WALAILAK JOURNAL
http://wjst.wu.ac.th Engineering and Physical Sciences

Thermo-Diffusion and Chemical Reaction Effects on MHD Three Dimensional Free Convective Couette Flow

Jagdish PRAKASH[1,*], Kuppareddy Subramanyam BALAMURUGAN[2] and Sibayala VIJAYAKUMAR VARMA[3]

[1]Department of Mathematics, University of Botswana, Gaborone, Botswana
[2]Department of Mathematics, RVR & JC College of Engineering, Andhra Pradesh, India
[3]Department of Mathematics, Sri Venkateswara University, Andhra Pradesh, India

([*]Corresponding author's e-mail: prakashj@mopipi.ub.bw)

Received: 20 November 2012, Revised: 29 April 2013, Accepted: 26 June 2013

Abstract

In this paper, the effects of thermo-diffusion and chemical reaction on a 3 dimensional free convection couette flow of a viscous incompressible and electrically conducting fluid, in the presence of a uniform magnetic field and heat absorption, are analyzed. The plates involved are vertical. The magnetic field is applied in a direction normal to the plates. The plate at rest is subjected to a transverse sinusoidal injection velocity distribution, while the plate in uniform motion is subjected to constant suction and slip boundary conditions. The equations governing the fluid flow are solved using simple perturbation technique. The expressions for skin friction, Nusselt number, and Sherwood number are also derived. Dimensionless velocity, temperature, and concentration profiles are displayed graphically for different values of the parameters entering into the problem, like *Pr*, *Re*, *M*, *Gr*, *Gm*, *h*, *A*, *Sc*, *So* and *Kr*. The variations in skin friction, Nusselt number, and Sherwood number for different physical parameters are presented.

Keywords: Magnetic field, heat transfer, mass transfer, thermo-diffusion, chemical reaction

Introduction

The science of magnetohydrodynamics (MHD) has been concerned with geophysical and astrophysical problems for a number of years. In recent years, the possibility has arisen of using MHD to affect a flow stream of an electrically conducting fluid, for the purpose of thermal protection, braking, propulsion, and control. In terms of applications, model studies on the effect of magnetic field on free convection flows have been made by several investigators [1-3].

The flow is also affected by the difference in concentrations on material constitution. In most of the works, the level of concentration of foreign mass is assumed to be very low, so that the Soret effects can be neglected. However, exceptions are observed therein. The Soret effect, for instance, has been utilized for isotope separation, and in mixtures between gases with very light molecular weight (H_2, H_e) and of medium molecular weight (N_2, air).

In view of the importance of this effect, several researchers in [4,5] initiated a few studies on Soret effects by taking various aspects of the flow phenomena into consideration. The authors in [6] studied thermal diffusion and diffusion-thermo effects on mixed free-forced convective and mass transfer boundary layer flow with temperature dependent viscosity.

The researcher in [7] analytically studied the MHD free-convective and mass-transfer flow past an infinite vertical plate moving on its own plane, taking into account the thermal diffusion effect when (i) the boundary surface impulsively started moving in its own plane and (ii) it was uniformly accelerated. Numerical studies were performed by the researchers in [8] to examine the steady MHD free convection and mass transfer fluid flow through a moving porous medium with thermal diffusion and diffusion-thermo past a semi-infinite vertical porous plate in a rotating system.

Thermal diffusion and heat generation effects on steady combined free-forced convection and mass transfer flow past a semi-infinite vertical porous flat plate embedded in a porous medium were studied numerically in [9]. The volumetric heat generation term may have exerted a strong influence on the heat transfer and also, as a consequence, on the fluid flow.

The Dufour and Soret effects on unsteady MHD free convection and mass transfer flow past an impulsively started infinite vertical porous flat plate embedded in a porous medium under the influence of transversely applied magnetic field were studied numerically in [10]. An analytical study of MHD free-convective and mass transfer flow past a moving infinite vertical plate in a rotating fluid was presented by a researcher in [11], taking the thermal diffusion effect into account. The plate was assumed to be moving on its own plane with arbitrary velocity. The effects of combined buoyancy forces from mass and thermal diffusion by natural convection flow, forming a vertical wavy surface, were investigated by researchers in [12]. The author in [13] derived numerical solutions of heat and mass transfer effects of an unsteady MHD free convective flow past an infinite vertical plate with constant suction.

The author in [14] studied the free convection and mass transfer flow of an incompressible, viscous, and electrically conducting fluid past a continuously moving infinite vertical porous plate, in the presence of large suction, and under the influence of uniform magnetic field, considering heat source and thermal diffusion. The problem of thermal diffusion and magnetic field effects on combined free-forced convection and mass transfer flow past a vertical porous flat plate in the presence of heat generation was studied by the author in [15]. The problem of coupled heat and mass transfer by natural convection from a semi-infinite inclined flat plate in the presence of an external magnetic field and internal heat generation or absorption effects was formulated and studied by the authors in [16]. The plate surface had a power-law variation of both wall temperature and concentration and was permeable, to allow for possible fluid wall suction or blowing.

However, in convective heat and mass transfer processes, diffusion rates can be altered tremendously by chemical reactions. The effect of a chemical reaction depends on whether the reaction is heterogeneous or homogeneous. This also depends on whether they occur at an interface or as a single phase volume reaction. A reaction is said to be of the order n, if the reaction rate is proportional to the n^{th} power of the concentration. In particular, a reaction is said to be of the first order if the rate of the reaction is directly proportional to the concentration itself. In nature, the presence of pure air or water is not possible. Some foreign mass may be present either naturally or mixed with the air or water that causes some kind of chemical reaction.

The study of such types of chemical reaction processes is useful for improving a number of chemical technologies, such as food processing and polymer production. Mass transfer effects on moving isothermal vertical plate in the presence of chemical reaction were studied by the authors in [17]. Several authors in [18] studied the effects of chemical reaction on unsteady MHD flow through an impulsively started semi-infinite vertical plate.

The effects of thermophoresis and chemical reaction on an unsteady hydro magnetic free convection and mass transfer flow past an impulsively started infinite inclined porous plate, in the presence of heat generation or absorption, were studied numerically by the authors in [19]. It was concluded that all of the hydrodynamic, thermal, and concentration boundary layers decrease with the increasing values of suction parameter. Both the velocity and temperature distribution increase with the increasing values of the heat generation parameter, but the opposite effects were observed in the case of Prandtl number. The concentration boundary layer increased with the increasing values of the chemical reaction parameter.

The researchers in [20] analyzed the effects of chemical reaction, thermal radiation, and heat generation or absorption on unsteady free convective heat and mass transfer along an infinite vertical porous plate, in the presence of a transverse magnetic field and Hall current. Using finite element method, the researcher in [21] studied the effects of chemical reaction, variable viscosity, thermophoresis, and heat generation/absorption on a boundary-layer hydro magnetic flow with heat and mass transfer over a heat surface.

However, most of the investigators confined themselves to 2-dimensional flows. There may be situations where the flow fields are essentially 3-dimensional, for example, when variation in the suction velocity distribution is transverse to the potential flow. The researchers in [22] developed a complete 3-dimensional mathematical model governing the steady, laminar flow of an incompressible fluid subjected to a magnetic field, including internal heating due to the Joule effect, heat transfer due to conduction, and thermally induced buoyancy forces. Results of test cases with thermally induced buoyancy demonstrate the stabilizing effect of the magnetic field on the re-circulating flows.

The authors in [23] studied the effect of periodic permeability on the free convective flow of a viscous incompressible fluid through a highly porous medium. The porous medium was bounded by an infinite vertical porous plate. The problem became three-dimensional due to permeability variation. The authors in [24] analyzed the effects of magnetic field on the three dimensional flow of an incompressible viscous fluid past a porous plate.

A theoretical analysis of couette flow of a viscous incompressible fluid through a porous medium between 2 infinite horizontal parallel porous flat plates was presented by the authors in [25]. The stationary plate and the plate in uniform motion were respectively, subjected to transverse sinusoidal injection and uniform suction of the fluid.

The effects of a transverse sinusoidal injection velocity distribution on the 3-dimensional free convection couette flow (non-magnetic case) of a viscous incompressible fluid (water) with transpiration cooling, in the presence of a heat source and first order velocity slip conditions, were investigated by the researchers in [26]. The authors in [27] investigated the three-dimensional free convection flow and heat transfer along a vertical porous plate with transverse sinusoidal suction velocity distribution.

Some authors in [28] analyzed the 3 dimensional couette flow of a viscous incompressible electrically conducting fluid between 2 infinite horizontal parallel porous flat plates in the presence of a transverse magnetic field. It was observed that the magnetic parameter acted as a retarding force on the main velocity and as an accelerating force on the cross velocity of the flow field. The suction parameter had a retarding effect on the main velocity as well as on the temperature field. The Prandtl number reduced the temperature and increased the rate of heat transfer at the wall.

The authors in [29] presented an analysis of 3-dimensional couette flow between 2 horizontal parallel porous flat plates of an electrically conducting, viscous incompressible fluid. The stationary plate was subjected to a transverse sinusoidal injection of the fluid and its corresponding removal by constant suction through the other plate in uniform motion. Soret effects on MHD three dimensional free convection couette flow with heat and mass transfer in the presence of heat absorption was presented by the researchers in [30].

No attention has been paid to the problem of chemical reaction and Soret effects on 3 dimensional hydro magnetic free convection flows with heat and mass transfer. The effects of chemical reaction and thermo-diffusion on a 3 dimensional free convection couette flow of a viscous, incompressible, and electrically conducting fluid, in the presence of a uniform magnetic field, are analyzed in this paper. The plates involved are vertical. The magnetic field is applied normal to the plates. The equations governing the fluid flow are solved by using perturbation technique. Dimensionless velocity, temperature, and concentration profiles are analyzed through graphs for different values of parameters entering into the problem, like Prandtl number (Pr), Reynolds number (Re), Magnetic parameter (M), Grashof number for heat transfer (Gr), Grashof number for mass transfer (Gm), Slip parameter (h), Heat absorption parameter (A), Schmidt number (Sc), Soret number (So), and Chemical reaction Parameter (Kr). Finally, the corresponding variations in skin friction, Nusselt number, and Sherwood number at rest plate, which are of physical interest, are derived and studied through the tables.

Mathematical formulation

The effects of thermo-diffusion and chemical reaction on a 3 dimensional free convection couette flow of a viscous incompressible and electrically conducting fluid, in the presence of heat absorption, are considered in this paper. A uniform magnetic field is applied normal to the plane of the plates. A coordinate system with the plates lying vertically along the $\bar{x}-\bar{z}$ plane is chosen, such that the $\bar{x}$ - axis is oriented along the length of the plates in the direction of the buoyancy force, and the $\bar{y}$ - axis is perpendicular to the plane of the plates and directed into the fluid region, as shown in **Figure 1**.

