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On the Homotopy Asymptotic Method of Quantum Zakharov-Kuznetsov Equation in Ion Acoustic Waves

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Abstract

Here, we investigate the effectiveness of the optimal homotopy asymptotic method (OHAM) with a symbolic computational method for constructing the approximate solution for quantum Zakharov-Kuznetsov equation that is derived to describe the in magnetized plasma in ion acoustic waves. The results reveal that the method is explicit, effective and easy to use. The proposed method is a strong and easy to use analytic tool for nonlinear problems and does not need small parameters in the equations. The results obtained here reveal that the proposed method is very effective and simple for solving nonlinear problems. The basic ideas of this approach can be widely employed to solve other strongly nonlinear evaluation form equations arising in physics.

Keywords: Optimal homotopy asymptotic method, quantum Zakharov-Kuznetsov equation, approximate solution

Introduction

Analytical methods have made a comeback in research methodology after taking a backseat to the numerical techniques for the latter half of the preceding century. The advantages of analytical methods are manifold, the main being that they give a much better insight than the numbers crunched by a computer using a purely numerical algorithm. As a result, the research on exact solutions of nonlinear evolution equations has become more and more important [1-24].

Perturbation techniques [1,2] have come to be considered as classical in solving nonlinear problems, specifically those that contain small parameters and therefore valid only for weakly nonlinear problems. These techniques are very effective in computing solutions, but the small parameters assumption, greatly restricts their applications.

Liao [10,11] employed the basic ideas of the homotopy in topology to propose a general analytical method for nonlinear problems, namely, homotopy analysis method (HAM). Based on homotopy of topology, the validity of the HAM is independent of the existence of small parameters in the considered equation. In recent years, much attention has been devoted to the newly developed methods for constructing an analytic solution of an equation. Marinca and Herisanu [7,8] introduced a new method known as the optimal homotopy asymptotic method (OHAM). The advantage of OHAM is in the built in convergence criteria similar to HAM but more flexible. Marinca have applied this method successfully to obtain the solution of currently important problems in science, and have also shown its effectiveness, generalization and reliability [7,8].

The investigation of ion-acoustic waves and structures in dense quantum plasmas has attracted much attention in recent years. It was shown that quantum effects play a crucial role in plasma dynamics when the de-Broglie wavelength of the charge carriers becomes comparable to the spatial scale of the system. In dense quantum plasma, the quantum hydrodynamic (QHD) model is one of the most popular models. The QHD model is a generalization of classical fluid model of plasmas where QHD transport equations are expressed in terms of the conservation laws of particles momentum and energy.

The rest of this paper is arranged as follows. In section 2, we simply provide the mathematical framework of the OHAM. In section 3, the governed equations of the plasma system is given and transformed into the quantum Zakharov-Kuznetsov (QZK) equation to illustrate the effectiveness and convenience of the proposed method. Finally, conclusions are given in section 4.

Methodology

In what follows, we summarize the OHAM [7,8]. For a given a nonlinear equations as;

$$L(u(x,t)) + N(u(x,t)) + g(x,t) = 0, \; x \in \Omega \qquad B\left(u, \frac{\partial u}{\partial t}\right) = 0, \tag{1}$$

where L is a linear operator and N is a nonlinear operator, B is boundary operator, $u(x,t)$ is an unknown function, and x and t denote spatial and time variables.

In view of OHAM, one can construct the optimal homotopy $\psi(x,t;q): \Omega \in [0,1] \to R$

$$(1-p)[L(\psi(x,t;q)) + g(x,t)] = H(q)[L(\psi(x,t;q)) + N(u(x,t;q)) + g(x,t)], \tag{2}$$

where $q \in [0,1]$ is an embedding parameter, $H(q)$ is a nonzero auxiliary function for $q \neq 0$, $H(0) = 0$. Eq. (2) called the optimal homotopy equation admits to;

$$L(\psi(x,t;q)) + g(x,t) = 0, \; q = 0 \qquad [L(\psi(x,t;q)) + N(\psi(x,t;q)) + g(x,t) = 0, \; q = 1 \tag{3}$$

As long as $q = 0$ and $q = 1$ it holds that $\psi(x,t;0) = u_0(x,t)$ and $\psi(x,t;1) = u(x,t)$. Then, as q varies from 0 to 1, the solution $\psi(x,t;q)$ approaches from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is obtained from Eq. (2) for $q = 0$;

$$L(u(x,t)) + g(x,t) = 0, \; B\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0 \tag{4}$$

