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New Results for Boundary Layer Flow and Convection Heat Transfer Over a Flat Plate by Using the Homotopy Perturbation Method

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Abstract

This work presents a boundary-layer analysis of an incompressible viscous steady flow and forced convection over a horizontal flat plate. The solution for velocity and temperature are calculated by applying the Homotopy perturbation method (HPM). A special technique is attempted by which one is able to obtain solutions that are close to the exact solution of the equation. The obtained results are compared to the exact solution and another results provided by previous works so that the high accuracy of the obtained results is clear. Also, the results reveal that this method is effective, simple, and can be applied for other nonlinear problems in different fields of science and engineering, especially some fluid mechanics and heat transfer equations.

Keywords: Boundary layer, heat transfer, Howarth number, Homotopy perturbation method

Introduction

There are few phenomena in different fields of science which occur linearly. Most problems and scientific phenomena, such as heat transfer, are inherently nonlinear. Except for a limited number of these problems, most of them do not have precise analytical solutions; therefore, these nonlinear equations should be solved using approximate analytical solutions. Analytical methods have always been of interest to scientists. Perturbation method is one of the most well-known methods to solve nonlinear problems; it is based on the existence of small or large parameters, the so-called perturbation quantity [1,2]. Since there are some limits with the common perturbation method, and because the basis of the common perturbation method is the existence of a small parameter, developing the method for different uses is difficult. Therefore, many different new methods have recently introduced to eliminate the small parameter, such as the Exp-function method [3], the artificial small parameter method [4], Adomian's decomposition method [5], the Variational iteration method (VIM) [6], and the Homotopy analysis method (HAM) [7]. One of the strong analytical methods for eliminating small parameters is applying Homotopy perturbation method (HPM) [8-11]. The HPM depends on coupling the classic perturbation method and the Homotopy method in topology. The basic idea of the HPM was proposed by He [12-14] and was successfully applied to various engineering problems. HPM is the most effective and convenient method for both linear and nonlinear equations. This method has eliminated limitations of the traditional perturbation techniques. In this paper, the mathematical model of this method is introduced, and its application in boundary-layer is studied. Also, the solution for velocity and temperature are calculated by applying the HPM. In addition, the Howarth number is calculated. The Howarth number is an important number in fluid mechanics. It is used for calculating drag coefficient. Howarth [15] obtained an accurate numerical solution for the Blasius equation in which $\alpha = f''(0) = 0.332057$. This value is acceptable for comparison because it has high accuracy. After that, several attempts are made for calculating this number from other methods. He [16] in 1998 solved this equation by applying Variational iteration method, and found $\alpha = 0.5436$ by a first approximation. This value has a 63.7 % relative error with respect to Howarth's calculation. In 2007, Wazwaz [17] used the same method, and found $\alpha = 0.37329$.

This value has 21.42 % relative error with respect to Howarth. In 2003 and 2004, by using the Homotopy method, He [18,19] obtained a first iteration step leading to $\alpha = 0.3095$ with a 6.8 % accuracy (relative error), and a second iteration step which yielded $\alpha = 0.3296$ with a 0.7 % accuracy of the initial slope. In 2007 Ganji [20] found $\alpha = 0.348505$ by the HPM with a three term approximation with a 4.9 % relative error. In 2009 Fathizadeh and Rashidi [21] found $\alpha = 0.348$ by the HPM with a 4.9 % relative error. In this paper, the problem is solved by using a special technique HPM, and the results compared with previous works. The approximations of f ″(0) obtained by this paper in comparison with previous HPM results provide the higher accuracy.

Basic idea of Homotopy perturbation method

To explain the basic ideas of this method, consider the following equation;

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{1}$$

with the boundary condition of;

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \Gamma. \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts, which are L and N, where L is linear and N is nonlinear Eq. (1) can therefore be rewritten as follows;

$$L(u) + N(u) - f(r) = 0,\ r \in \Omega. \tag{3}$$

Homotopy perturbation structure is shown as follows;

$$\begin{aligned} H(\nu, p) &= (1-p)\left[L(\nu) - L(u_0)\right] \\ &+ p\left[A(\nu) - f(r)\right] = 0. \end{aligned} \tag{4}$$

where

$$\nu(r, p):\ \Omega \times [0,1] \to R. \tag{5}$$

In Eq. (4), $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. It can be assumed the solution of Eq. (4) can be written as a power series in p, as follows;

$$\nu = \nu_0 + p^1 \nu_1 + p^2 \nu_2 + \cdots = \sum_{i=0}^{n} \nu_i p^i, \tag{6}$$

and the best approximation for the solution is;

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \cdots \tag{7}$$

The above convergence is discussed in [22].