In the analysis of the problem, the following assumptions are made:

1) The channel is long enough in the $\bar{x}$ - direction so that all the fluid properties are assumed to be independent of $\bar{x}$, except for the pressure.

2) Induced magnetic field and applied electric field are neglected.

3) Viscous dissipation and Joules dissipation are neglected.

4) The plate at rest is subjected to a transverse sinusoidal injection velocity distribution of the form: $\bar{v}(\bar{z}) = V\left(1 + \varepsilon \cos\frac{\pi \bar{z}}{d}\right)$, $v > 0$ and $0 < \varepsilon < 1$. Here V is the undisturbed part of the injection velocity, d is the wavelength of the periodic injection velocity, and ε is a reference parameter.

5) The other plate has a uniform motion U and is subjected to a constant suction V under first order velocity slip conditions.

6) Without loss of generality, the distance between the plates may be taken equal to the wave length d of the injection velocity.

However, the flow becomes 3 dimensional due to the form of the injection velocity distribution and the constant suction at the respective plates, as assumed above. Let $\bar{q} = \bar{i}u + \bar{j}v + \bar{k}w$ be the velocity of the fluid at the point $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{i}, \bar{j}, \bar{k}$ are the unit vectors along the $\bar{x}$ - axis, $\bar{y}$ - axis, and $\bar{z}$ - axis, respectively.

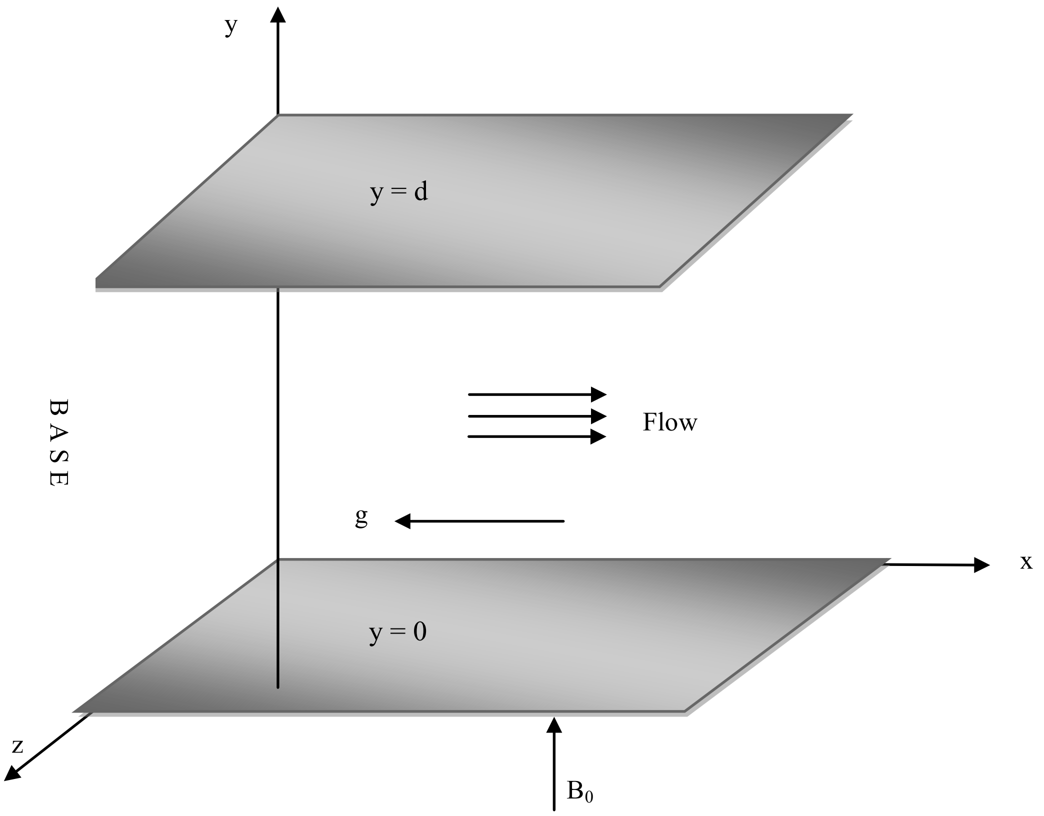

Figure 1 The flow configuration.

With the foregoing assumptions, following are the equations governing the flow;

Continuity equation: $\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial \bar{w}}{\partial \bar{z}}=0$ (1)

Momentum equations;

$$\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}=g\beta(\bar{T}-\bar{T}_e)+g\bar{\beta}(\bar{C}-\bar{C}_e)+\upsilon\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right)-\frac{\sigma B_0^2 \bar{u}}{\rho} \quad (2)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{v}}{\partial \bar{z}}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}}+\upsilon\left(\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right) \quad (3)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{v}}{\partial \bar{z}}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}}+\upsilon\left(\frac{\partial^2 \bar{w}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right)-\frac{\sigma B_0^2 \bar{w}}{\rho} \tag{4}$$

Energy equation; $$\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{T}}{\partial \bar{z}}=\frac{k}{\rho C_p}\left(\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right)+\frac{Q}{\rho C_p}(\bar{T}-\bar{T}_e) \tag{5}$$

Mass transfer equation;

$$\bar{v}\frac{\partial \bar{C}}{\partial \bar{y}}+\bar{w}\frac{\partial \bar{C}}{\partial \bar{z}}=D\left(\frac{\partial^2 \bar{C}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{C}}{\partial \bar{z}^2}\right)+D_1\left(\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}+\frac{\partial^2 \bar{T}}{\partial \bar{z}^2}\right)+D_2(\bar{C}-\bar{C}_e) \tag{6}$$

where $\bar{u}, \bar{v}$ and $\bar{w}$ are the components of dimensional velocities along the $\bar{x}, \bar{y}$ and $\bar{z}$ directions, respectively. g is the acceleration due to gravity. β is the coefficient of volume expansion for the heat transfer and $\bar{\beta}$ is the volumetric coefficient of expansion with species concentration. $\bar{T}$ is the fluid temperature and $\bar{T}_e$ is the equilibrium temperature of the fluid. $\bar{C}$ is the molar species concentration of the fluid and $\bar{C}_e$ is the equilibrium molar species concentration of the fluid. C_p is the specific heat at constant pressure. $\bar{P}$ is the fluid pressure. υ is the kinematic viscosity. ρ is the fluid density and σ is the fluid electrical conductivity. B_0 is the magnetic field component along y axis. k is the thermal conductivity. Q is the dimensionless heat absorption coefficient. D is the chemical molecular diffusivity. D_1 is the coefficient of thermal diffusivity and D_2 is the chemical reaction rate constant. The corresponding boundary conditions are;

$$\left.\begin{array}{l}\bar{y}=0;\bar{u}=0;\bar{v}=V\left(1+\varepsilon\cos\frac{\pi\bar{z}}{d}\right);\bar{w}=0;\bar{T}=\bar{T}_0;\bar{C}=\bar{C}_0\\ \bar{y}=d;\bar{u}=U+L_1\frac{\partial\bar{u}}{\partial\bar{y}};\bar{v}=V;\bar{w}=0;\bar{T}=\bar{T}_1;\bar{C}=\bar{C}_1\end{array}\right\} \tag{7}$$

where $\bar{T}_0$ is the temperature of the fluid at the plate at rest and $\bar{T}_1$ is the temperature of the fluid at the plate in motion. $\bar{C}_0$ is the molar species concentration of the fluid at the rest plate and $\bar{C}_1$ is the molar species concentration of the fluid at the plate in motion. $L_1=\left(\frac{2-m_1}{m_1}\right)L$; where *L* is the mean free path and m_1 is Maxwell's reflection coefficient.

The following dimensionless quantities are introduced;

$$\left.\begin{aligned} & y=\frac{\bar{y}}{d}; z=\frac{\bar{z}}{d}; u=\frac{\bar{u}}{U}; V=\frac{\bar{v}}{V}; p=\frac{\bar{p}}{\rho V^2}; \mathrm{Re}=\frac{Vd}{\upsilon}; \mathrm{Pr}=\frac{\mu C_p}{k}; w=\frac{\bar{w}}{v}; \\ & Sc=\frac{\upsilon}{D}; So=\frac{D_1(\bar{T}_0-\bar{T}_e)}{\upsilon(\bar{C}_0-\bar{C}_e)}; \theta=\frac{(\bar{T}-\bar{T}_e)}{(\bar{T}_0-\bar{T}_e)}; \phi=\frac{(\bar{C}-\bar{C}_e)}{(\bar{C}_0-\bar{C}_e)}; M=\frac{\sigma B_0^2 d}{\rho V}; \\ & h=\frac{L_1}{d}; \upsilon=\frac{\mu}{\rho}; Gr=\frac{gd\beta(\bar{T}_0-\bar{T}_e)}{UV}; Gm=\frac{gd\bar{\beta}(\bar{C}_0-\bar{C}_e)}{UV}; n=\frac{(\bar{T}_1-\bar{T}_e)}{(\bar{T}_0-\bar{T}_e)}; \\ & r=\frac{(\bar{C}_1-\bar{C}_e)}{(\bar{C}_0-\bar{C}_e)}; A=\frac{Qd\nu}{Vk}; Kr=\frac{d^2 D_2}{\mathrm{Re}\,\upsilon} \end{aligned}\right\} \tag{8}$$

where p is the dimensionless fluid pressure. Re is the Reynolds number. Pr is the Prandtl number. Sc is the Schmidt number. M is the magnetic field parameter. h is the slip parameter. A is the heat absorption parameter. Gr is the Grashof number. Gm is the solutal Grashof number. So is the Soret number. Kr is the Chemical reaction parameter. n is the wall temperature ratio and r is the wall concentration ratio. θ is the dimensionless temperature and ϕ is the dimensionless concentration.