The auxiliary function $H(x,q)$ can be expressed as;

$$H(x,q) = qC_1(x) + q^2 C_2(x) + \ldots\ldots\ldots\ldots + q^m C_m(x) \tag{5}$$

In view the proposed method, to get an approximate solution, we expand $\psi(x,t;q,C_i)$ in Taylor's series about q as;

$$\psi(x,t;q,C_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t;C_i) q^k, \; i = 1,2,\ldots\ldots \tag{6}$$

Inserting (6) into (2) and equating the coefficients of same power of q, we obtain governing equations for $u_k(x,t)$ as;

$$L(u_1(x,t)) = C_1 N_0(u_0(x,t)); B(u_0, \frac{\partial u_0}{\partial t}) = 0, \tag{7}$$

$$L(u_2(x,t)) - L(u_1(x,t)) = C_2 N_0(u_0(x,t)) + C_1[L(u_1(x,t)) + N_1(u_0(x,t), u_1(x,t))], B(u_2, \frac{\partial u_2}{\partial t}) = 0, \tag{8}$$

$$L(u_k(x,t)) - L(u_{k-1}(x,t)) = C_k N_0(u_0(x,t)) + \sum_{i=0}^{k-1} C_i [L(u_{k-i}(x,t)) + N_{k-i}(u_0(x,t), u_{k-i}(x,t)],$$
$$B(u_k, \frac{\partial u_k}{\partial t}) = 0,\ k = 2,3,...... \tag{9}$$

where $N_{k-i}(u_0(x,t), u_1(x,t)), ..., u_{k-i}(x,t)$ is the coefficients of q^{k-i} in the expansion of $N(\psi(x,t;q)$ as;

$$N(\psi(x,t;q) = N_0(u_0(x,t) + \sum_{k>1} N_k(u_0, u_1, ..., u_k) q^k \tag{10}$$

The convergence of the series in (6) depends upon the auxiliary constants C_i. If it is convergent at $q = 1$, one has;

$$u^*(x,t,C_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t,C_i) \tag{11}$$

Substituting Eq. (11) into (1), we have the residual *R*;

$$R(x,t,C_i) = L(u^*(x,t,C_i)) + g(x,t) + N^*(x,t,C_i) \tag{12}$$

To compute the auxiliary constants $C_i, i = 1,2,..,.m$, there are many methods such as Galerkin's Method, Ritz Method, Least Squares Method and Collocation method to obtain the values of C_i. By applying the method of least squares as;

$$J(C_i) = \int_0^t \int_{\Omega} R^2(x,t,C_i) dx dt, \tag{13}$$

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = ... = \frac{\partial J}{\partial C_m} = 0 \tag{14}$$

Governed equations of the problem

The importance of quantum effects in ultra-small electronic devices, in dense astrophysical plasma systems and in laser plasma have produced interest on the investigation of the quantum counterpart of some of the classical plasma physics phenomena. For instance, quantum plasma has attracted much attention as waves in collisionless unmagnetized quantum plasma have potential applications in different scientific areas either in the laboratory or in astrophysics.

The nonlinear propagation of the electrostatic waves, in a dense Thomas-Fermi magneto-plasma whose constituents are the electrons and singly charged ions confined in an external magnetic field of strength B_0 along the *x*-axis, is governed by the dimensionless ion continuity and momentum equations represented by;

$$\frac{\partial n_i}{\partial t}+\frac{\partial (n_i u_{ix})}{\partial x}+\frac{\partial (n_i u_{iy})}{\partial x}=0, \tag{15}$$

$$\frac{\partial u_{ix}}{\partial t}+[u_{ix}\frac{\partial}{\partial x}+u_{iy}\frac{\partial}{\partial y}]\,u_{ix}+\frac{3}{2}\frac{\partial \phi}{\partial x}=0, \tag{16}$$

$$\frac{\partial u_{iy}}{\partial t}+[u_{ix}\frac{\partial}{\partial x}+u_{iy}\frac{\partial}{\partial y}]\,u_{iy}+\frac{3}{2}\frac{\partial \phi}{\partial y}-u_{iz}=0, \tag{17}$$

$$\frac{\partial u_{iz}}{\partial t}+[u_{ix}\frac{\partial}{\partial x}+u_{iy}\frac{\partial}{\partial y}]\,u_{iz}+u_{iy}=0, \tag{18}$$

with Tomas-Fermi law for degenerate electrons;

$$n_e=(1+\phi)^{\frac{3}{2}} \tag{19}$$

Eqs. (15) - (18) are closed by the Poisson's equation;

$$\Omega[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}]\phi=\frac{2}{3}(n_e-n_i) \tag{20}$$

