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New Analytic Solution of Ito Equation by the Sine-Cosine Method

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Abstract

In this paper, the sine-cosine method is used to construct exact traveling wave solutions of the Ito equation. As a result, many new periodic and solitary wave solutions are derived to generalized (2+1)-dimensional Ito equation. Throughout the paper, all the calculations are made with the aid of the Maple packet program.

Keywords: Sine-cosine method, (2+1)-dimensional Ito equation, periodic solution, exact solution

Introduction

In the field of nonlinear science, nonlinear evolution equations (NLEEs) used to describe complex phenomena in various fields of science, especially in physics and plasma. Moreover, studies of finding periodic and solitary wave solutions of the nonlinear equations attracted huge number of works in a variety of fields. Many effective methods of obtaining explicit solutions of nonlinear partial differential equations (NPDEs) have been presented such as the tanh-method [1], the extended tanh-method [2], the sine-cosine method [3], the homogeneous balance method [4], the homotopy analysis method [5-6], the expansion method [7], the three-wave method [8-10], the extended homoclinic test approach (EHTA) [11-13], the expansion method [14] and the exp-function method [15-18]. After all, an exact solution is an essential necessity in this field.

While numerical simulations provide a visual effect of such NLEEs, exact solutions always provide a better analytical insight into these equations such as stability issues, constraint relations between the parameters that the numerical codes are unable to depict. Therefore, it is imperative to address the analytical aspects of NLEEs in parallel to numerical studies.

In this paper, the traveling wave solutions obtained via the sine-cosine method are expressed by hyperbolic functions and the trigonometric functions. In addition the solitary wave solutions are obtained from the generalized (2+1)-dimensional Ito equation when the choice of parameters are taken at special values. In the following section we have a brief review on the sine-cosine method and in Section 3, we apply the Sine-cosine method to obtain analytic solutions of the generalized (2+1)-dimensional Ito equation. Finally, the paper is concluded in Section 4.

The Sine-cosine method

We introduce the wave variable $\xi = x - ct$ into the PDE;

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \ldots) = 0, \tag{1}$$

where $u(x,t)$ is the traveling wave solution. This enables us to use the following changes;

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \ldots \tag{2}$$

One can immediately reduce the NPDE (1) into a nonlinear ordinary differential equation (NODE);

$$Q(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \tag{3}$$

The ordinary differential Eq. (3) is then integrated as long as all terms contain derivatives, where we neglect integration constants.

The solutions of many nonlinear equations can be expressed in the form;

$$u(x,t) = \begin{cases} \lambda sin^\beta(\mu\xi), & |\xi| \le \dfrac{\pi}{\mu}, \\ 0 & otherwise, \end{cases} \tag{4}$$

or in the form;

$$u(x,t) = \begin{cases} \lambda cos^\beta(\mu\xi), & |\xi| \le \dfrac{\pi}{2\mu}, \\ 0 & otherwise, \end{cases} \tag{5}$$

where λ, μ and $\beta \neq 0$ are parameters that will be determined. μ and c are the wave number and the wave speed respectively. We use;

$$\begin{aligned} &u(\xi) = \lambda sin^\beta(\mu\xi), \\ &u^n(\xi) = \lambda^n sin^{n\beta}(\mu\xi), \\ &(u^n)_\xi = n\mu\beta\lambda^n \cos(\mu\xi)\sin^{n\beta-1}(\mu\xi), \\ &(u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) \\ &\qquad + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi), \end{aligned} \tag{6}$$

and the derivatives of (5) becomes;

$$u(\xi) = \lambda cos^{\beta}(\mu\xi),$$

$$u^n(\xi) = \lambda^n cos^{n\beta}(\mu\xi),$$

$$(u^n)_{\xi} = -n\mu\beta\lambda^n \sin(\mu\xi)\cos^{n\beta-1}(\mu\xi), \tag{7}$$

$$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n\cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi),$$

and so on for other derivatives.

We substitute (6) or (7) into the reduced equation obtained above in (3), balance the terms of the cosine functions when (7) is used, or balance the terms of the sine functions when (6) is used, and solving the resulting system of algebraic equations by using computerized symbolic calculations. We next collect all terms with the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns μ, β and λ. We obtained all possible values of the parameters μ, β and λ [3].

The generalized (2+1)-dimensional Ito equation

In this section we apply the sine-cosine method to the (2+1)-dimensional Ito equation [19];

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + uu_{xt}) + 3u_{xx}\int_{-\infty}^{x} u_t dx' + au_{yt} + bu_{xt} = 0, \tag{8}$$

We next use the transformation $u(x, y, t) = v_x(x, y, t)$, this transformation changes the (2+1)-dimensional Ito Eq. (8) to the

$$v_{ttx} + v_{xxxxt} + 3(2v_{xx}v_{xt} + v_x v_{xxt}) + 3v_{xxx}v_t + au_{xyt} + bu_{xxt} = 0. \tag{9}$$