Using the above substitutions (8) in Eqs. (1) - (6), following are the equations in non- dimensional form;

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{9}$$

$$v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=(Gr\theta+Gm\phi)+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-Mu \tag{10}$$

$$v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{11}$$

$$v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-Mw \tag{12}$$

$$v\frac{\partial \theta}{\partial y}+w\frac{\partial \theta}{\partial z}=\frac{1}{\mathrm{Pr}\,\mathrm{Re}}\left(\frac{\partial^2 \theta}{\partial y^2}+\frac{\partial^2 \theta}{\partial z^2}\right)+\frac{A\theta}{\mathrm{Pr}} \tag{13}$$

$$v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}=\frac{1}{\mathrm{Re}}\left[\frac{1}{Sc}\left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)+So\left(\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right)\right]+Kr\phi \tag{14}$$

The relevant boundary conditions obtained from (7) using the dimensionless quantities (8) are;

$$\left.\begin{aligned} &y=0: u=0, v=(1+\varepsilon\cos\pi z), w=0, \theta=1, \phi=1\\ &y=1: u=1+h\frac{\partial u}{\partial y}, v=1, w=0, \theta=n, \phi=r \end{aligned}\right\} \tag{15}$$

Method of solution

The amplitude ε of the injection velocity is very small, hence, using perturbation technique, the solution to this flow problem may be assumed to be as the following form, neglecting the higher powers of ε.

$$\left.\begin{aligned} v(y,z)&=v_0(y)+\varepsilon v_1(y,z)\\ u(y,z)&=u_0(y)+\varepsilon u_1(y,z)\\ w(y,z)&=w_0(y)+\varepsilon w_1(y,z)\\ \theta(y,z)&=\theta_0(y)+\varepsilon \theta_1(y,z)\\ \phi(y,z)&=\phi_0(y)+\varepsilon \phi_1(y,z)\\ p(y,z)&=p_0(y)+\varepsilon p_1(y,z) \end{aligned}\right\} \tag{16}$$

when $\boldsymbol{\varepsilon}$ = *0,* the problem reduces to a 2 dimensional flow, and is governed by the following equations obtained from Eqs. (9) - (14) using Eq. (16);

$$\frac{dv_o}{dy}=0 \tag{17}$$

$$\frac{d^2u_0}{dy^2}-\mathrm{Re}\frac{du_0}{dy}-\mathrm{Re}\,Mu_0=-\mathrm{Re}(Gr\theta_0+Gm\phi_0) \tag{18}$$

$$\frac{dp_o}{dy}=0 \tag{19}$$

$$\frac{d^2 w_0}{dy^2} - \mathrm{Re}\frac{dw_0}{dy} - \mathrm{Re}\, M w_0 = 0 \tag{20}$$

$$\frac{d^2 \theta_0}{dy^2} - \mathrm{Pr}\,\mathrm{Re}\frac{d\theta_0}{dy} + \mathrm{Re}\, A\theta_0 = 0 \tag{21}$$

$$\frac{d^2 \phi_0}{dy^2} - Sc\,\mathrm{Re}\frac{d\phi_0}{dy} + SoSc\frac{d^2\theta_0}{dy^2} + \mathrm{Re}\, ScKr\phi_0 = 0 \tag{22}$$

where θ_0 and ϕ_0 are dimensionless temperature and concentration of rest.
Using (16) in (15), the corresponding boundary conditions are;

$$\left.\begin{array}{l} y = 0 : u_0 = 0; v_0 = 1; w_0 = 0; \theta_0 = 1; \phi_0 = 1 \\ y = 1 : u_0 = 1 + h\dfrac{du_0}{dy}; v_0 = 1; w_0 = 0; \theta_0 = n;\ \phi_0 = r \end{array}\right\} \tag{23}$$

subject to the boundary conditions (23). Solving the Eqs. (17) - (22), the following solutions are obtained.

$W_0 = 0$; $V_0 = 1$; P_0 = Constant;

$$u_0 = c_7 e^{t_5 y} + c_8 e^{t_6 y} + c_9 e^{t_1 y} + c_{10} e^{t_2 y} + c_{11} e^{t_3 y} + c_{12} e^{t_4 y} \tag{24}$$

$$\phi_0 = c_3 e^{t_3 y} + c_4 e^{t_4 y} + c_5 e^{t_1 y} + c_6 e^{t_2 y} \tag{25}$$

$$\theta_0 = c_1 e^{t_1 y} + c_2 e^{t_2 y} \tag{26}$$

When $\varepsilon \neq 0$, substituting (16) into the Eqs. (9) - (14), respectively, and equating the coefficients of like powers of ε on both sides and neglecting those of $\varepsilon^2, \varepsilon^3$ etc., the following equations are obtained.

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{27}$$

$$\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_0}{\partial y} = Gr\theta_1 + Gm\phi_1 + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) - Mu_1 \tag{28}$$

$$\frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right) \tag{29}$$

$$\frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}\right) - Mw_1 \tag{30}$$

$$\frac{\partial \theta_1}{\partial y} + v_1\frac{\partial \theta_0}{\partial y} = \frac{1}{\mathrm{Pr}\,\mathrm{Re}}\left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2}\right) + \frac{A\theta_1}{\mathrm{Pr}} \tag{31}$$

$$\frac{\partial \phi_1}{\partial y} + v_1\frac{\partial \phi_0}{\partial y} = \frac{1}{\mathrm{Re}}\left[\frac{1}{Sc}\left(\frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_1}{\partial z^2}\right) + So\left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2}\right)\right] + Kr\phi_1 \tag{32}$$

where θ_1 and ϕ_1 are dimensionless temperature and concentration of moving plates.
The corresponding boundary conditions are;

$$\left.\begin{array}{l} y = 0 : u_1 = 0; v_1 = \cos \pi z; w_1 = 0; \theta_1 = 0; \phi_1 = 0 \\ y = 1 : u_1 = h\dfrac{\partial u_1}{\partial y}; v_1 = 0; w_1 = 0; \theta_1 = 0; \phi_1 = 0 \end{array}\right\} \tag{33}$$

Cross flow solution

The equations governing the cross flow are given by (27), (29), and (30). The solutions for $v_1(y, z)$, $w_1(y, z)$ and $p_1(y, z)$ are assumed to be the following form.

$$\left.\begin{array}{l} v_1(y,z) = v_{11}(y)\cos \pi z \\ w_1(y,z) = -\dfrac{1}{\pi} v_{11}^1(y)\sin \pi z \\ p_1(y,z) = p_{11}(y)\cos \pi z \end{array}\right\} \tag{34}$$

where $v_{11}^1(y)$ denotes the differentiation of $v_{11}(y)$ with respect to y. Using the substitutions (34) in (33), the following boundary conditions are obtained.

$$\begin{array}{l} y = 0; v_{11} = 1; v_{11}^1 = 0 \\ y = 1; v_{11} = 0; v_{11}^1 = 0 \end{array} \tag{35}$$

Applying the above substitutions (34) for v_1, w_1 and p_1 in Eqs. (27), (29), and (30), and then solving them subject to the boundary conditions (35), the following solutions are obtained;

$$v_1 = (c_{13}e^{t_7 y} + c_{14}e^{t_8 y} + c_{15}e^{t_9 y} + c_{16}e^{t_{10} y})\cos \pi z \tag{36}$$

$$w_1 = (c_{17}e^{t_7 y} + c_{18}e^{t_8 y} + c_{19}e^{t_9 y} + c_{20}e^{t_{10} y})\sin \pi z \tag{37}$$

Solutions of the main flow temperature and species concentration fields

The equations governing the main flow, the temperature and the species concentration fields are given by (28), (31) and (32) respectively.

The solutions for $u_1(y,z)$, $\theta_1(y,z)$, $\phi_1(y,z)$ are assumed to be of the form;

$$\left.\begin{aligned} u_1(y,z) &= u_{11}(y)\cos \pi z \\ \theta_1(y,z) &= \theta_{11}(y)\cos \pi z \\ \phi_1(y,z) &= \phi_{11}(y)\cos \pi z \end{aligned}\right\} \tag{38}$$

The following boundary conditions are obtained using the above substitutions (38) in (33);

$$\left.\begin{aligned} y = 0;\, u_{11} = 0,\, \theta_{11} = 0,\, \phi_{11} = 0 \\ y = 1;\, u_{11} = hu_{11}^1,\, \theta_{11} = 0,\, \phi_{11} = 0 \end{aligned}\right\} \tag{39}$$

Applying the above substitutions (38) for u_1, θ_1 and ϕ_1 in Eqs. (28), (31), and (32), and then solving them subject to the boundary conditions (39), the following solutions are obtained;

$$\begin{aligned} u_1 = (&c_{51}e^{t_{11}y} + c_{52}e^{t_{12}y} + c_{53}e^{t_{13}y} + c_{54}e^{t_{14}y} + c_{55}e^{t_{15}y} \\ &+ c_{56}e^{t_{16}y} + c_{57}e^{t_{17}y} + c_{58}e^{t_{18}y} + c_{59}e^{t_{19}y} + c_{60}e^{t_{20}y} \\ &+ c_{61}e^{t_{21}y} + c_{62}e^{t_{22}y} + c_{63}e^{t_{23}y} + c_{64}e^{t_{24}y} + c_{65}e^{t_{25}y} \\ &+ c_{66}e^{t_{26}y} + c_{67}e^{t_{27}y} + c_{68}e^{t_{28}y} + c_{69}e^{t_{29}y} + c_{70}e^{t_{30}y} \\ &+ c_{71}e^{t_{31}y} + c_{72}e^{t_{32}y} + c_{73}e^{t_{33}y} + c_{74}e^{t_{34}y} + c_{75}e^{t_{35}y} \\ &+ c_{76}e^{t_{36}y} + c_{77}e^{t_{37}y} + c_{78}e^{t_{38}y} + c_{79}e^{t_{39}y} + c_{80}e^{t_{40}y})\cos \pi z \end{aligned} \tag{40}$$

$$\begin{aligned} \theta_1 = (&c_{21}e^{t_{21}y} + c_{22}e^{t_{22}y} + c_{23}e^{t_{23}y} + c_{24}e^{t_{24}y} + c_{25}e^{t_{25}y} \\ &+ c_{26}e^{t_{26}y} + c_{27}e^{t_{27}y} + c_{28}e^{t_{28}y} + c_{29}e^{t_{29}y} + c_{30}e^{t_{30}y})\cos \pi z \end{aligned} \tag{41}$$

$$\begin{aligned}\phi_1 = (&c_{31}\, e^{t_{11}y} + c_{32}\, e^{t_{12}y} + c_{33}\, e^{t_{13}} + c_{34}\, e^{t_{14}y} + c_{35}\, e^{t_{15}y} \\ &+ c_{36}\, e^{t_{16}y} + c_{37}\, e^{t_{17}y} + c_{38}\, e^{t_{18}y} + c_{39}\, e^{t_{19}y} + c_{40}\, e^{t_{20}y} \\ &+ c_{41}\, e^{t_{21}y} + c_{42}\, e^{t_{22}y} + c_{43}\, e^{t_{23}y} + c_{44}\, e^{t_{24}y} + c_{45}\, e^{t_{25}y} \\ &+ c_{46}\, e^{t_{26}y} + c_{47}\, e^{t_{27}y} + c_{48}\, e^{t_{28}y} + c_{49}\, e^{t_{29}y} + c_{50}\, e^{t_{30}y})\cos\pi z\end{aligned} \tag{42}$$

Substituting (24) - (26) and (40) - (42) in (16), the solutions for velocity, temperature, and concentration of the fluid are obtained.

