WALAILAK JOURNAL

http://wjst.wu.ac.th Mathematics

An Efficient Method for Solving the Brusselator System

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Received: 9 December 2012, Revised: 2 May 2013, Accepted: 13 July 2013

Abstract

In this paper, a new efficient recurrence relation is constructed to solve a nonlinear Brusselator equation. The system, known as the reaction-diffusion Brusselator, arises in the modeling of certain diffusion processes. The Laplace transform method and the new homotopy perturbation method (NHPM) are used to solve these equations. Since mathematical modeling of numerous scientific and engineering experiment lead to the Brusselator equation, it is worthwhile to try new methods to solve this system. Comparison of the results with those of the homotopy perturbation method, the Adomian decomposition method and the dual-reciprocity boundary element method leads to significant consequences. The method is tested using various examples and the results show that the new method is more effective and convenient to use, and has an evident high accuracy rate.

Keywords: Laplace transform method, new homotopy perturbation method (NHPM), Brusselator equation, reaction-diffusion systems

Introduction

Many physical, chemical, biological, environmental and even sociological processes are driven by reaction-diffusion systems. These are multi component models involving 2 different mechanisms: one is diffusion, a random particle movement, and the other, chemical, biological or sociological reactions representing instantaneous interactions, which depend on the state variables themselves and, possibly explicitly, on the position of particles [1].

An example of an important reaction-diffusion equation, both in biology and in chemistry, is known as the Brusselator system, which is used to describe a mechanism of chemical reaction-diffusion with nonlinear oscillations [2-4]. The reaction-diffusion Brusselator system contains a pair of variable intermediates, with reactant and product chemicals whose concentrations are controlled. It consists of the following four intermediate reaction steps;

$$A \to X, \qquad B + X \to Y + D, \qquad 2X + Y \to 3X, \qquad X \to E.$$

The global reaction is $A + B \to D + E$ and corresponds to the transformation of input products A and B into output products D and E. The reaction-diffusion Brusselator [5-8] prepares a useful model for the study of cooperative processes in chemical kinetics, such as trimolecular reaction steps arising from the formation of ozone by atomic oxygen via a triple collision. This system also governs in enzymatic reactions and in plasma and laser physics in multiple couplings between certain modes.

Solving partial differential equations is very important in mathematical sciences and engineering. Partial differential equations which arise in real-world physical problems are often too complicated to be precisely solved. Even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or it might be difficult to interpret the outcome. In recent years, an increasing amount of interest of scientists and engineers has been devoted to analytical asymptotic techniques for solving problems. Many new numerical techniques have been widely applied to these problems. The homotopy method is a powerful device for solving functional equations [9]. Based on homotopy, which is a basic concept in topology, the general analytical method, namely the homotopy perturbation method (HPM), was established by He [9-15] in 1998 to obtain series solutions of differential equations. The He HPM has been already used to solve various functional equations. In this method, the problem is transferred to an infinite number of sub-problems and then the solution is approximated by the sum of the solutions of the first several sub-problems. This simple method has been applied to solve linear and nonlinear equations of heat transfer [16-18], fluid mechanics [19], nonlinear Schrodinger equations [20], integral equations [21], boundary value problems [22], fractional KdV-Burgers equation [23], and nonlinear system of second order boundary value problems [24]. Also, there are new powerful analytical methods, such as variational itration method (VIM), homotopy analysis method (HAM), differential transform method (DTM) etc, which can propose a semi exact solution for nonlinear models[25-28].

The Brusselator system has been extensively investigated in the last decade from both analytical and numerical points of view (see, for instance, [29-37]). Numerical methods which are commonly used, such as finite difference, finite element or characteristics methods, need a large size of computational works and are usually affected by round-off errors which can cause a loss of accuracy in the results. Analytical methods commonly used for solving the Brusselator equation are very restrictive and are used in very special cases, so they cannot be used to solve equations of numerous realistic scenarios. The dual-reciprocity boundary element method [38], ADM method [39,40] VIM [41] and HPM [42] are applied for solving the Brusselator system.

In this work, an analytical approximation to the solution is constructed using a combination of the Laplace transform method and the new homotopy perturbation method (LTNHPM). The two-dimensional Brusselator system [38] has the following form;

$$\begin{cases} \dfrac{\partial u}{\partial t} = B + u^2 v - (A+1)u + \alpha\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} = Au - u^2 v + \alpha\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{1}$$

For $u(x,y,t)$ and $v(x,y,t)$ in a two dimensional region **R** bounded by a simple closed curve C subject to the initial conditions;

$$(u(x,y,t),v(x,y,t)) = (f(x,y),g(x,y)) for\,(x,y) \in R \tag{2}$$

and the boundary conditions;

$$(u(x,y,t),v(x,y,t)) = (w(x,y,t),z(x,y,t)) \;\; for\,(x,y) \in C_1 \;\; and \;\; t > 0, \tag{3}$$

$$\left(\frac{\partial u}{\partial n},\frac{\partial v}{\partial n}\right) = \left(p(x,y,t),q(x,y,t)\right) \;\; for\,(x,y) \in C_2 \;\; and \;\; t > 0, \tag{4}$$

where A, B and α are suitably given constants f, g, w, z, p and q are suitably prescribed functions, C_1 and C_2 are nonintersecting curves such that $C_1 \cup C_2 = C, \dfrac{\partial u}{\partial n} = \vec{n}.\nabla v$ and $\vec{n}$ is the unit normal outward vector R at the point (x,y) on C.