Through these Eqs. (15) - (20), $n_i(x,y,t)$ and $n_e(x,y,t)$ are the number densities of the ion and electron species, respectively and are normalized by the unperturbed electron/ion number density $(n_{i0}=n_{e0}=2E_F/(3m_i)$ where $E_F=2k_B T_F$, k_B is the Boltzmann constant, T_F is the Fermi electron temperature and mi is the ion mass.

$\phi(x,y,t)$ is the electrostatic potential normalized by E_F/e where e is the magnitude of the electron charge. The time and space variables are in units of the ion gyro-frequency $\omega_{ci}=eB_0/(m_i c)$ and ion sound gyro-radius $\rho_s=C_{si}/\omega_{ci}$, respectively, where B_0 is the strength of the magnetic field taken along the *x*-axis and *c* is the speed of light in vacuum. Furthermore, $\Omega=(3c9_{ci}/3c9_{pi})^2$ where $\omega_{pi}=\sqrt{4\pi e^2 n_0/m_i}$ is the ion plasma frequency.

The reductive perturbation method introduces the stretching space-time coordinates;

$$X = 3b5^{1/2}(x - t), Y = 3b5^{1/2} y \;\; and \;\; T = 3b5^{3/2} t \tag{21}$$

where ε is a smallness parameter measuring the weakness of the amplitude or dispersion. The dependent variables $n_{i(e)}$, u_{ix} ;

$$\frac{\partial\phi}{\partial T} + A\phi\frac{\partial\phi}{\partial^3\phi}\partial X\,\partial Y = 0, \tag{22}$$

$$A = 2, B = \frac{1}{2}\Omega, C = \frac{1}{2}(1+\Omega)$$

Making use of the transformation;

$$u(\xi) = \varphi(X, Y, T), \xi = L_x X + L_y Y + vT, \tag{23}$$

where v is a constant speed, L_x and L_y are the directional cosine of the propagation wave vector along the X and Y axes, respectively, (22) reduces to;

$$\alpha\frac{du(\xi)}{d\xi} + \beta u(\xi)\frac{du(\xi)}{d\xi} + \frac{d^3u(\xi)}{d\xi} = 0, \tag{24}$$

with boundary and initial conditions as;

$$u(0) = 1, u(1) = 0, \frac{\partial u(0)}{d\xi} = 0 \tag{25}$$

$$\phi_1(\xi) = u(\xi), \xi = L_x X + L_y Y + vT, \; \alpha = \frac{v}{B_0}, \beta = \frac{A_0}{B_0}, \tag{26}$$

$$A_0 = AL_x, B_0 = BL_x^3 + CL_x L_y^2 \tag{27}$$

New application of OHAM to the quantum Zakharov-Kuznetsov equation

To solve the reduced Eq. (24) by the proposed method, we choose the auxiliary linear operators as;

$$L[\varphi(\xi;q)] = \frac{\partial^3\varphi(\xi;q)}{\partial\xi^3}, g(\xi) = 0 \tag{28}$$

$$N[\varphi(\xi;q)] = \alpha\frac{\partial\varphi(\xi;q)}{\partial\xi} + \beta\varphi(\xi;q)\frac{\partial\varphi(\xi;q)}{\partial\xi}, \tag{29}$$

According to the OHAM, with the aid of Eqs. (7) - (9), admits to the following;

Zeroth-order problems are given by;

$$\frac{\partial^3 u_0(\xi)}{\partial \xi^3} = 0, \tag{30}$$

$$u_0(0) = 1, u_0(1) = 0, \frac{\partial u_0(0)}{\partial \xi} = 0 \tag{31}$$

Its solutions as;

$$u_0(\xi) = 1 - \xi^2 \tag{32}$$

First-order problems are given by Eq. (7);

$$L(u_1(\xi)) = (1 + C_1)(L(u_0(\xi)) + C_1(\alpha u_{0\xi} + \beta u_0 u_{0\xi}), \tag{33}$$

$$u_1(0) = 1, u_1(1) = 0, \frac{\partial u_1(0)}{\partial \xi} = 0 \tag{34}$$

Its solutions as;

$$u_1(\xi) = \frac{C_1 \beta}{60}\xi^6 - \frac{2C_1}{24}(\beta + \alpha)\xi^4 + C_1(\frac{\beta}{15} + \frac{\alpha}{15})\xi^2 \tag{35}$$