Skin friction

The non-dimensional skin friction τ_{xo} in the main flow direction at the plate $y = 0$ is given as;

$$\tau_{xo} = \left(\frac{\partial u}{\partial y}\right)_{y=0} = u_0^{'}(0) + \varepsilon u_{11}^{'}(0)\cos\pi z \tag{43}$$

Nusselt number

The dimensionless heat flux Nu_0 at the plate $y = 0$ is given as;

$$N_u = \left(\frac{\partial \theta}{\partial y}\right)_{y=0} = \theta_0^{'}(0) + \varepsilon\theta_{11}^{'}(0)\cos\pi z \tag{44}$$

Sherwood number

The non-dimensional mass flux at the plate $y = 0$ in terms of Sherwood number Sh is given as;

$$Sh = \left(\frac{\partial \phi}{\partial y}\right)_{y=0} = \phi_0^{'}(0) + \varepsilon\phi_{11}^{'}(0)\cos\pi z \tag{45}$$

Results and discussion

This paper aims at explaining the couette flow; i.e., when the lower plate is at rest, the fluid is injected with sinusoidal velocity, while the upper plate moves with uniform velocity subjected to a constant suction under first order velocity slip conditions.

In the present investigation, the effects of Soret number (*So*), Chemical reaction (*Kr*), Schmidt number (*Sc*), Reynolds number (*Re*), Grashof number for heat transfer (*Gr*), Grashof number for mass transfer (*Gm*), heat absorption parameter $|A|$, slip parameter (*h*), Prandtl number (*Pr*) and Magnetic parameter (*M*) on main flow velocity *'u'*, temperature *'*θ*'*, and concentration ϕ are observed. The Prandtl number for air at 298^{o} *K* and 1atm is given by $Pr = 0.71$. The values of ε and *z* are fixed at $\varepsilon = 0.01$ and $z = 0.3$, unless otherwise stated. The effects of the above stated parameters are graphically shown in **Figures 2 - 14**. The influence of *Kr* on skin friction and the rate of mass transfer are studied, and the results are shown in **Tables 1** and **2**. The effect of |A| on dimensionless rate of heat transfer is studied. and the results are exhibited through **Table 3**.

Generally, the fluid velocity is higher near the moving surface, and decreases to zero value far away from the plate surface, satisfying the far field boundary condition for all parameter values. In **Figure 2**, the effect of increasing the magnetic field strength on the momentum boundary-layer thickness is illustrated. It is now a well-established fact that the magnetic field presents a damping effect on the velocity field by creating a drag force that opposes the fluid motion, causing the velocity to decrease. A complete reverse phenomenon is observed in the region very close to the moving plate.

The influence of thermal Gr and solutal Gm on the velocity is shown in **Figure 3**. The thermal Grashof number signifies the relative effect of the thermal buoyancy force to the viscous hydrodynamic force. The flow is accelerated due to the enhancement in buoyancy force, corresponding to an increase in the thermal Grashof number. The solutal Gm defines the ratio of the species buoyancy force to the viscous hydrodynamic force. It is noticed that the velocity increases with increasing values of the Gm. A complete reverse phenomenon is observed in the region very close to the moving plate.

Usually *So* accelerates the fluid velocity. It is observed that *u* increases as *So* increases in the region of the channel approximately given by y € (0, 0.85). However, this trend reverses completely in the region very close to the moving plate, as shown in **Figure 4**.

Figure 5 shows the velocity for different values of chemical reaction parameter *Kr*. It is observed that the velocity increases with the increase of the chemical reaction parameter in the region of the channel approximately given by $y \in (0, 0.85)$. However, this trend reverses in the region very close to the moving plate.

The flow velocity increases for increases in *Sc* or *Re*, as exhibited through **Figure 6**, in the region of the channel approximately given by $y \in (0, 0.85)$. However, this trend reverses in the region very close to the moving plate.

From **Figure 7**, it is observed that an increase in *h* leads to a decrease in *'u'*. From **Figure 8**, the effects of wall temperature and concentration ratios on velocity of the fluid *'u'* are studied. As wall temperature ratio *n* or wall concentration ratio *r* increases, the velocity increases. However, this trend reverses in the region very close to the moving plate.

The effects of *A*, *Re* and *Pr* on the temperature of the fluid are depicted through **Figures 9 - 11**. From these figures, it is clear that an increase in the magnitude of heat absorption parameter *A*, *Re* and *Pr* result in a decrease of temperature θ. The reason is that smaller values of *Pr* are equivalent to increasing the thermal conductivities and, therefore, heat is able to diffuse away from the heated surface more rapidly than for higher values of *Pr*. Hence, in the case of smaller Prandtl numbers, as the boundary layer is thicker, the rate of heat transfer is reduced.

The effects of *So*, *Kr*, *Sc*, *Re*, *A* and *Pr* on concentration of the fluid ϕ are studied, and the results are exhibited through **Figures 12 - 14**. ϕ increases with increments in *So*, *Kr*, *Sc*, *Re*, |A| and *Pr*, as shown in **Figures 12 - 14**.

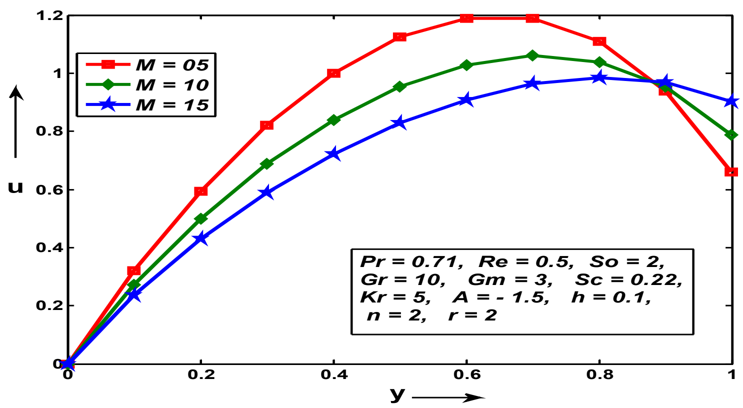

Figure 2 Velocity profiles with variations in *M*.

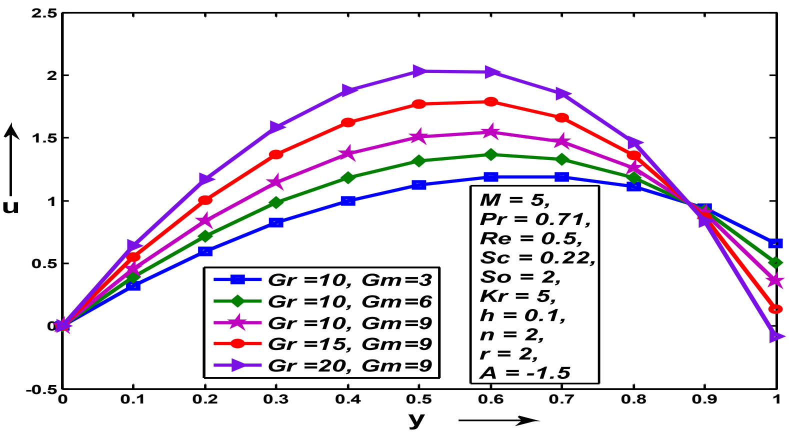

Figure 3 Velocity profiles with variations in *Gr* and *Gm*.

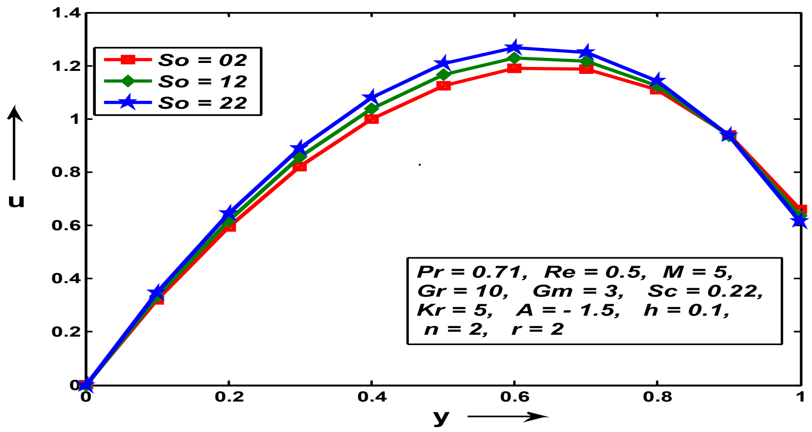

Figure 4 Velocity profiles with variations in *So*.

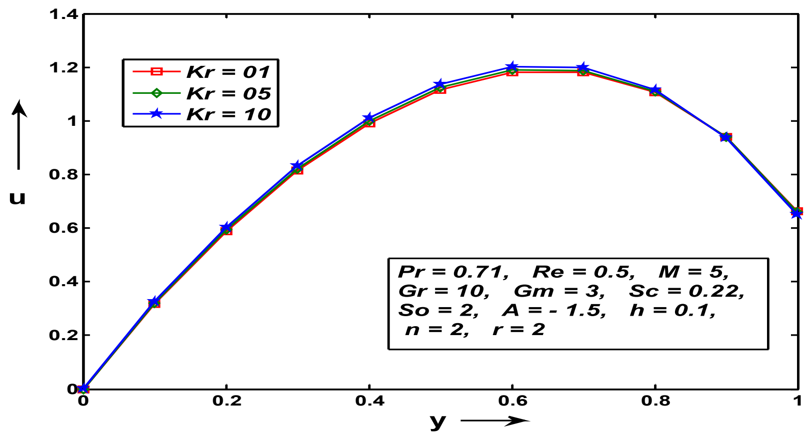

Figure 5 Velocity profiles with variations in *Kr*.

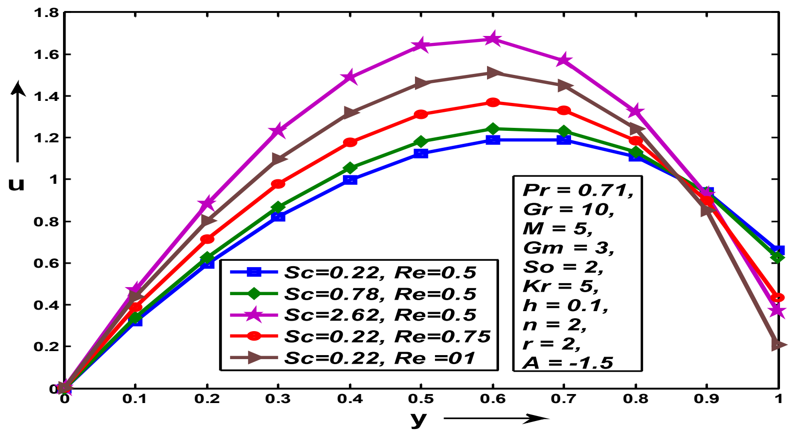

Figure 6 Velocity profiles with variations in *Sc* and *Re*.