In this paper, the proposed method is tested on some examples. The results obtained via LTNHPM confirm the validity of the method. The rest of this paper is organized as follows: In section two, basic ideas of LTNHPM and the homotopy perturbation method are presented. In section three, the uses of

LTNHPM for solving the Brusselator system are presented. Some examples are solved by the proposed method in section four. The conclusion appear in the last section.

Materials and methods

To illustrate the basic ideas of this method, the following nonlinear differential equation is considered:

$$A(u)-f(r)=0, \quad r\in\Omega, \tag{5}$$

with the following initial conditions;

$$u(0)=\alpha_0, u'(0)=\alpha_1,\dots,u^{(n-1)}(0)=\alpha_{n-1} \tag{6}$$

where A is a general differential operator and $f(r)$ is a known analytical function. The operator A can be divided into two parts, L and N, where L is a linear and N is a nonlinear operator. Therefore, (5) can be rewritten as;

$$L(u)+N(u)-f(r)=0 \tag{7}$$

Based on NHPM [43,44], a homotopy $U(r,p):\Omega\times[0,1]\to \mathrm{R}$, is constructed which satisfies;

$$H(U,p)=(1-p)[L(U)-u_0]+p[A(U)-f(r)]=0, p\in[0,1], r\in\Omega, . \tag{8}$$

or equivalently;

$$H(U,p)=L(U)-u_0+pu_0+p[N(U)-f(r)]=0, \tag{9}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of (5). Clearly, (8) and (9) give;

$$H(U,0)=L(U)-u_0=0, \tag{10}$$
$$H(U(x),1)=A(U)-f(r)=0. \tag{11}$$

Applying the Laplace transform method to both sides of (9), we have;

$$\mathrm{L}\{L(U)-u_0+pu_0+p[N(U)-f(r)]\}=0 \tag{12}$$

Using the differential property of the Laplace transform method shows;

$$s^n\mathrm{L}\{U\}-s^{n-1}U(0)-s^{n-2}U'(0)-\cdots-U^{(n-1)}(0)=\mathrm{L}\{u_0-pu_0+p\left[N(U)-f(r)\right]\} \tag{13}$$

or

$$\mathrm{L}\{U\}=\frac{1}{s^n}\left\{s^{n-1}U(0)+s^{n-2}U'(0)+\cdots+U^{(n-1)}(0)+\mathrm{L}\{u_0-pu_0+p\left[N(U)-f(r)\right]\}\right\} \tag{14}$$

Finally, applying the inverse Laplace transform method to both sides of (14), one can successfully reach the following;

$$U = \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left\{s^{n-1}U(0)+s^{n-2}U'(0)+\cdots+U^{(n-1)}(0)+\mathrm{L}\left\{u_0-pu_0+p\left[N(U)-f(r)\right]\right\}\right\}\right\} \tag{15}$$

According to the HPM, the embedding parameter p as a small parameter can be used first, and the solutions of (15) can be represented as a power series in p as;

$$U(x)=\sum_{n=0}^{\infty}p^{n}U_{n}. \tag{16}$$

Eq. (15) can be rewritten using Eq. (16) as;

$$\sum_{n=0}^{\infty}p^{n}U_{n} = \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left\{s^{n-1}U(0)+s^{n-2}U'(0)+\cdots+U^{(n-1)}(0)+\mathrm{L}\left\{u_0-pu_0+p\left[N(\sum_{n=0}^{\infty}p^{n}U_{n})-f(r)\right]\right\}\right\}\right\} \tag{17}$$

Therefore, equating the coefficients of p with the same power leads to;

$$\begin{aligned}
p^{0}: U_0 &= \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left(s^{n-1}U(0)+s^{n-2}U'(0)+\cdots+U^{(n-1)}(0)+\mathrm{L}\{u_0\}\right)\right\},\\
p^{1}: U_1 &= \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left(\mathrm{L}\{N(U_0)-u_0-f(r)\}\right)\right\},\\
p^{2}: U_2 &= \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left(\mathrm{L}\{N(U_0,U_1)\}\right)\right\},\\
p^{3}: U_3 &= \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left(\mathrm{L}\{N(U_0,U_1,U_2)\}\right)\right\},\\
&\vdots\\
p^{j}: U_j &= \mathrm{L}^{-1}\left\{\frac{1}{s^{n}}\left(\mathrm{L}\{N(U_0,U_1,U_2,\ldots,U_{j-1})\}\right)\right\},\\
&\vdots
\end{aligned} \tag{18}$$

Supposing that the initial approximation has the form $U(0)=u_0=\alpha_0, U'(0)=\alpha_1,\ldots,U^{(n-1)}(0)=\alpha_{n-1}$, the exact solution may be obtained as the following;

$$u=\lim_{p\to 1}U = U_0+U_1+U_2+\cdots \tag{19}$$

To show the capability of the method, the NHPM is applied to some examples in the next section.