After that we use the transformation;

$$v = \varphi(\xi) \quad , \quad \xi = x + y - ct \tag{10}$$

where c is a constant. Therefore Eq. (9) converts to;

$$c^2\varphi''' - c\varphi^{(5)} - 6c(\varphi'')^2 - 6c\varphi'\varphi''' - ac\varphi''' - bc\varphi''' = 0, \tag{11}$$

or equivalently;

$$c(c-(a+b))\varphi''' - c\varphi^{(5)} - 3\left[c(\varphi')^2\right]'' = 0, \tag{12}$$

where by integrating twice we obtain;

$$(c-(a+b))\varphi' - \varphi''' - 3(\varphi')^2 = 0, \tag{13}$$

setting $\varphi'(\xi) = \phi(\xi)$, Eq. (13) becomes;

$$(c-(a+b))\phi - \phi'' - 3\phi^2 = 0. \tag{14}$$

Substituting (4) into (14) gives;
$$\begin{aligned}&(c-(a+b))\lambda sin^{\beta}(\mu\xi) + \mu^2\beta^2\lambda \sin^{\beta}(\mu\xi)\\ &-\mu^2\lambda\beta(\beta-1)\sin^{\beta-2}(\mu\xi) - 3\lambda^2 sin^{2\beta}(\mu\xi) = 0.\end{aligned}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following system of algebraic equations;

$$\begin{aligned}&(\beta-1)\neq 0,\\ &\beta-2 = 2\beta,\\ &(c-(a+b))\lambda + \mu^2\beta^2\lambda = 0,\\ &-\lambda\mu^2\beta(\beta-1) - 3\lambda^2 = 0,\end{aligned} \tag{15}$$

Solving the system (15) yields;

$$\beta = -2, \mu = \frac{1}{2}\sqrt{(a+b)-c}, \lambda = \frac{1}{2}(c-(a+b)),$$

where c is a free parameter. Hence, for $c < 0$, the following periodic solutions;

$$\phi_1(\xi) = \frac{c-(a+b)}{2}\csc^2\left[\frac{\sqrt{(a+b)-c}}{2}\xi\right] \tag{16}$$

where $0 < \frac{1}{2}\sqrt{(a+b)-c}\,\xi < \pi$, and

$$\phi_2(\xi) = \frac{c-(a+b)}{2}\sec^2\left[\frac{\sqrt{(a+b)-c}}{2}\xi\right] \tag{17}$$

where $|\frac{1}{2}\sqrt{(a+b)-c}\,\xi| < \frac{\pi}{2}$.

In view of these results and recall $\varphi'(\xi) = \phi(\xi)$, integrating (16) and (17) with respect to ξ and considering the zero constants for integration we obtain;

$$\begin{aligned}\varphi_1(\xi) &= \sqrt{(a+b)-c}\cot\left[\frac{\sqrt{(a+b)-c}}{2}\xi\right],\\ \varphi_2(\xi) &= -\sqrt{(a+b)-c}\tan\left[\frac{\sqrt{(a+b)-c}}{2}\xi\right].\end{aligned} \tag{18}$$

Recall that $v(x,y,t) = \varphi(\xi)$ and $u(x,y,t) = v_x(x,y,t)$ we get;

$$\begin{aligned}u_1(\xi) &= \frac{c-(a+b)}{2}\left[1+\cot^2\left(\frac{\sqrt{(a+b)-c}}{2}\xi\right)\right]\\ &= \frac{c-(a+b)}{2}csc^2\left[\frac{\sqrt{-c}}{2}(\xi)\right],\\ u_2(\xi) &= \frac{c-(a+b)}{2}\left[1+\tan^2\left(\frac{\sqrt{(a+b)-c}}{2}\xi\right)\right]\\ &= \frac{c-(a+b)}{2}sec^2\left[\frac{\sqrt{-c}}{2}(\xi)\right].\end{aligned} \tag{19}$$

Working with Maple interactively, we proved our solutions (19) is exact. However, for $c > 0$, the following periodic solutions;

$$\phi_3(\xi) = -\frac{c-(a+b)}{2}csch^2\left[\frac{\sqrt{c-(a+b)}}{2}\xi\right] \tag{20}$$

$$\phi_4(\xi) = \frac{c-(a+b)}{2}sech^2\left[\frac{\sqrt{c-(a+b)}}{2}\xi\right] \tag{21}$$

In view of these results and recall $\varphi'(\xi) = \phi(\xi)$, integrating (20) and (21) with respect to ξ and considering the zero constants for integration we obtain;

$$\varphi_3(\xi) = \sqrt{c-(a+b)}coth\left[\frac{\sqrt{c-(a+b)}}{2}\xi\right],$$

$$\varphi_4(\xi) = \sqrt{c-(a+b)}tanh\left[\frac{\sqrt{c-(a+b)}}{2}\xi\right]. \tag{22}$$

Recall that $v(x,y,t) = \varphi(\xi)$ and $u(x,y,t) = v_x(x,y,t)$
we get;

$$u_3(\xi) = \frac{c-(a+b)}{2}\left[1-coth^2\left(\frac{\sqrt{c-(a+b)}}{2}\xi\right)\right]$$
$$= -\frac{c-(a+b)}{2}csch^2\left[\frac{\sqrt{c-(a+b)}}{2}\xi\right],$$

$$u_4(\xi) = \frac{c-(a+b)}{2}\left[1-tanh^2\left(\frac{\sqrt{c-(a+b)}}{2}\xi\right)\right]$$
$$= \frac{c-(a+b)}{2}sech^2\left[\frac{\sqrt{c-(a+b)}}{2}\xi\right]. \tag{23}$$

Working with Maple interactively, we proved our solutions (23) is exact. The solutions are compared with the solutions obtained by Wazwaz [19], and it is found that the solutions obtained are exactly same as determined by Khani [20].

Conclusions

In this paper, we obtained exact solutions for the generalized (2+1)-dimensional Ito equation by means of the sine-cosine method. This paper is shown that the sine-cosine method provides a very effective and powerful mathematical tool to seek more new exact solutions of NPDEs in mathematical physics.

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