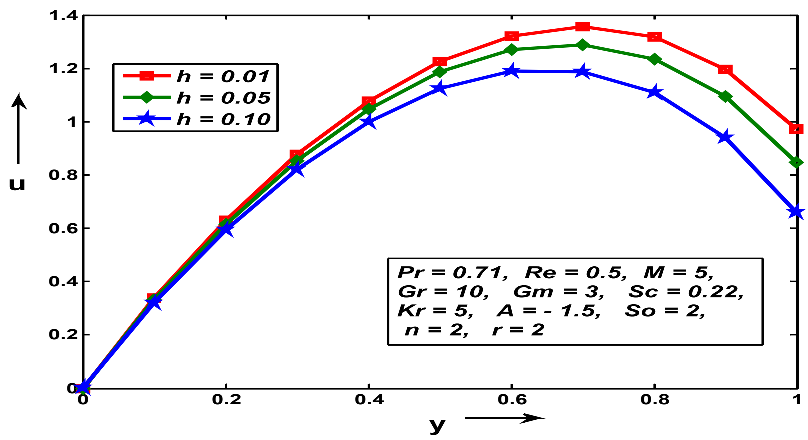

Figure 7 Velocity profiles with variations in *h*.

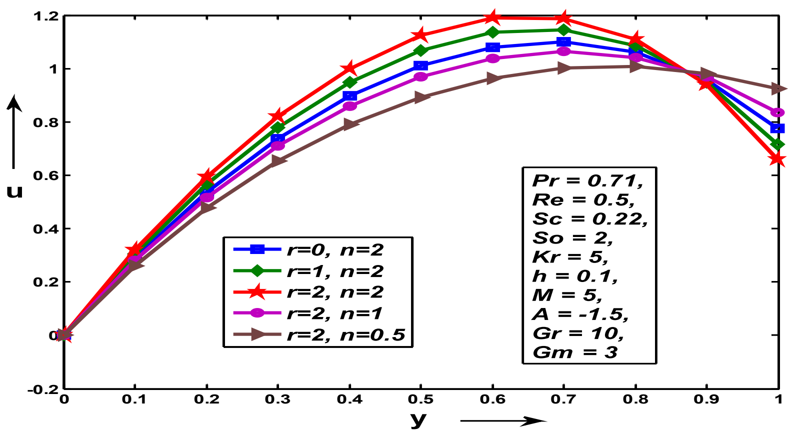

Figure 8 Velocity profiles with variations in *r* and *n*.

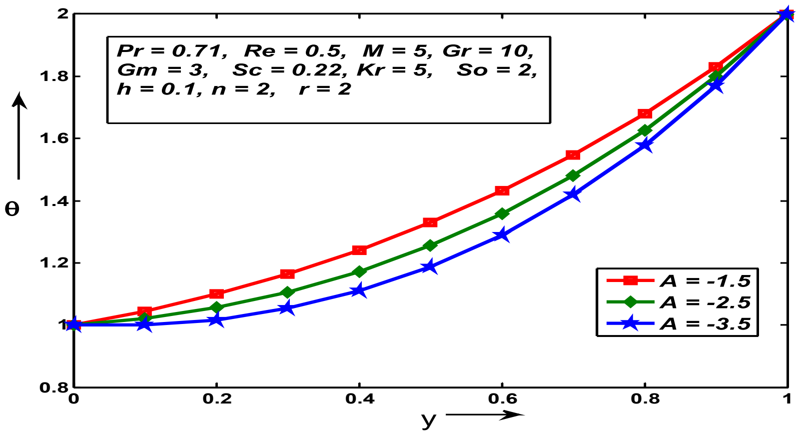

Figure 9 Temperature profiles with variations in *A*.

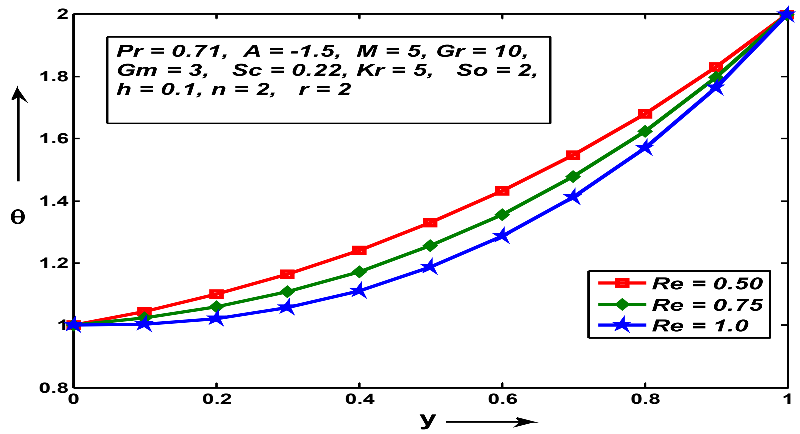

Figure 10 Temperature profiles with variations in *Re*.

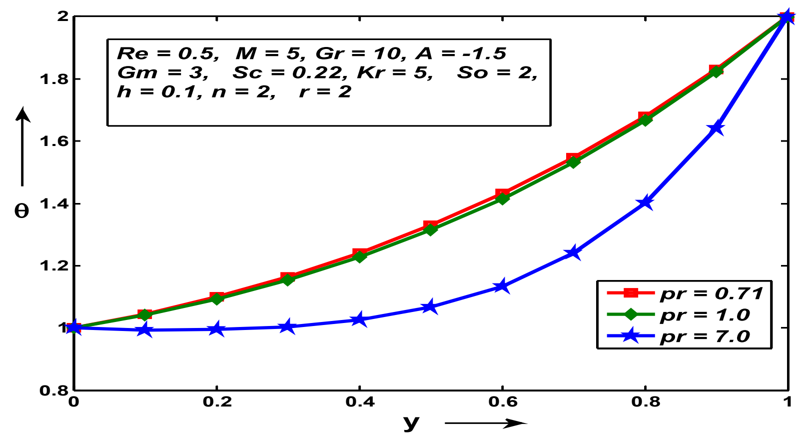

Figure 11 Temperature profiles with variations in *Pr*.

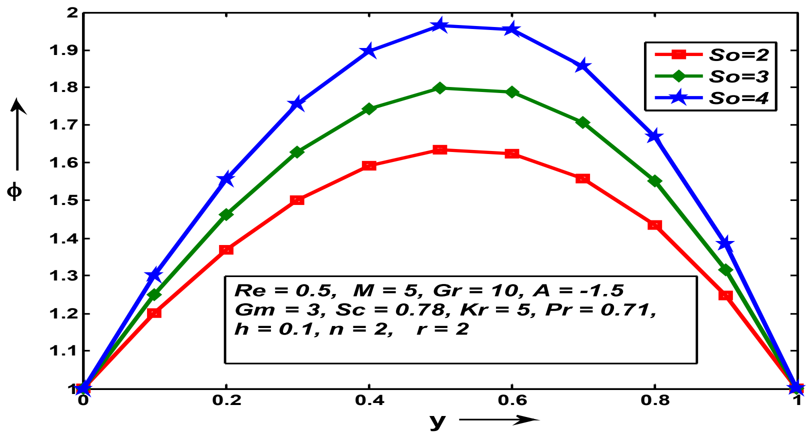

Figure 12 Concentration profiles with variations in *So*.

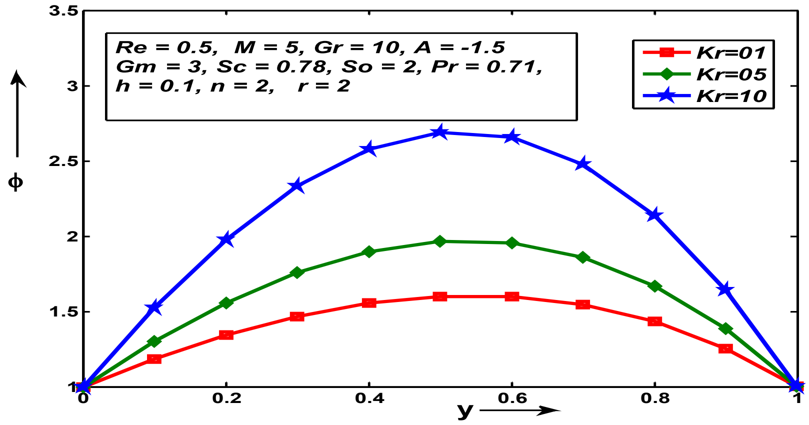

Figure 13 Concentration profiles with variations in *Kr*.

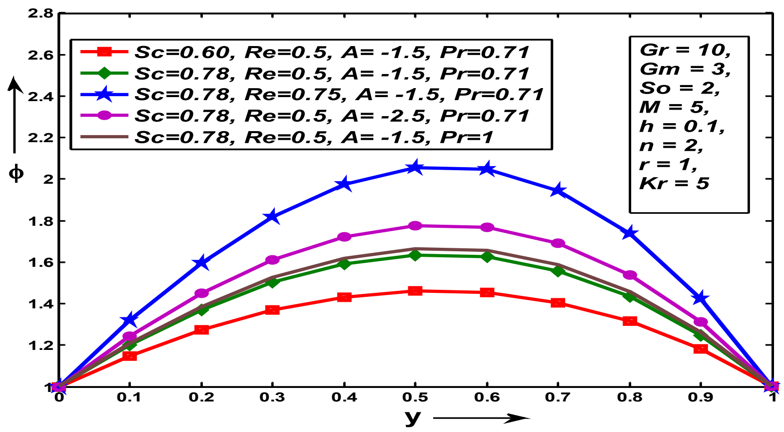

Figure 14 Concentration profiles with variations in *Sc, Re, A, Pr*.

Table 1 Numerical values of skin friction coefficient for different values of *Re* and *Kr*.

Re	τ_{XO} (*Kr* = 0.01)	τ_{XO} (*Kr* = 0.1)	τ_{XO} (*Kr* = 0.4)
0.2	2.2086	2.2087	2.2091
0.4	3.0551	3.0555	3.0569
0.6	3.7177	3.7185	3.7213
0.8	4.2426	4.2440	4.2486
1.0	4.6643	4.6663	4.6728

Table 2 Numerical values of Sherwood Number for different values of *Re* and *Kr*.

