, c_1 \cos(\frac{\sqrt{-\Delta_1}}{2}\xi) - c_2 \sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\, c_1 \sin(\frac{\sqrt{-\Delta_1}}{2}\xi) + c_2 \cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)}]^{-1} + [\frac{4A^2}{4AC - B^2 - 4A}][\frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)} \frac{i\, c_1 \cos(\frac{\sqrt{-\Delta_1}}{2}\xi) - c_2 \sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{i\, c_1 \sin(\frac{\sqrt{-\Delta_1}}{2}\xi) + c_2 \cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)}]^{-2} \tag{55}$$

$$\xi = x - [\pm\sqrt{\frac{4AC - B^2 - 4A - \beta^2}{4AC - B^2 - 4A}}]t \tag{56}$$

Family -3: when $B = 0$ and $\Delta_2 = A(C-1) > 0$, then;

$$\nu_{1c}(\xi) = [\frac{4A^2}{4AC - B^2 - 4A}][\frac{\sqrt{\Delta_2}}{2(1-C)} \frac{c_1 \cos(\sqrt{\Delta_2}\xi) + c_2 \sin(\sqrt{\Delta_2}\xi)}{c_1 \sin(\sqrt{\Delta_2}\xi) - c_2 \cos(\sqrt{\Delta_2}\xi)}]^{-2} \tag{57}$$

$$\xi = x - [\pm\sqrt{\frac{4AC^2 - 4A - \beta^2}{4AC - 4A}}]t \tag{58}$$

Family -4: when $B = 0$ and $\Delta_2 = A(C-1) < 0,$ then;

$$\nu_{1c}(\xi) = [\frac{4A^2}{4AC - B^2 - 4A}][\frac{\sqrt{-\Delta_2}}{2(1-C)}\frac{i\,c_1\cosh(\sqrt{-\Delta_2}\xi) - c_2\sinh(\sqrt{-\Delta_2}\xi)}{i\,c_1\sinh(\sqrt{-\Delta_2}\xi) - c_2\cosh(\sqrt{-\Delta_2}\xi)}]^{-2} \tag{59}$$

$$\xi = x - [\pm\sqrt{\frac{4AC^2 - 4A - \beta^2}{4AC - 4A}}]t \tag{60}$$

With similarly, through a combination of Case (2), and Eqs. (7) - (10) and (49), we can construct exact solutions to Eq. (42), which are omitted here for simplicity.

Conclusions

In this paper, a new analytical technique, namely, $\frac{G'(\xi)}{G(\xi)}$-expansion method, with a computerized symbolic computation using Maple, to establish new exact travelling wave solutions for isothermal magnetostatic atmosphere equations arising in physics. The proposed method has been successfully used to obtain some exact travelling wave solutions for the Liouville and sinh-Poisson equations.

As a result, many exact travelling wave solutions have been obtained, which include the hyperbolic, triangular, and exponential function. Finally, it is worthwhile to mention that the proposed method is reliable, effective, and gives more solutions. The applied method will be used in further works to establish more, entirely new, exact travelling wave solutions, for other kinds of nonlinear evolution equations arising in physics.

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