Governing equations

Boundary layer flow over a flat plate is governed by the continuity and the Navier-Stokes equations. Under the boundary layer assumptions, for a 2-dimensional, steady state, incompressible flow with a zero pressure gradient over a flat plate, the governing equations are simplified to;

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2} \tag{8}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{9}$$

Subjected to boundary conditions;

$$y=0\to u=0, v=0 \tag{10}$$

$$y=\infty\to \mathrm{u}=\mathrm{U}_\infty \tag{11}$$

Under boundary layer assumptions, the energy transport equation is also simplified.

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{12}$$

The thermal boundary conditions for Eq. (12) are;

$$\mathrm{y}=0 \;\to\; \mathrm{T}=\mathrm{T}_w, \tag{13}$$

$$y=\infty\to T=\mathrm{T}_\infty. \tag{14}$$

Here u and v are the velocity components along the flow direction (x-direction) and normal to the flow direction (y-direction), υ is the kinematic viscosity, α is the thermal diffusivity, T is the temperature across the thermal boundary layer, T_w is the constant temperature of the wall, T_∞ is the constant temperature of the ambient fluid and U_∞ is the constant free stream velocity. By applying a dimensionless variable (η) defined as;

$$\eta=\frac{y}{x}\mathrm{Re}^{0.5} \tag{15}$$

Re is the Reynolds number and defined as;

$$(\mathrm{Re}=\frac{\mathrm{U}_\infty x}{\upsilon})$$

The governing Eqs. (8) and (9) can be reduced to the well-known Blasius equation where f is a function of variable (η);

$$f''' + \frac{1}{2} ff'' = 0, \tag{16}$$

with boundary conditions;

$$\eta = 0 \rightarrow f = 0, f' = 0, \tag{17}$$
$$\eta = \infty \rightarrow f' = 1. \tag{18}$$

where f' is related to the u velocity by;

$$f' = \frac{u}{u_\infty}, \tag{19}$$

and the "prime" denotes the derivatives with respect to η.
Defining the non-dimensional temperature $\theta(\eta)$ and the Prandtl number, Pr, as follows;

$$\theta(\eta) = \frac{T - T_W}{T_\infty - T_W}, \Pr = \frac{\upsilon}{\alpha}. \tag{20}$$

Upon using these variables, the energy equation can be written in non-dimensional form as;

$$\theta'' + \frac{\Pr}{2} f\theta' = 0. \tag{21}$$

The transformed thermal boundary conditions for the energy Eq. (21) are;

$$\eta = 0 \rightarrow \theta = 1, \tag{22}$$
$$\eta = \infty \rightarrow \theta = 0. \tag{23}$$

HPM solution for flow over a flat plate

For a sufficiently large number, M, the conditions (18) and (23) can be replaced by the conditions;

$$f'(M) = 1 \quad , \quad \theta(M) = 0. \tag{24}$$

Under the transformation $z = \frac{\eta}{M}$, equations of momentum and energy are transformed to;

$$\begin{aligned} &g'''(z) + \frac{M^2}{2} g(z)g''(z) = 0, \\ &h''(z) + \frac{M^2}{2} \Pr g(z)h'(z) = 0, \end{aligned} \tag{25}$$

Where $g(z) = \frac{f(\eta)}{M}$, $h(z) = \frac{\theta(\eta)}{M}$ and the "prime" denotes the derivatives with respect to $z \in [0,1]$.The boundary conditions (17), (18), (22) and (23) are transformed to;

$$z = 0 \rightarrow g = 0, g' = 0, \tag{26}$$
$$z = 1 \rightarrow g' = 1, \tag{27}$$

$$z = 0 \to h = \frac{1}{M}, \tag{28}$$

$$z = 1 \to h = 0. \tag{29}$$

In this section, the HPM is applied to nonlinear ordinary differential Eqs.(25). According to the HPM, a Homotopy of Eqs. (25) can be constructed as follows;

$$\begin{aligned} &(1-p)(v'''-g_0''') + p\left(v''' + \frac{M^2}{2} v v''\right) = 0, \\ &(1-p)(u''-h_0'') + p\left(u'' + \frac{M^2}{2} \Pr v u'\right) = 0. \end{aligned} \tag{30}$$

v and u are considered as follows;

$$\begin{aligned} &v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \ldots = \sum_{i=0}^{6} v_i p^i, \\ &u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \ldots = \sum_{i=0}^{6} u_i p^i. \end{aligned} \tag{31}$$