Re	*Sh* (*Kr* = 0.01)	*Sh* (*Kr* = 0.1)	*Sh* (*Kr* = 0.4)
0.2	0.1083	0.1103	0.1169
0.4	0.2123	0.2164	0.2300
0.6	0.3007	0.3063	0.3270
0.8	0.3788	0.3871	0.4150
1.0	0.4468	0.4572	0.4921

The behavior of the functions of τ_{XO} and *Sh*, under the influence of *Kr* against increasing *Re* is demonstrated through **Tables 1** and **2**. We can conclude that an increase in *Kr* causes both τ_{XO} and *Sh* to increase. For a given value of *Kr*, τ_{XO} and *Sh* increase with increasing *Re*. Further, **Table 3** depicts the nature of *Nu* under the influence of *A* against increasing *Re*. It is seen that an increase in the magnitude of heat absorption parameter *A*, i.e., an increase in |A|, causes *Nu* to decrease against increasing *Re*. It can also be noted that the influence of |A| on *Nu* is significant for small values of *Re*. However, the effect of |A| on *Nu* assumes significance when the inertial forces and viscous forces are of nearly comparable magnitudes. Also, *Nu* is higher in the absence of heat absorption.

Table 3 Numerical values of Nusselt number for different values of *Re* and *A*.

Re	*Nu* (*A*= 0)	*Nu* (*A*= −1.5)	*Nu* (*A*= −3.5)
0.2	0.9305	0.7530	0.5070
0.4	0.8644	0.5101	0.0924
0.6	0.8016	0.3007	−0.2579
0.8	0.7422	0.1117	−0.5540
1.0	0.6860	−0.0589	−0.8004

Conclusions

In this paper:

1) We have studied thermo-diffusion and chemical reaction effects on a 3 dimensional free convection couette flow of a viscous incompressible and electrically conducting fluid, in the presence of a uniform magnetic field and heat absorption.

2) A perturbation technique was employed to solve the resulting coupled partial differential equations.

3) It was found that the velocity profiles increased due to increase in thermo-diffusion parameter, chemical reaction parameter, the Schmidt number, thermal Grashof number, mass Grashof number, Reynolds number and Prandtl number.

4) An increase in the *Re* and *Pr* led to a decrease in the temperature profile and an increase in the concentration profile.

5) Also, it was found that the concentration profile increased due to increases in the thermo-diffusion parameter *So* and the *Kr*.

6) An increase in chemical reaction parameter causes both skin friction and Sherwood number to increase.

The results of this study can be applied in many chemical engineering processes, such as drying, evaporation, condensation, sublimation, and crystal growth, as well as deposition of thin films. These processes take place in numerous industrial applications, e.g., polymer production, manufacturing of ceramics or glassware and food processing.

Acknowledgements

The authors thank their respective departments of Mathematics at the University of Botswana, RVR & JC College of Engineering, and Sri Venkateswara University, for the excellent research-stimulating environment.

References

[1] M Acharya, GC Dash and LP Singh. Magnetic field effects on the free convection and mass transfer flow through porous medium with constant suction and constant heat flux. *Indian J. Pure Appl. Math.* 2000; **31**, 1-18.

[2] BK Jha and R Prasad. MHD free-convection and mass transfer flow through a porous medium with heat source. *Astrophys. Space Sci.* 1991; **181**, 117-23.

[3] P K Sahoo, N Datta and S Biswal. Magnetohydrodynamic unsteady free convection flow past an infinite vertical plate with constant suction and heat sink. *Indian J. Pure Appl. Math.* 2003; **34**, 145-55.

[4] B Gebhart and L Pera. The nature of vertical convection flows resulting from combined buoyancy effects of the thermal and mass diffusion. *Int. J. Heat Mass Tran.* 1971; **14**, 2025-50.

[5] GA Georgantopoulos. Effects of free convection on the hydro magnetic accelerated flow past a vertical porous limiting surface. *Astrophys. Space Sci.* 1979; **65**, 433-41.

[6] NG Kaufoussias and EW Williams. Thermal-diffusion and diffusion-thermo effects on mixed free-forced convective and mass transfer boundary layer flow with temperature dependent viscosity. *Int. J. Eng. Sci.* 1995; **33**, 1369-84.

[7] NG Kaufoussias. MHD Thermal-diffusion effects on free-convective and mass transfer flow over an infinite vertical moving plate. *Astrophys. Space Sci.* 1992; **192**, 11-9.

[8] I Nazmul and MM Alam. Dufour and Soret effects on steady MHD free convection and mass transfer fluid flow through a porous medium in a rotating system. *J. Nav. Architect. Mar. Eng.* 2007; **4**, 43-55.

[9] MS Alam, MM Rahman and MA Samad. Numerical study of the combined free forced convection and mass transfer flow past a vertical porous plate in a porous medium with heat generation and thermal diffusion. *Nonlinear Anal. Model. Contr*. 2006; **11**, 331-43.

[10] MS Alam, MM Rahman and MA Samad. Dufour and Soret effects on unsteady MHD free convection and mass transfer flow past a vertical porous plate in a porous medium. *Nonlinear Anal. Model. Contr.* 2006; **11**, 217-26.
[11] N Nanousis. Thermal diffusion effects on MHD free-convective and mass Transfer flow past a moving vertical plate in a rotating fluid. *Astrophys. Space Sci.* 1992; **191**, 313-22.
[12] MA Hossain and DA S Rees. Combined heat and mass transfer in natural convection flow from a vertical wavy surface. *Acta Mech.* 1999; **136**, 133-41.
[13] V Ambethkar. Numerical solutions of heat and mass transfer effects of an unsteady MHD free convective flow past an infinite vertical plate with constant suction. *J. Nav. Architect. Mar. Eng.* 2008; **5,** 28-36.
[14] A K Singh. MHD free convection and mass transfer flow with heat source and thermal diffusion. *J. Energ. Heat Mass Tran.* 2001; **23**, 227-49.
[15] GM Abdel-Rahman. Thermal diffusion and MHD effects on combined free- forced convection and mass transfer of a viscous fluid flow through a porous medium with heat generation. *Chem. Eng. Tech.* 2008; **31**, 554-9.
[16] AJ Chamka, A Rahim and A Khaled. Similarity solutions for hydro magnetic simultaneous heat and mass transfer by natural convection form an inclined plate with internal heat generation absorption. *Heat Mass Tran.* 2001; **37**, 117-23.
[17] UN Das, RK Deka and VM Soundalgekar. Effects of mass transfer on flow past an impulsively started infinite vertical plate with chemical reaction. *Forschung im Ingenieurwesen* 1994; **60**, 284-7.
[18] R Muthucumarswamy, J Maheswari and J Pandurangan. Unsteady MHD flow past an impulsively started semi-infinite vertical plate in the presence of chemical reaction. *Int. Rev. Pure Appl. Math.* 2008; **4**, 119-33.
[19] MS Alam, MM Rahman and MA Sattar. Effects of thermophoresis and chemical reaction on unsteady hydro magnetic free convection and mass transfer flow past an impulsively started infinite inclined porous plate in the presence of heat generation/absorption. *Thammasat Int. J. Sci. Tech.* 2007; **12**, 44-52.
[20] AJ Chamka, MA Mansour and A Aly. Unsteady MHD free convective heat and mass transfer from a vertical porous plate with Hall current, thermal radiation and chemical reaction effects. *Int. J. Numer. Meth. Fluid.* 2011; **65**, 432-47.
[21] MA Seddeek. Finite-element method for the effects of chemical reaction, variable viscosity, thermophoresis and heat generation/ absorption on a boundary layer hydro magnetic flow with heat and mass transfer over a heat surface. *Acta Mech.* 2005; **177**, 1-18.
[22] S Lee and S D George. Magneto hydrodynamic steady flow computations in three dimensions. *Int. J. Num. Meth. Fluid.* 1991; **13**, 917-36.
[23] KD Singh and R Sharma. Three dimensional free convective flow and heat transfer through a porous medium with periodic permeability. *Indian J. Pure Appl. Math.* 2002; **33**, 941-9.
[24] GD Gupta and R Johari. MHD three dimensional flow past a porous plate. *Indian J. Pure Appl. Math.* 2001; **32**, 377-86.
[25] KD Singh and R Sharma. Three dimensional couette flow through porous medium with heat transfer. *Indian J. Pure Appl. Math.* 2001; **32**, 1819-929.
[26] NC Jain and P Gupta. Three dimensional free convection couette flow with transpiration cooling. *J. Zhejiang Univ. Sci. A* 2006; **7**, 340-6.
[27] P Singh, VP Sharma and UN Misra. Three-dimensional free convection flow and heat transfer along porous vertical plate. *Appl. Sci. Res.* 1978; **34**, 105-15.
[28] SS Das, M Mohanty, JP Panda and SK Sahoo. Hydromagnetic three dimensional couette flow and heat transfer. *J. Nav. Architect. Mar. Eng.* 2008; **5**, 1-10.
[29] KD Singh and R Sharma. MHD three dimensional couette flow with transpiration cooling. *ZAMM-J. Appl Math. Mech.* 2001; **81**, 715-20.
[30] NA Reddy, MC Raju and SVK Varma. Soret effects on MHD three dimensional free convection couette flow with heat and mass transfer in presence of heat sink. *Int. J. Fluid Mech*. 2010; **2**, 51-60.

Appendix

$a_0 = P_r R_e; \quad b_0 = R_e A; \quad c_0 = S_c R_e; \quad d_0 = R_e S_c K_r; \quad e_0 = R_e; \quad f_0 = 4R_e M;$

$g_0 = \sqrt{a_0^2 - 4(b_0^2 - \pi^2)}; \quad h_0 = S_c S_0;$

$A_1 = \sqrt{e_0^2 + f_0}; \quad B = \frac{1}{2}[A_1 - e_0]; C = \frac{1}{2}[A_1 + e_0]; \; F = \frac{e_0^2}{2} + \frac{f_0}{4} + 4\pi^2;$