Results and discussion

To solve Eq. (1) with initial condition (2), according to the LTNHPM, the following homotopy is constructed.

$$\begin{cases}(1-p)\left(\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial U}{\partial t}-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)\right)=0,\\(1-p)\left(\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}\right)+p\left(\frac{\partial V}{\partial t}-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\right)=0,\end{cases}\tag{20}$$

or

$$\begin{cases}H(U,p)=\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}+p\left(-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)=0,\\H(V,p)=\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}+p\left(-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)=0,\end{cases}\tag{21}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of the solution of the system. Clearly, from Eq. (21), we obtain;

$$\begin{cases}H(U,0)=\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}=0,\\H(V,0)=\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}=0,\end{cases}\tag{22}$$

$$\begin{cases}H(U,1)=-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}=0,\\H(V,1)=-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}=0,\end{cases}\tag{23}$$

By applying the Laplace transform method on both sides of (21);

$$\begin{cases}\mathrm{L}\{H(U,p)\}=\mathrm{L}\left\{\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}+p\left(-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\},\\\mathrm{L}\{H(V,p)\}=\mathrm{L}\left\{\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}+p\left(-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\},\end{cases}\tag{24}$$

Using the differential property of the Laplace transform method;

$$\begin{aligned}&s\mathrm{L}\{U(x,y,t)\}-U(x,y,0)=\mathrm{L}\left\{\frac{\partial u_0}{\partial t}-p\left(-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\},\\&s\mathrm{L}\{V(x,y,t)\}-V(x,y,0)=\mathrm{L}\left\{\frac{\partial v_0}{\partial t}-p\left(-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\},\end{aligned}\tag{25}$$

or

$$\begin{aligned} &\mathrm{L}\{U(x,y,t)\}=\frac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\frac{\partial u_0}{\partial t}-p\left(-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\}\right),\\ &\mathrm{L}\{V(x,y,t)\}=\frac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\frac{\partial v_0}{\partial t}-p\left(-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\}\right), \end{aligned} \tag{26}$$

By applying the inverse Laplace transform method to both sides of (26);

$$\begin{aligned} &U(x,y,t)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\frac{\partial u_0}{\partial t}-p\left(-B-U^2V+(A+1)U-\alpha\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\}\right)\right\},\\ &V(x,y,t)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\frac{\partial v_0}{\partial t}-p\left(-AU+U^2V-\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\}\right)\right\}, \end{aligned} \tag{27}$$

According to the HPM, the embedding parameter p is used as a small parameter, and it is assumed that the solutions of Eq. (27) can be represented as a power series in p as;

$$\begin{aligned} &U(x,y,t)=\sum_{n=0}^{\infty}p^n U_n(x,y,t)\\ &V(x,y,t)=\sum_{n=0}^{\infty}p^n V_n(x,y,t) \end{aligned} \tag{28}$$

Substituting Eq. (28) into Eq. (27), and equating the terms with the identical powers of p, leads to the calculations $U_j(x,y,t), V_j(x,y,t),\ j=0,1,2,\ldots$

$$\begin{aligned} &p^0:\begin{cases} U_0(x,y,t)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\frac{\partial u_0}{\partial t}\right\}\right)\right\}\\ V_0(x,y,t)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\frac{\partial v_0}{\partial t}\right\}\right)\right\} \end{cases}\\ &p^1:\begin{cases} U_1(x,y,t)=\mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-B-U_0^2V_0+(A+1)U_0-\alpha\left(\frac{\partial^2 U_0}{\partial x^2}+\frac{\partial^2 U_0}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right\}\right\}\\ V_1(x,y,t)=\mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-AU_0+U_0^2V_0-\alpha\left(\frac{\partial^2 V_0}{\partial x^2}+\frac{\partial^2 V_0}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right\}\right\} \end{cases}\\ &p^2:\begin{cases} U_2(x,y,t)=\mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_0^2V_1+2U_0U_1V_0)+(A+1)U_1-\alpha\left(\frac{\partial^2 U_1}{\partial x^2}+\frac{\partial^2 U_1}{\partial y^2}\right)\right\}\right\}\\ V_2(x,y,t)=\mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-AU_1+(U_0^2V_1+2U_0U_1V_0)-\alpha\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}\right)\right\}\right\} \end{cases}\\ &\vdots \end{aligned} \tag{29}$$

$$p^j:\begin{cases} U_j(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}V_iU_kU_{j-i-k-1})+(A+1)U_{j-1}-\alpha\left(\dfrac{\partial^2 U_{j-1}}{\partial x^2}+\dfrac{\partial^2 U_{j-1}}{\partial y^2}\right)\right\}\right\} \\ V_j(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}V_iU_kU_{j-i-k-1})-AU_{j-1}-\alpha\left(\dfrac{\partial^2 V_{j-1}}{\partial x^2}+\dfrac{\partial^2 V_{j-1}}{\partial y^2}\right)\right\}\right\} \end{cases}$$

$\vdots$

For simplicity,

$$u_0=U_0=f(x,y)+Bt, v_0=V_0=g(x,y). \tag{30}$$

The exact or approximate solution of Eq. (20) can be obtained by setting $p=1$,

$$\begin{aligned} u&=\lim_{p\to1}U=U_0+pU_1+p^2U_2+\cdots \\ v&=\lim_{p\to1}V=V_0+pV_1+p^2V_2+\cdots \end{aligned} \tag{31}$$