Second-order problems are given by Eq. (8) for $m = 2$;

$$L(u_2(\xi)) = (1 + C_1)(L(u_1(\xi)) + C_2(\alpha u_{0\xi} + \beta u_0 u_{0\xi}) + C_2 L(u_0(\xi)) + C_1((\alpha u_{1\xi}) + \beta(u_0 u_{1\xi} + u_1 u_{0\xi})), \tag{36}$$

$$u_2(0) = 0, u_2(1) = 0, \frac{\partial u_2(0)}{\partial \xi} = 0 \tag{37}$$

Third-order problems are given by Eq. (8) for $m = 3$;

$$\begin{aligned} L(u_3(\xi)) &= (1 + C_1)(L(u_2(\xi)) + C_3(\alpha u_{0\xi} + \beta u_0 u_{0\xi}) + C_1(\alpha u_{2\xi} + \beta(u_2 u_{2\xi} + u_1 u_{1\xi} + u_0 u_{2\xi}) \\ &+ C_2 L(u_1(\xi)) + C_3 L(u_0(\xi)) + C_2((\alpha u_{1\xi}) + \beta(u_0 u_{1\xi} + u_1 u_{0\xi})), \end{aligned} \tag{38}$$

$$u_3(0) = 0, u_3(1) = 0, \frac{\partial u_3(0)}{\partial \xi} = 0 \tag{39}$$

by solving Eqs. (33) - (39), the explicit form for $u_0(\xi)$, $u_1(\xi)$ and $u_2(\xi)$ can be directly obtained (see Appendix (A.1)). For simplicity it is omitted here. Inserting the results of solutions of these equations into (11), we get the third order approximate solution of $u_{approx}(x,t)$ as (see Appendix (A.2)).

$$u^*(\xi) = u_0(\xi) + u_1(\xi) + u_2(\xi) \tag{40}$$

and inserting this approximation solution into Eq. (12) yields the residual and functional;

$$R(\xi, C_1, C_2) = \frac{\partial^3 u^*(\xi)}{\partial \xi^3} + \alpha \frac{\partial u^*(\xi)}{\partial \xi} + \beta u^*(\xi) \frac{\partial u^*(\xi)}{\partial \xi}, \tag{41}$$

$$J(\xi, C_1, C_2) = \int_0^{\xi} R^2(\xi, C_1, C_2) d\xi, \tag{42}$$

$$\xi = L_x X + L_y Y + vT \tag{43}$$

For fixed values of α and β with $\xi = 0.1..0.5$, admits to a different values of C_1 and C_2 (see Appendix (A.2 - A.4)). Using the obtained auxiliary constants, we will get the third order approximate solutions using OHAM.

Conclusions

In this paper, the optimal homotopy asymptotic method with the aid of a symbolic computational method is used for constructing the approximate solution for a nonlinear problem arising in plasma physics. The validity and reliability of the OHAM is tested by their applications for nonlinear problems, namely, Zakharov-Kuznetsov equation that is derived to describe the ion acoustic waves in magnetized plasma.

The advantages of the OHAM with respect to homotopy perturbation method are illustrated, the OHAM provides a convenient way to control the convergence by optimal determining of the auxiliary constants and is converges rapidly to the solution and requires less computational work.

The results obtained here will enrich previous results and help us further understand the physical structures and analyze the nonlinear propagation of the quantum ion-acoustic waves in quantum magneto-plasma. Finally, it is worthwhile to mention that the proposed method is straightforward, concise and can also be applied to other nonlinear problems in science and engineering. This is our task in future works.

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Appendix

$$u_0(\xi) = -\xi^2 + 1$$

$$u_1(\xi) := \frac{1}{60} C_1 \beta \xi^6 - 2 C_1 \left(\frac{1}{24}\beta + \frac{1}{24}\alpha \right) \xi^4 + \left(\frac{1}{15} C_1 \beta + \frac{1}{12} C_1 \alpha \right) \xi^2$$