$G = \frac{1}{2} e_0 A; \; D = \sqrt{F - G}; \quad E = \sqrt{F + G};$

$$t_1 = \frac{a_0 + \sqrt{a_0^{\,2} - 4b_0}}{2}; t_2 = \frac{a_0 - \sqrt{a_0^{\,2} - 4b_0}}{2}; \quad t_3 = \frac{c_0 - \sqrt{c_0^{\,2} - 4d_0}}{2};$$

$$t_4 = \frac{c_0 + \sqrt{c_0^{\,2} - 4d_0}}{2}; \; t_5 = \frac{e_0 + \sqrt{e_0^{\,2} + f_0}}{2}; \; t_6 = \frac{e_0 - \sqrt{e_0^{\,2} + f_0}}{2};$$

$$t_7 = \frac{1}{2}[D - B]; \quad t_8 = \frac{1}{2}[D + B]; \quad t_9 = \frac{1}{2}[E + C]; \quad t_{10} = \frac{1}{2}[E - C];$$

$$t_{11} = \frac{c_0 + \sqrt{c_0^{\,2} - 4(d_0 - \pi^2)}}{2}; \quad t_{12} = \frac{c_0 - \sqrt{c_0^{\,2} - 4(d_0 - \pi^2)}}{2};$$

$$t_{13} = t_3 + t_7; \quad t_{14} = t_4 + t_7; \quad t_{15} = t_3 + t_8; \quad t_{16} = t_4 + t_8;$$

$$t_{17} = t_3 + t_9; \quad t_{18} = t_4 + t_9; \quad t_{19} = t_4 + t_{10}; \quad t_{20} = t_4 + t_{10};$$

$$t_{21} = (a_0 + g_0)/2; \quad t_{22} = (a_0 - g_0)/2; \quad t_{23} = t_1 + t_7;$$

$$t_{24} = t_2 + t_7; \quad t_{25} = t_1 + t_8; \quad t_{26} = t_2 + t_8; \quad t_{27} = t_1 + t_9;$$

$$t_{28} = t_2 + t_9; \quad t_{29} = t_1 + t_{10}; \; t_{30} = t_2 + t_{10}; \quad t_{31} = t_5 + t_7;$$

$$t_{32} = t_6 + t_7; \; t_{33} = t_5 + t_8; \quad t_{34} = t_6 + t_8; \quad t_{35} = t_5 + t_9;$$

$$t_{36} = t_6 + t_9; \quad t_{37} = t_5 + t_{10}; \quad t_{38} = t_6 + t_{10};$$

$$t_{39} = \frac{e_0 + \sqrt{e_0^2 + 4\pi^2 + f_0}}{2}; \qquad t_{40} = \frac{e_0 - \sqrt{e_0^2 + 4\pi^2 + f_0}}{2};$$

$$c_1 = \frac{e^{t_2} - n}{e^{t_2} - e^{t_1}};\ c_2 = \frac{n - e^{t_2}}{e^{t_2} - e^{t_1}};\ c_5 = \frac{-ScSoC_1t_1^2}{t_1^2 - c_0t_1 + d_0};\ c_6 = \frac{-ScSoC_2t_2^2}{t_2^2 - c_0t_2 + d_0};$$

$$K_1 = 1 - C_5 - C_6;\ K_2 = r - C_5e^{t_1} - C_6e^{t_2};\ c_3 = \frac{K_1e^{t_4} - K_2}{e^{t_4} - e^{t_3}};\ c_4 = \frac{K_2 - K_1e^{t_3}}{e^{t_4} - e^{t_3}};$$

$$C_9 = \frac{\mathrm{Re}\,GrC_1 + \mathrm{Re}\,GmC_5}{\mathrm{Re}\,M + \mathrm{Re}\,t_1 - t_1^2};\ C_{10} = \frac{\mathrm{Re}\,GrC_2 + \mathrm{Re}\,GmC_6}{\mathrm{Re}\,M + \mathrm{Re}\,t_2 - t_2^2};\quad C_{11} = \frac{\mathrm{Re}\,GmC_3}{\mathrm{Re}\,M + \mathrm{Re}\,t_3 - t_3^2};$$

$$C_{12} = \frac{\mathrm{Re}\,GmC_4}{\mathrm{Re}\,M + \mathrm{Re}\,t_4 - t_4^2};\quad L_1 = C_9 + C_{10} + C_{11} + C_{12};$$

$$L_2 = C_9e^{t_1}(1 - ht_1) + C_{10}e^{t_2}(1 - ht_2) + C_{11}e^{t_3}(1 - ht_3) + C_{12}e^{t_4}(1 - ht_4) - 1;$$

$$L_{21} = e^{t_5}(1 - ht_5);\ L_{22} = e^{t_6}(1 - ht_6);\ c_7 = \frac{L_2 - L_1L_{22}}{L_{22} - L_{21}};\quad c_8 = \frac{L_1L_{21} - L_2}{L_{22} - L_{21}};$$

$$D_r = (e^{t_7} - e^{t_8})(e^{t_{10}} - e^{t_9})(t_7t_8 + t_9t_{10}) + (e^{t_9} - e^{t_7})(e^{t_{10}} - e^{t_8})(t_7t_9 + t_8t_{10}) + (e^{t_8} - e^{t_9})(e^{t_{10}} - e^{t_7})(t_9t_8 + t_7t_{10});$$

$$N_rC_{13} = e^{t_9 + t_{10}}(t_8t_{10} - t_8t_9) + e^{t_8 + t_{10}}(t_8t_9 - t_{10}t_9) + e^{t_9 + t_8}(t_9t_{10} - t_8t_{10});$$

$$N_rC_{14} = e^{t_9 + t_{10}}(t_7t_9 - t_7t_{10}) + e^{t_7 + t_{10}}(t_{10}t_9 - t_7t_9) + e^{t_9 + t_7}(t_7t_{10} - t_9t_{10});$$

$$N_rC_{15} = e^{t_8 + t_{10}}(t_7t_{10} - t_8t_7) + e^{t_7 + t_{10}}(t_8t_7 - t_{10}t_8) + e^{t_7 + t_8}(t_8t_{10} - t_7t_{10});$$

$$N_rC_{16} = e^{t_9 + t_8}(t_8t_7 - t_7t_9) + e^{t_7 + t_9}(t_8t_9 - t_7t_8) + e^{t_7 + t_8}(t_9t_7 - t_8t_9);$$

$$C_{13} = (N_rC_{13})/D_r;\quad C_{14} = (N_rC_{14})/D_r;\quad C_{15} = (N_rC_{15})/D_r;\quad C_{16} = (N_rC_{16})/D_r;$$

$$a = a_0C_1C_{13}t_1;\quad b = a_0C_2C_{13}t_2;\quad c = a_0C_1C_{14}t_1;\quad d = a_0C_{12}C_{14}t_2;$$

$$e = a_0C_1C_{15}t_1;\ f = a_0C_2C_{15}t_2;\quad g = a_0C_1C_{16}t_1;\quad h = a_0C_2C_{16}t_2;$$

$$c_{17} = \frac{-t_7C_{13}}{\pi};\ c_{18} = \frac{-t_8C_{14}}{\pi};\ c_{19} = \frac{-t_9C_{15}}{\pi};\ c_{20} = \frac{-t_{10}C_{16}}{\pi};$$

$$c_{23} = \frac{a}{t_{23}^2 - a_0t_{23} + (b_0 - \pi^2)};\ C_{24} = \frac{b}{t_{24}^2 - a_0t_{24} + (b_0 - \pi^2)};$$

$$C_{25} = \frac{c}{t_{25}^2 - a_0t_{25} + (b_0 - \pi^2)};\ C_{26} = \frac{d}{t_{26}^2 - a_0t_{26} + (b_0 - \pi^2)};$$

$$C_{27} = \frac{e}{t_{27}^2 - a_0 t_{27} + (b_0 - \pi^2)}; \; C_{28} = \frac{f}{t_{28}^2 - a_0 t_{28} + (b_0 - \pi^2)};$$

$$C_{29} = \frac{g}{t_{29}^2 - a_0 t_{29} + (b_0 - \pi^2)}; \quad C_{30} = \frac{h}{t_{30}^2 - a_0 t_{30} + (b_0 - \pi^2)};$$

$$M_2 = C_{23}e^{t_{23}} + C_{24}e^{t_{24}} + C_{25}e^{t_{25}} + C_{26}e^{t_{26}} + C_{27}e^{t_{27}} + C_{28}e^{t_{28}} + C_{29}e^{t_{29}} + C_{30}e^{t_{30}};$$

$$c_{21} = \frac{M_2 - M_1 e^{t_{22}}}{e^{t_{22}} - e^{t_{21}}}; \qquad c_{22} = \frac{M_1 e^{t_{21}} - M_2}{e^{t_{22}} - e^{t_{21}}};$$

$$C_{33} = \frac{\mathrm{Re}\, Sc C_3 C_{13} t_3}{t_{13}^2 - \mathrm{Re}\, Sc t_{13} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{34} = \frac{\mathrm{Re}\, Sc C_4 C_{13} t_4}{t_{14}^2 - \mathrm{Re}\, Sc t_{14} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{35} = \frac{\mathrm{Re}\, Sc C_3 C_{14} t_3}{t_{15}^2 - \mathrm{Re}\, Sc t_{15} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{36} = \frac{\mathrm{Re}\, Sc C_4 C_{14} t_4}{t_{16}^2 - \mathrm{Re}\, Sc t_{63} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{37} = \frac{\mathrm{Re}\, Sc C_3 C_{15} t_3}{t_{17}^2 - \mathrm{Re}\, Sc t_{17} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{38} = \frac{\mathrm{Re}\, Sc C_4 C_{15} t_4}{t_{18}^2 - \mathrm{Re}\, Sc t_{18} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{39} = \frac{\mathrm{Re}\, Sc C_3 C_{16} t_3}{t_{19}^2 - \mathrm{Re}\, Sc t_{19} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{40} = \frac{\mathrm{Re}\, Sc C_4 C_{16} t_4}{t_{20}^2 - \mathrm{Re}\, Sc t_{20} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{41} = \frac{- Sc So C_{21} (t_{21}^2 - \pi^2)}{t_{21}^2 - \mathrm{Re}\, Sc t_{21} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{42} = \frac{- Sc So C_{22} (t_{22}^2 - \pi^2)}{t_{22}^2 - \mathrm{Re}\, Sc t_{22} + (Kr \,\mathrm{Re}\, Sc - \pi^2)};$$

$$C_{43} = \frac{\operatorname{Re} ScC_5C_{13}t_1 - ScSoC_{23}(t_{23}^2 - \pi^2)}{t_{23}^2 - \operatorname{Re} Sct_{23} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{44} = \frac{\operatorname{Re} ScC_6C_{13}t_2 - ScSoC_{24}(t_{24}^2 - \pi^2)}{t_{24}^2 - \operatorname{Re} Sct_{24} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{45} = \frac{\operatorname{Re} ScC_5C_{14}t_1 - ScSoC_{25}(t_{25}^2 - \pi^2)}{t_{25}^2 - \operatorname{Re} Sct_{25} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{46} = \frac{\operatorname{Re} ScC_6C_{14}t_2 - ScSoC_{26}(t_{26}^2 - \pi^2)}{t_{26}^2 - \operatorname{Re} Sct_{26} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{47} = \frac{\operatorname{Re} ScC_5C_{15}t_1 - ScSoC_{27}(t_{27}^2 - \pi^2)}{t_{27}^2 \operatorname{Re} Sct_{273} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{48} = \frac{\operatorname{Re} ScC_6C_{15}t_2 - ScSoC_{28}(t_{28}^2 - \pi^2)}{t_{28}^2 - \operatorname{Re} Sct_{28} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{49} = \frac{\operatorname{Re} ScC_5C_{16}t_1 - ScSoC_{29}(t_{29}^2 - \pi^2)}{t_{29}^2 - \operatorname{Re} Sct_{29} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$C_{50} = \frac{\operatorname{Re} ScC_6C_{16}t_2 - ScSoC_{30}(t_{30}^2 - \pi^2)}{t_{30}^2 - \operatorname{Re} Sct_{30} + (Kr \operatorname{Re} Sc - \pi^2)};$$