Examples

To show the efficiency and ability of the proposed method, 2 examples are presented in this section.
Example 1. Consider the two-dimensional Brusselator system [38];

$$\begin{aligned} \frac{\partial u}{\partial t}&=u^2v-2u+\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t}&=u-u^2v+\frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \end{aligned} \tag{32}$$

In region $R=\{(x,y):x^2+y^2<1, x>0, y>0\}$ subject to the initial and boundary conditions;

$$\begin{aligned} &(u(x,y,0),v(x,y,0))=e^{-x-y},e^{x+y})\, for\ (x,y)\in R \\ &(u(0,y,t),v(0,y,t))=(e^{-\frac{t}{2}-y},e^{\frac{t}{2}+y})\, for\ 0<y<1\ and\ t>0, \\ &(u(x,0,t),v(x,0,t))=(e^{-\frac{t}{2}-x},e^{\frac{t}{2}+x})\, for\ 0<x<1\ and\ t>0, \\ &\left(\frac{\partial u}{\partial n},\frac{\partial v}{\partial n}\right)=(x+y)\left(-e^{-\frac{t}{2}-x-y},e^{\frac{t}{2}+x+y}\right) for\ x^2+y^2=1\ and\ t>0. \end{aligned} \tag{33}$$

To solve Eq. (32) by the LTNHPM, the following homotopy is constructed:

$$\begin{cases} (1-p)\left(\dfrac{\partial U}{\partial t}-\dfrac{\partial u_0}{\partial t}\right)+p\left(\dfrac{\partial U}{\partial t}-U^2V+2U-\dfrac{1}{4}\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)\right)=0, \\ (1-p)\left(\dfrac{\partial V}{\partial t}-\dfrac{\partial v_0}{\partial t}\right)+p\left(\dfrac{\partial V}{\partial t}-U+U^2V-\dfrac{1}{4}\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)\right)=0, \end{cases} \tag{34}$$

or

$$\begin{cases} H(U,p)=\dfrac{\partial U}{\partial t}-\dfrac{\partial u_0}{\partial t}+p\left(-U^2V+2U-\dfrac{1}{4}\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)+\dfrac{\partial u_0}{\partial t}\right)=0, \\ H(V,p)=\dfrac{\partial V}{\partial t}-\dfrac{\partial v_0}{\partial t}+p\left(-U+U^2V-\dfrac{1}{4}\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)+\dfrac{\partial v_0}{\partial t}\right)=0, \end{cases} \tag{35}$$

Applying the Laplace transform method to both sides of Eq. (35);

$$\begin{cases} \mathrm{L}\{H(U,p)\}=\mathrm{L}\left\{\dfrac{\partial U}{\partial t}-\dfrac{\partial u_0}{\partial t}+p\left(-U^2V+2U-\dfrac{1}{4}\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)+\dfrac{\partial u_0}{\partial t}\right)\right\}, \\ \mathrm{L}\{H(V,p)\}=\mathrm{L}\left\{\dfrac{\partial V}{\partial t}-\dfrac{\partial v_0}{\partial t}+p\left(-U+U^2V-\dfrac{1}{4}\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)+\dfrac{\partial v_0}{\partial t}\right)\right\}, \end{cases} \tag{36}$$

Using the differential property of the Laplace transform method;

$$\begin{cases} s\mathrm{L}\{U(x,y,t)\}-U(x,y,0)=\mathrm{L}\left\{\dfrac{\partial u_0}{\partial t}-p\left(-U^2V+2U-\dfrac{1}{4}\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)+\dfrac{\partial u_0}{\partial t}\right)\right\}, \\ s\mathrm{L}\{V(x,y,t)\}-V(x,y,0)=\mathrm{L}\left\{\dfrac{\partial v_0}{\partial t}-p\left(-U+U^2V-\dfrac{1}{4}\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)+\dfrac{\partial v_0}{\partial t}\right)\right\}, \end{cases} \tag{37}$$

or

$$\begin{cases} \mathrm{L}\{U(x,y,t)\}=\dfrac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\dfrac{\partial u_0}{\partial t}-p\left(-B-U^2V+2U-\dfrac{1}{4}\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)+\dfrac{\partial u_0}{\partial t}\right)\right\}\right), \\ \mathrm{L}\{V(x,y,t)\}=\dfrac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\dfrac{\partial v_0}{\partial t}-p\left(-U+U^2V-\dfrac{1}{4}\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)+\dfrac{\partial v_0}{\partial t}\right)\right\}\right), \end{cases} \tag{38}$$

By applying the inverse Laplace transform method to both sides of Eq. (38);

$$\begin{cases} U(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\dfrac{\partial u_0}{\partial t}-p\left(-U^2V+2U-\dfrac{1}{4}\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)+\dfrac{\partial u_0}{\partial t}\right)\right\}\right)\right\}, \\ V(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\dfrac{\partial v_0}{\partial t}-p\left(-U+U^2V-\dfrac{1}{4}\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)+\dfrac{\partial v_0}{\partial t}\right)\right\}\right)\right\}, \end{cases} \tag{39}$$

Supposing the solution of Eq. (39) has the following form.