$$\begin{aligned} u_2(\xi) := & -\frac{1}{5400} C_1^{\,2} \beta^2 \xi^{10} + \left(\frac{1}{560} C_1^{\,2} \beta^2 + \frac{1}{560} \beta C_1^{\,2} \alpha \right) \xi^8 \\ & + \left(\frac{1}{60} C_1 \beta + \frac{1}{60} C_2 \beta + \frac{1}{60} C_1^{\,2} \beta - \frac{1}{200} C_1^{\,2} \beta^2 - \frac{1}{120} \beta C_1^{\,2} \alpha - \frac{1}{360} C_1^{\,2} \alpha^2 \right) \xi^6 + \Big(\\ & -\frac{1}{12} C_1^{\,2} \beta - \frac{1}{12} C_1 \beta - \frac{1}{12} C_1 \alpha - \frac{1}{12} C_2 \beta - \frac{1}{12} C_2 \alpha - \frac{1}{12} C_1^{\,2} \alpha + \frac{1}{180} C_1^{\,2} \beta^2 \\ & + \frac{1}{80} \beta C_1^{\,2} \alpha + \frac{1}{144} C_1^{\,2} \alpha^2 \Big) \xi^4 + \Big(-\frac{163}{75600} C_1^{\,2} \beta^2 - \frac{1}{168} \beta C_1^{\,2} \alpha + \frac{1}{15} C_1 \beta \\ & + \frac{1}{15} C_2 \beta + \frac{1}{15} C_1^{\,2} \beta - \frac{1}{240} C_1^{\,2} \alpha^2 + \frac{1}{12} C_1 \alpha + \frac{1}{12} C_2 \alpha + \frac{1}{12} C_1^{\,2} \alpha \Big) \xi^2 \end{aligned}$$

$$\begin{aligned} u_{approx} := & -\xi^2 + 1 + \frac{1}{60} C_1 \beta \xi^6 - 2 C_1 \left(\frac{1}{24}\beta + \frac{1}{24}\alpha \right) \xi^4 + \left(\frac{1}{15} \beta C_1 + \frac{1}{12} C_1 \alpha \right) \xi^2 \\ & - \frac{1}{5400} C_1^{\,2} \beta^2 \xi^{10} + \left(\frac{1}{560} C_1^{\,2} \beta^2 + \frac{1}{560} \beta C_1^{\,2} \alpha \right) \xi^8 \\ & + \left(\frac{1}{60} \beta C_1 + \frac{1}{60} C_2 \beta + \frac{1}{60} C_1^{\,2} \beta - \frac{1}{200} C_1^{\,2} \beta^2 - \frac{1}{120} \beta C_1^{\,2} \alpha - \frac{1}{360} C_1^{\,2} \alpha^2 \right) \xi^6 + \Big(\\ & -\frac{1}{12} C_1^{\,2} \beta - \frac{1}{12} \beta C_1 - \frac{1}{12} C_1 \alpha - \frac{1}{12} C_2 \beta - \frac{1}{12} C_2 \alpha - \frac{1}{12} C_1^{\,2} \alpha + \frac{1}{180} C_1^{\,2} \beta^2 \\ & + \frac{1}{80} \beta C_1^{\,2} \alpha + \frac{1}{144} C_1^{\,2} \alpha^2 \Big) \xi^4 + \Big(-\frac{163}{75600} C_1^{\,2} \beta^2 - \frac{1}{168} \beta C_1^{\,2} \alpha + \frac{1}{15} \beta C_1 \\ & + \frac{1}{15} C_2 \beta + \frac{1}{15} C_1^{\,2} \beta - \frac{1}{240} C_1^{\,2} \alpha^2 + \frac{1}{12} C_1 \alpha + \frac{1}{12} C_2 \alpha + \frac{1}{12} C_1^{\,2} \alpha \Big) \xi^2 \end{aligned} \tag{A.1}$$

$$\begin{aligned} u_{approx} := & \ 1. - 1.277651216\ \xi^2 - .1822479272\ \xi^6 + .4235352844\ \xi^4 - .005415894059\ \xi^{10} \\ & + .04177975418\ \xi^8 \end{aligned} \tag{A.2}$$

$$\begin{aligned} u_{approx} := & -1.269667194\ \xi^2 + 1. - .3329981208\ \xi^6 + .5240972353\ \xi^4 - .02055987113\ \xi^{10} \\ & + .09912795011\ \xi^8 \end{aligned} \tag{A.3}$$

$$\begin{aligned} u_{approx} := & -.364001237\ \xi^2 + 1. - 5.176208523\ \xi^6 + 2.586388732\ \xi^4 - .2347939433\ \xi^{10} \\ & + 2.188614972\ \xi^8 \end{aligned} \tag{A.4}$$