$$\begin{aligned} n_1 = {} & C_{33} + C_{34} + C_{35} + C_{36} + C_{37} + C_{38} + C_{39} + C_{40} + C_{41} + \\ & C_{42} + C_{43} + C_{44} + C_{45} + C_{46} + C_{47} + C_{48} + C_{49} + C_{50}; \end{aligned}$$

$$\begin{aligned} n_2 = {} & C_{33}e^{t_{13}} + C_{34}e^{t_{14}} + C_{35}e^{t_{15}} + C_{36}e^{t_{16}} + C_{37}e^{t_{17}} + C_{38}e^{t_{18}} + C_{39}e^{t_{19}} \\ & + C_{40}e^{t_{20}} + C_{41}e^{t_{21}} + C_{42}e^{t_{22}} + C_{43}e^{t_{23}} + C_{44}e^{t_{24}} + C_{45}e^{t_{25}} \\ & + C_{46}e^{t_{26}} + C_{47}e^{t_{27}} + C_{48}e^{t_{28}} + C_{49}e^{t_{29}} + C_{50}e^{t_{30}} \end{aligned}$$

$$C_{31} = \frac{n_2 - n_1 e^{t_{12}}}{e^{t_{12}} - e^{t_{11}}};\ C_{32} = \frac{n_1 e^{t_{11}} - n_2}{e^{t_{12}} - e^{t_{11}}};\ C_{51} = \frac{-Gm\,\mathrm{Re}\,C_{31}}{t_{11}^2 - \mathrm{Re}\,t_{11} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{52} = \frac{-Gm\,\mathrm{Re}\,C_{32}}{t_{12}^2 - \mathrm{Re}\,t_{12} - (\pi^2 + M\,\mathrm{Re})}; \qquad C_{53} = \frac{\mathrm{Re}\,C_{13}C_{11}t_3 - Gm\,\mathrm{Re}\,C_{33}}{t_{13}^2 - \mathrm{Re}\,t_{13} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{55} = \frac{\mathrm{Re}\,C_{14}C_{11}t_3 - Gm\,\mathrm{Re}\,C_{35}}{t_{15}^2 - \mathrm{Re}\,t_{15} - (\pi^2 + M\,\mathrm{Re})}; \qquad C_{56} = \frac{\mathrm{Re}\,C_{14}C_{12}t_4 - Gm\,\mathrm{Re}\,C_{36}}{t_{16}^2 - \mathrm{Re}\,t_{16} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{57} = \frac{\mathrm{Re}\,C_{15}C_{11}t_3 - Gm\,\mathrm{Re}\,C_{37}}{t_{17}^2 - \mathrm{Re}\,t_{17} - (\pi^2 + M\,\mathrm{Re})}; \qquad C_{58} = \frac{\mathrm{Re}\,C_{15}C_{12}t_4 - Gm\,\mathrm{Re}\,C_{38}}{t_{18}^2 - \mathrm{Re}\,t_{18} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{59} = \frac{\mathrm{Re}\,C_{16}C_{11}t_3 - Gm\,\mathrm{Re}\,C_{39}}{t_{19}^2 - \mathrm{Re}\,t_{19} - (\pi^2 + M\,\mathrm{Re})}; \qquad C_{60} = \frac{\mathrm{Re}\,C_{16}C_{12}t_4 - Gm\,\mathrm{Re}\,C_{40}}{t_{20}^2 - \mathrm{Re}\,t_{20} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{61} = \frac{-Gr\,\mathrm{Re}\,C_{21} - Gm\,\mathrm{Re}\,C_{41}}{t_{21}^2 - \mathrm{Re}\,t_{21} - (\pi^2 + M\,\mathrm{Re})}; \qquad C_{62} = \frac{-Gr\,\mathrm{Re}\,C_{22} - Gm\,\mathrm{Re}\,C_{42}}{t_{22}^2 - \mathrm{Re}\,t_{22} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{63} = \frac{\mathrm{Re}\,C_{13}C_9t_1 - Gr\,\mathrm{Re}\,C_{23} - Gm\,\mathrm{Re}\,C_{43}}{t_{23}^2 - \mathrm{Re}\,t_{23} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{64} = \frac{\mathrm{Re}\,C_{13}C_{10}t_2 - Gr\,\mathrm{Re}\,C_{24} - Gm\,\mathrm{Re}\,C_{44}}{t_{24}^2 - \mathrm{Re}\,t_{24} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{65} = \frac{\mathrm{Re}\,C_{14}C_9t_1 - Gr\,\mathrm{Re}\,C_{25} - Gm\,\mathrm{Re}\,C_{45}}{t_{25}^2 - \mathrm{Re}\,t_{25} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{66} = \frac{\mathrm{Re}\,C_{14}C_{10}t_2 - Gr\,\mathrm{Re}\,C_{26} - Gm\,\mathrm{Re}\,C_{46}}{t_{26}^2 - \mathrm{Re}\,t_{26} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{67} = \frac{\mathrm{Re}\,C_{15}C_9t_1 - Gr\,\mathrm{Re}\,C_{27} - Gm\,\mathrm{Re}\,C_{47}}{t_{27}^2 - \mathrm{Re}\,t_{27} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{68} = \frac{\mathrm{Re}\,C_{15}C_{10}t_2 - Gr\,\mathrm{Re}\,C_{28} - Gm\,\mathrm{Re}\,C_{48}}{t_{28}^2 - \mathrm{Re}\,t_{28} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{69} = \frac{\mathrm{Re}\,C_{16}C_9t_1 - Gr\,\mathrm{Re}\,C_{29} - Gm\,\mathrm{Re}\,C_{49}}{t_{29}^2 - \mathrm{Re}\,t_{29} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{70} = \frac{\mathrm{Re}\,C_{16}C_{10}t_2 - Gr\,\mathrm{Re}\,C_{30} - Gm\,\mathrm{Re}\,C_{50}}{t_{30}^2 - \mathrm{Re}\,t_{30} - (\pi^2 + M\,\mathrm{Re})};$$

$$C_{71} = \frac{\mathrm{Re}\, C_{13} C_7 t_5}{t_{31}^2 - \mathrm{Re}\, t_{31} - (\pi^2 + M\, \mathrm{Re})}; \quad C_{72} = \frac{\mathrm{Re}\, C_{13} C_8 t_6}{t_{32}^2 - \mathrm{Re}\, t_{32} - (\pi^2 + M\, \mathrm{Re})};$$

$$C_{73} = \frac{\mathrm{Re}\, C_{14} C_7 t_5}{t_{33}^2 - \mathrm{Re}\, t_{33} - (\pi^2 + M\, \mathrm{Re})}; \quad C_{74} = \frac{\mathrm{Re}\, C_{14} C_8 t_6}{t_{34}^2 - \mathrm{Re}\, t_{34} - (\pi^2 + M\, \mathrm{Re})};$$

$$C_{75} = \frac{\mathrm{Re}\, C_{15} C_7 t_5}{t_{35}^2 - \mathrm{Re}\, t_{35} - (\pi^2 + M\, \mathrm{Re})}; \quad C_{76} = \frac{\mathrm{Re}\, C_{15} C_8 t_6}{t_{36}^2 - \mathrm{Re}\, t_{36} - (\pi^2 + M\, \mathrm{Re})};$$

$$C_{77} = \frac{\mathrm{Re}\, C_{16} C_7 t_5}{t_{37}^2 - \mathrm{Re}\, t_{37} - (\pi^2 + M\, \mathrm{Re})}; \quad C_{78} = \frac{\mathrm{Re}\, C_{16} C_8 t_6}{t_{38}^2 - \mathrm{Re}\, t_{38} - (\pi^2 + M\, \mathrm{Re})};$$

$$\begin{aligned} R_1 = {} & C_{51} + C_{52} + C_{53} + C_{54} + C_{55} + C_{56} + C_{57} + C_{58} + C_{59} + C_{60} + C_{61} + C_{62} + C_{63} \\ & + C_{64} + C_{65} + C_{66} + C_{67} + C_{68} + C_{69} + C_{70} + C_{71} + C_{72} + C_{73} + C_{74} + C_{75} \\ & + C_{76} + C_{77} + C_{78}; \end{aligned}$$

$$\begin{aligned} R_2 = {} & C_{51} e^{t_{11}} (1 - h t_{11}) + C_{52} e^{t_{12}} (1 - h t_{12}) + C_{53} e^{t_{13}} (1 - h t_{13}) \\ & + C_{54} e^{t_{14}} (1 - h t_{14}) + C_{55} e^{t_{15}} (1 - h t_{15}) + C_{56} e^{t_{16}} (1 - h t_{16}) \\ & + C_{57} e^{t_{17}} (1 - h t_{17}) + C_{58} e^{t_{18}} (1 - h t_{18}) + C_{59} e^{t_{19}} (1 - h t_{19}) \\ & + C_{60} e^{t_{20}} (1 - h t_{20}) + C_{61} e^{t_{21}} (1 - h t_{21}) + C_{62} e^{t_{22}} (1 - h t_{22}) \\ & + C_{63} e^{t_{23}} (1 - h t_{23}) + C_{64} e^{t_{24}} (1 - h t_{24}) + C_{65} e^{t_{25}} (1 - h t_{25}) \\ & + C_{66} e^{t_{26}} (1 - h t_{26}) + C_{67} e^{t_{27}} (1 - h t_{27}) + C_{68} e^{t_{28}} (1 - h t_{28}) \\ & + C_{69} e^{t_{29}} (1 - h t_{29}) + C_{70} e^{t_{30}} (1 - h t_{30}) + C_{71} e^{t_{31}} (1 - h t_{31}) \\ & + C_{72} e^{t_{32}} (1 - h t_{32}) + C_{73} e^{t_{33}} (1 - h t_{33}) + C_{74} e^{t_{34}} (1 - h t_{34}) \\ & + C_{75} e^{t_{35}} (1 - h t_{35}) + C_{76} e^{t_{36}} (1 - h t_{36}) + C_{77} e^{t_{37}} (1 - h t_{37}) \\ & + C_{78} e^{t_{38}} (1 - h t_{38}) \end{aligned}$$

$$c_{79} = \frac{R_2 - R_1 e^{t_{40}} (1 - h t_{40})}{e^{t_{40}} (1 - h t_{40}) - e^{t_{39}} (1 - h t_{39})}; \quad c_{80} = \frac{R_1 e^{t_{39}} (1 - h t_{39}) - R_2}{e^{t_{40}} (1 - h t_{40}) - e^{t_{39}} (1 - h t_{39})}.$$