$$U(x,y,t)=\sum_{n=0}^{\infty}p^{n}U_{n}\left(x,y,t\right)$$
$$V(x,y,t)=\sum_{n=0}^{\infty}p^{n}V_{n}\left(x,y,t\right) \tag{40}$$

where $U_n(x,y,t), V_n(x,y,t)$ are unknown functions which should be determined. Substituting Eq. (40) into Eq. (39), and equating the terms with the identical powers of p, leads to the calculations $U_j(x,y,t), V_j(x,y,t),\ j=0,1,2,\ldots$

$$p^0:\begin{cases}U_0(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\dfrac{\partial u_0}{\partial t}\right\}\right)\right\}\\ V_0(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\dfrac{\partial v_0}{\partial t}\right\}\right)\right\}\end{cases}$$

$$p^1:\begin{cases}U_1(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-U_0^2V_0+2U_0-\dfrac{1}{4}\left(\dfrac{\partial^2U_0}{\partial x^2}+\dfrac{\partial^2U_0}{\partial y^2}\right)+\dfrac{\partial u_0}{\partial t}\right\}\right\}\\ V_1(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-U_0+U_0^2V_0-\dfrac{1}{4}\left(\dfrac{\partial^2V_0}{\partial x^2}+\dfrac{\partial^2V_0}{\partial y^2}\right)+\dfrac{\partial v_0}{\partial t}\right\}\right\}\end{cases}$$

$$p^2:\begin{cases}U_2(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-(U_0^2V_1+2U_0U_1V_0)+2U_1-\dfrac{1}{4}\left(\dfrac{\partial^2U_1}{\partial x^2}+\dfrac{\partial^2U_1}{\partial y^2}\right)\right\}\right\}\\ V_2(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-U_1+(U_0^2V_1+2U_0U_1V_0)-\dfrac{1}{4}\left(\dfrac{\partial^2V_1}{\partial x^2}+\dfrac{\partial^2V_1}{\partial y^2}\right)\right\}\right\}\end{cases}$$

$$\vdots \tag{41}$$

$$p^j:\begin{cases}U_j(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}V_iU_kU_{j-i-k-1})+2U_{j-1}-\alpha\left(\dfrac{\partial^2U_{j-1}}{\partial x^2}+\dfrac{\partial^2U_{j-1}}{\partial y^2}\right)\right\}\right\}\\ V_j(x,y,t)=\mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}V_iU_kU_{j-i-k-1})-U_{j-1}+-\dfrac{1}{4}\left(\dfrac{\partial^2V_{j-1}}{\partial x^2}+\dfrac{\partial^2V_{j-1}}{\partial y^2}\right)\right\}\right\}\end{cases}$$

$$\vdots$$

Assuming $u_0(x,y,t)=e^{-x-y}, v_0(x,y,t)=e^{x+y}$, and solving the above equation for $U_j(x,y,t)$, $V_j(x,y,t), j=0,1,2,\ldots$ leads to the result.

$$\begin{aligned}
U_0(x,y,t) &= e^{-x-y}\left(1+t\right),\\
U_1(x,y,t) &= \frac{1}{4}e^{-x-y}\left(-6t+3t^2+4t^3+t^4\right),\\
U_2(x,y,t) &= \frac{1}{280}e^{-x-y}t^2\left(-595t+140t^4+35t^2+329t^3+20t^5\right)\\
&\quad -175-\frac{1}{420}e^{-3y-3x}t^3\left(140+105t^3+315t+273t^2+15t^4\right),\\
&\vdots
\end{aligned} \tag{42}$$

$$\begin{aligned}
V_0(x,y,t) &= e^{x+y}\left(1+t\right),\\
V_1(x,y,t) &= \frac{-1}{4}e^{-x-y}t^2\left(t+2\right)^2+\frac{1}{4}e^{x+y}t\left(t-2\right),\\
V_2(x,y,t) &= \frac{1}{420}e^{-3y-3x}t^3\left(140+105t^3+315t+273t^2+15t^4\right)\\
&-\frac{1}{840}t^2\left(\left(-1540t+420t^4+315t^2+1029t^3+60t^5-840\right)e^{-x-y}-35\left(t-3\right)e^{x+y}\right),\\
&\vdots
\end{aligned} \tag{43}$$

Therefore using algebra with the aid of a symbolic computation tool, the solution of Eq. (32) is determined as;

$$\begin{aligned}
u(x,y,t) &= U_0(x,y,t)+U_1(x,y,t)+U_2(x,y,t)+U_3(x,y,t)+\cdots\\
&= e^{-x-y}\left(1-\frac{1}{2}t+\frac{1}{8}t^2-\frac{1}{48}t^3+\frac{1}{384}t^4-\frac{1}{3840}t^5+\cdots\right)\\
&= e^{-x-y}\left(1-\frac{1}{2\times 1!}t+\frac{1}{2^2\times 2!}t^2-\frac{1}{2^3\times 3!}t^3+\frac{1}{2^4\times 4!}t^4-\frac{1}{2^5\times 5!}t^5+\cdots\right)\\
&= e^{-x-y-\frac{t}{2}}\\
v(x,y,t) &= V_0(x,y,t)+V_1(x,y,t)+V_2(x,y,t)+V_3(x,y,t)+\cdots\\
&= e^{x+y}\left(1+\frac{1}{2}t+\frac{1}{8}t^2+\frac{1}{48}t^3+\frac{1}{384}t^4+\frac{1}{3840}t^5+\cdots\right)\\
&= e^{x+y}\left(1+\frac{1}{2\times 1!}t+\frac{1}{2^2\times 2!}t^2+\frac{1}{2^3\times 3!}t^3+\frac{1}{2^4\times 4!}t^4+\frac{1}{2^5\times 5!}t^5+\cdots\right)\\
&= e^{x+y+\frac{t}{2}}
\end{aligned} \tag{44}$$

which is the exact solution of problem.

Example 2. Consider the two-dimensional Brusselator system [42];

$$\begin{aligned}
\frac{\partial u}{\partial t} &= 1+u^2v-\frac{22}{5}u+\frac{1}{500}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\
\frac{\partial v}{\partial t} &= \frac{17}{5}u-u^2v+\frac{1}{500}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).
\end{aligned} \tag{45}$$

subject to the initial conditions of;

$$(u(x,y,0),v(x,y,0))=(2+\frac{y}{4},1+\frac{4x}{5}) \tag{46}$$

To solve Eq. (45) by the LTNHPM, the following homotopy is constructed;

$$\begin{cases}(1-p)\left(\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial U}{\partial t}-U^2V+\frac{22}{5}U-\frac{1}{500}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)-1\right)=0,\\(1-p)\left(\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}\right)+p\left(\frac{\partial V}{\partial t}-\frac{17}{5}U+U^2V-\frac{1}{500}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\right)=0,\end{cases} \tag{47}$$

or

$$\begin{cases}H(U,p)=\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}+p\left(-1-U^2V+\frac{22}{5}U-\frac{1}{500}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)=0,\\H(V,p)=\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}+p\left(-\frac{17}{5}U+U^2V-\frac{1}{500}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)=0,\end{cases} \tag{48}$$

applying the Laplace transform method to both sides of Eq. (48);

$$\begin{cases}\mathrm{L}\{H(U,p)\}=\mathrm{L}\left\{\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}+p\left(-1-U^2V+\frac{22}{5}U-\frac{1}{500}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\},\\\mathrm{L}\{H(V,p)\}=\mathrm{L}\left\{\frac{\partial V}{\partial t}-\frac{\partial v_0}{\partial t}+p\left(-\frac{17}{5}U+U^2V-\frac{1}{500}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\},\end{cases} \tag{49}$$

Using the differential property of the Laplace transform method;

$$\begin{cases}s\mathrm{L}\{U(x,y,t)\}-U(x,y,0)=\mathrm{L}\left\{\frac{\partial u_0}{\partial t}-p\left(-1-U^2V+\frac{22}{5}U-\frac{1}{500}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\},\\s\mathrm{L}\{V(x,y,t)\}-V(x,y,0)=\mathrm{L}\left\{\frac{\partial v_0}{\partial t}-p\left(-\frac{17}{5}U+U^2V-\frac{1}{500}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\},\end{cases} \tag{50}$$

or

$$\begin{cases}\mathrm{L}\{U(x,y,t)\}=\frac{1}{s}\left(U(x,y,0)+\mathrm{L}\left\{\frac{\partial u_0}{\partial t}-p\left(-1-U^2V+\frac{22}{5}U-\frac{1}{500}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+\frac{\partial u_0}{\partial t}\right)\right\}\right),\\\mathrm{L}\{V(x,y,t)\}=\frac{1}{s}\left(V(x,y,0)+\mathrm{L}\left\{\frac{\partial v_0}{\partial t}-p\left(-\frac{17}{5}U+U^2V-\frac{1}{500}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\frac{\partial v_0}{\partial t}\right)\right\}\right),\end{cases} \tag{51}$$

By applying the inverse Laplace transform method to both sides of Eq. (51);

$$\begin{cases} U(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(U(x,y,0) + \mathrm{L}\left\{\dfrac{\partial u_0}{\partial t} - p\left(-1 - U^2V + \dfrac{22}{5}U - \dfrac{1}{500}\left(\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}\right) + \dfrac{\partial u_0}{\partial t}\right)\right\}\right)\right\}, \\ V(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(V(x,y,0) + \mathrm{L}\left\{\dfrac{\partial v_0}{\partial t} - p\left(-\dfrac{17}{5}U + U^2V - \dfrac{1}{500}\left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}\right) + \dfrac{\partial v_0}{\partial t}\right)\right\}\right)\right\}, \end{cases} \tag{52}$$

Supposing the solution of Eq. (52) has the following form;

$$\begin{aligned} U(x,y,t) &= \sum_{n=0}^{\infty} p^n U_n(x,y,t) \\ V(x,y,t) &= \sum_{n=0}^{\infty} p^n V_n(x,y,t) \end{aligned} \tag{53}$$

where $U_n(x,y,t), V_n(x,y,t)$ are unknown functions which should be determined. Substituting Eq. (53) into Eq. (52), and equating the terms with the identical powers of p, leads to the calculations $U_j(x,y,t), V_j(x,y,t), j = 0,1,2,\ldots$

$$p^0: \begin{cases} U_0(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(U(x,y,0) + \mathrm{L}\left\{\dfrac{\partial u_0}{\partial t}\right\}\right)\right\} \\ V_0(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{1}{s}\left(V(x,y,0) + \mathrm{L}\left\{\dfrac{\partial v_0}{\partial t}\right\}\right)\right\} \end{cases}$$

$$p^1: \begin{cases} U_1(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-1 - U_0^2V_0 + \dfrac{22}{5}U_0 - \dfrac{1}{500}\left(\dfrac{\partial^2 U_0}{\partial x^2} + \dfrac{\partial^2 U_0}{\partial y^2}\right) + \dfrac{\partial u_0}{\partial t}\right\}\right\} \\ V_1(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-\dfrac{17}{5}U_0 + U_0^2V_0 - \dfrac{1}{500}\left(\dfrac{\partial^2 V_0}{\partial x^2} + \dfrac{\partial^2 V_0}{\partial y^2}\right) + \dfrac{\partial v_0}{\partial t}\right\}\right\} \end{cases}$$

$$p^2: \begin{cases} U_2(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-(U_0^2V_1 + 2U_0U_1V_0) + \dfrac{22}{5}U_1 - \dfrac{1}{500}\left(\dfrac{\partial^2 U_1}{\partial x^2} + \dfrac{\partial^2 U_1}{\partial y^2}\right)\right\}\right\} \\ V_2(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-\dfrac{17}{5}U_1 + (U_0^2V_1 + 2U_0U_1V_0) - \dfrac{1}{500}\left(\dfrac{\partial^2 V_1}{\partial x^2} + \dfrac{\partial^2 V_1}{\partial y^2}\right)\right\}\right\} \end{cases}$$

$$\vdots$$

$$p^j: \begin{cases} U_j(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-\left(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} V_i U_k U_{j-i-k-1}\right) + \dfrac{22}{5}U_{j-1} - \dfrac{1}{500}\left(\dfrac{\partial^2 U_{j-1}}{\partial x^2} + \dfrac{\partial^2 U_{j-1}}{\partial y^2}\right)\right\}\right\} \\ V_j(x,y,t) = \mathrm{L}^{-1}\left\{\dfrac{-1}{s}\mathrm{L}\left\{-\dfrac{17}{5}U_{j-1} + \left(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} V_i U_k U_{j-i-k-1}\right) - \dfrac{1}{500}\left(\dfrac{\partial^2 V_{j-1}}{\partial x^2} + \dfrac{\partial^2 V_{j-1}}{\partial y^2}\right)\right\}\right\} \end{cases} \tag{54}$$

$$\vdots$$

Assuming $u_0(x,y,t)=2+\frac{y}{4}, v_0(x,y,t)=1+\frac{4x}{5}$, and solving the above equation for $U_j(x,y,t),\ U_j(x,y,t),\ j=0,1,2,\dots$ leads to the result.

$$\begin{aligned}
U_0(x,y,t) &= \frac{1}{4}(8+y)(1+t),\\
U_1(x,y,t) &= \frac{1}{320}t\,\big(256xt^2y+24txy^2+1536tx+320t^3+1280t^2\\
&\quad +1024xt^2+256xt^3+20y^2+30ty^2+320t^2y+20t^2y^2+80t^3y\\
&\quad +5t^3y^2+256xy+16xy^2-1856+64xt^3y+4xt^3y^2+512t\\
&\quad -112y+1024x+16xt^2y^2+384txy+304ty\big),\\
&\vdots
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
V_0(x,y,t) &= \frac{1}{5}(5+4x)(1+t),\\
V_1(x,y,t) &= -\frac{1}{320}t\,\big(256xt^2y+24txy^2-576+1536tx+320t^3+1280t^2\\
&\quad +1024xt^2+256xt^3+20y^2+30ty^2+320t^2y+20t^2y^2+80t^3y\\
&\quad +5t^3y^2+256xy+16xy^2+64xt^3y+4xt^3y^2+832t+48y\\
&\quad +1280x+16xt^2y^2+384txy+344ty\big),\\
&\vdots
\end{aligned} \tag{56}$$

Using algebra with the aid of a symbolic computation tool, to make a direct comparison with [40,42], $x=y=0.1$ is considered. The 6-term LTNHPM solutions to the Brusselator model for this case is given by;

$$\begin{aligned}
u(x,y,t) &= U_0(x,y,t)+U_1(x,y,t)+U_2(x,y,t)+U_3+(x,y,t)+\cdots\\
&= 2.025-3.4813t+5.0816t^2-11.7847t^3+32.7015t^4-92.347t^5\\
&\quad +266.6238t^6-749.2341t^7+283.5752t^8+2466.1745t^9+O(t^{10})\\
v(x,y,t) &= V_0(x,y,t)+V_1(x,y,t)+V_2(x,y,t)+V_3+(x,y,t)+\cdots\\
&= 1.08+2.4563t-3.3407t^2+10.1045t^3-29.8288t^4+86.118t^5\\
&\quad -252.4259t^6+710.6276t^7-285.0708t^8-2391.8422t^9+O(t^{10})
\end{aligned} \tag{57}$$

The 3-term HPM solutions are shown in **Figures 1 - 3**.

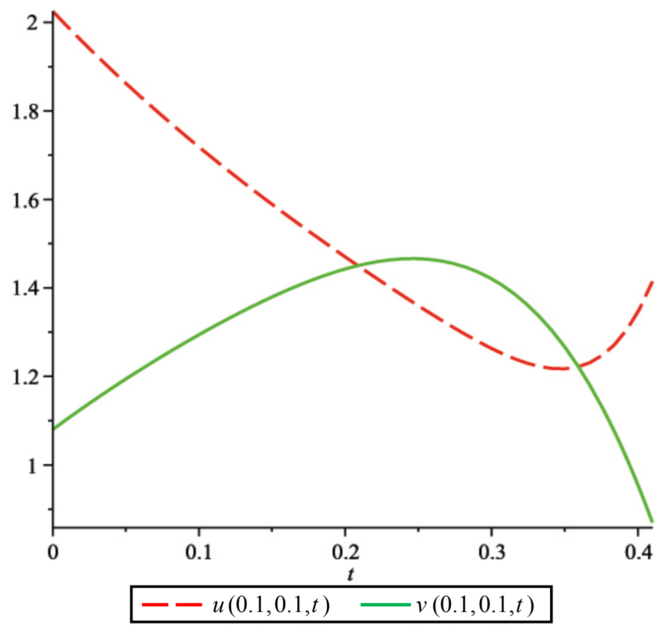

Figure 1 Numerical results for $u(0.1,0.1,t)$ and $v(0.1,0.1,t)$.

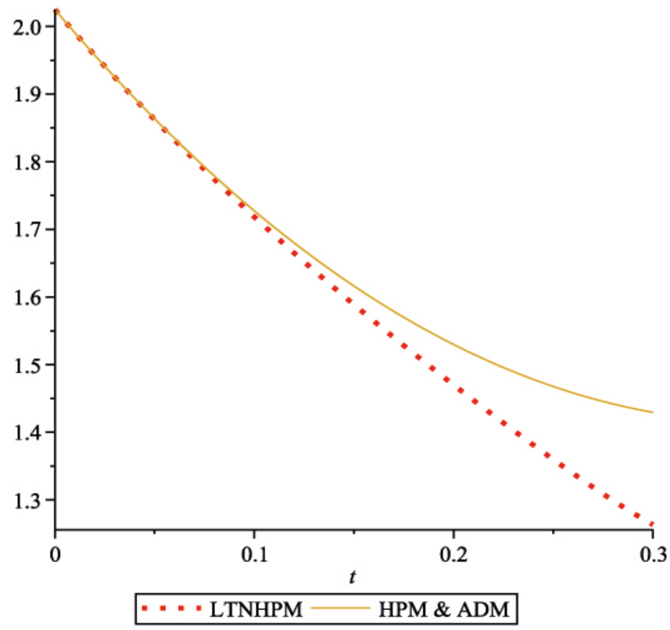

Figure 2 Comparison of numerical results of $u(0.1,0.1,t)$ with ADM ([40]) and HPM ([42]).

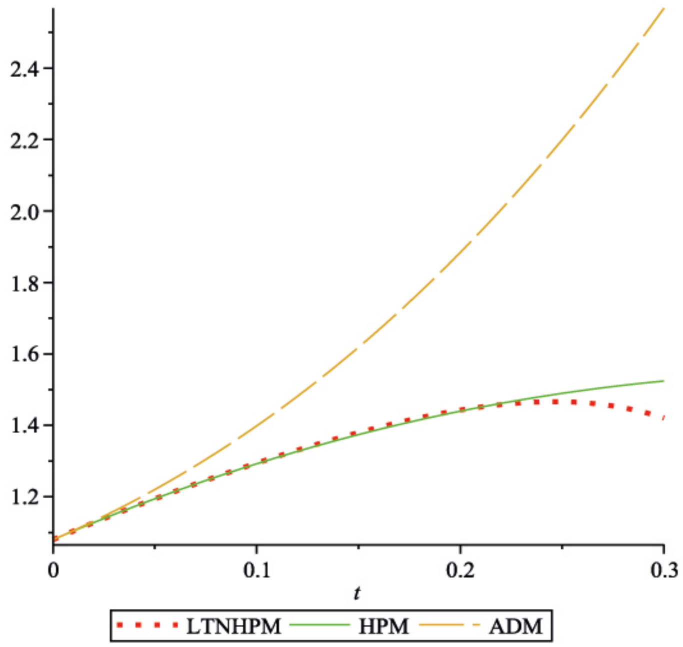

Figure 3 Comparison of numerical results of $v(0.1, 0.1, t)$ with ADM ([40]) and HPM ([42]).

Conclusion

Mathematical physics and population growth models characterized by systems of partial differential equations, such as the Brusselator model, are of wide applicability. In the present work, a combination of the Laplace transform method and the homotopy perturbation method is proposed to solve the Brusselator equation. This method, unlike most numerical techniques, provides a closed form of the solution. By using this method, a new efficient recurrence relation to solve nonlinear Brusselator equations is obtained. The results show that the LTNHPM is a powerful mathematical tool for solving systems of nonlinear partial differential equations, having wide applications in sciences and engineering. Compared with the Adomian decomposition method and the homotopy perturbation method, the present method does not require specific algorithms or complex calculations, such as ADM. In comparison with the boundary element method, computational size has been reduced and rapid convergence has been guaranteed.

Acknowledgements

The authors thank the reviewers of this paper for their insightful comments.

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