Assuming $g'''_0 = h''_0 = 0$ and substituting v, u from Eq. (31) into Eq. (30) and some simplification and rearranging based on powers of p-terms;

$$\begin{aligned} &P^0 : v_0''' = 0, \\ &\quad v_0(0) = 0, v_0'(0) = 0, v_0'(1) = 1, \\ &\quad u_0'' = 0, \\ &\quad u_0(0) = \frac{1}{M}, u_0(1) = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} &P^1 : v_1''' + \frac{M^2}{2}(v_0 v_0'') = 0, \\ &\quad v_1(0) = 0, v_1'(0) = 0, v_1'(1) = 0, \\ &\quad u_1'' + \frac{M^2}{2} \Pr(v_0 u_0') = 0, \\ &\quad u_1(0) = 0, u_1(1) = 0, \end{aligned} \tag{33}$$

$$
\begin{aligned}
&P^2: v_2''' + \frac{M^2}{2}(v_1 v_0'' + v_0 v_1'') = 0,\\
&\quad v_2(0) = 0, v_2'(0) = 0, v_2'(1) = 0,\\
&\quad u_2'' + \frac{M^2}{2}\Pr(v_0 u_1' + v_1 u_0') = 0,\\
&\quad u_2(0) = 0, u_2(1) = 0,
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&P^3: v_3''' + \frac{M^2}{2}(v_2 v_0'' + v_0 v_2'' + v_1 v_1'') = 0,\\
&\quad v_3(0) = 0, v_3'(0) = 0, v_3'(1) = 0,\\
&\quad u_3'' + \frac{M^2}{2}\Pr(v_0 u_2' + v_1 u_1' + v_2 u_0') = 0,\\
&\quad u_3(0) = 0, u_3(1) = 0,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&P^4: v_4''' + \frac{M^2}{2}(v_3 v_0'' + v_0 v_3'' + v_2 v_1''\\
&\quad + v_1 v_2'') = 0,\\
&\quad v_4(0) = 0, v_4'(0) = 0, v_4'(1) = 0,\\
&\quad u_4'' + \frac{M^2}{2}\Pr(v_0 u_3' + v_1 u_2' + v_2 u_1'\\
&\quad + v_3 u_0') = 0,\\
&\quad u_4(0) = 0, u_4(1) = 0,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&P^5: v_5''' + \frac{M^2}{2}(v_3 v_1'' + v_1 v_3'' + v_4 v_0''\\
&\quad + v_0 v_4'' + v_2 v_2'') = 0,\\
&\quad v_5(0) = 0, v_5'(0) = 0, v_5'(1) = 0,\\
&\quad u_5'' + \frac{M^2}{2}\Pr(v_0 u_4' + v_1 u_3' + v_2 u_2'\\
&\quad + v_3 u_1' + v_4 u_0') = 0,\\
&\quad u_5(0) = 0, u_5(1) = 0,
\end{aligned}
\tag{37}
$$

$$P^6: v_6''' + \frac{M^2}{2}(v_4 v_1'' + v_1 v_4'' + v_5 v_0''$$
$$+ v_0 v_5'' + v_3 v_2'' + v_2 v_3'') = 0,$$
$$v_6(0) = 0, v_6'(0) = 0, v_6'(1) = 0, \tag{38}$$
$$u_6'' + \frac{M^2}{2}\Pr(v_0 u_5' + v_1 u_4' + v_2 u_3'$$
$$+ v_3 u_2' + v_4 u_1' + v_5 u_0') = 0,$$
$$u_6(0) = 0, u_6(1) = 0,$$

Solving Eqs. (32) - (38) with boundary conditions;

$$v_0(z) = \frac{1}{2}z^2, \tag{39}$$

$$v_1(z) = \frac{-1}{240}M^2 z^5 + \frac{1}{96}M^2 z^2, \tag{40}$$

$$v_2(z) = \frac{11}{161280}M^4 z^8 - \frac{1}{5760}M^4 z^5$$
$$+ \frac{13}{80640}M^4 z^2, \tag{41}$$

$$v_3(z) = \frac{-1}{107520}M^6\left(\frac{25}{198}z^{11} - \frac{11}{24}z^8 + \frac{29}{60}z^5\right)$$
$$+ \frac{1}{1548288}M^6 z^2, \tag{42}$$

$$v_4(z) = \frac{1}{425779200}M^8\left(\frac{9299}{1092}z^{14} - \frac{125}{3}z^{11}\right.$$
$$\left. + \frac{7381}{112}z^8 - \frac{341}{12}z^5\right) - \frac{50249}{929901772800}M^8 z^2, \tag{43}$$

$$v_5(z) = \frac{-1}{1859803545600}M^{10}\left(\frac{1272379}{2040}z^{17}\right.$$
$$\left. - \frac{46495}{12}z^{14} + \frac{51025}{6}z^{11} - \frac{162019}{24}z^8 - \frac{27139}{60}z^5\right) \tag{44}$$
$$- \frac{763571}{446352850944000}M^{10} z^2,$$

$$v_6(z) = \frac{1}{2529332822016000} M^{12}\left(\frac{19241647}{1368} z^{20} - \frac{1272379}{12} z^{17} + \frac{6323320}{21} z^{14} - \frac{3259750}{9} z^{11} + \frac{16488725}{168} z^{8} + \frac{6647629}{60} z^{5}\right) - \frac{59045561}{6055222775906304000} M^{12} z^{2}, \tag{45}$$

$$u_0(z) = \frac{-1}{M} z + \frac{1}{M}, \tag{46}$$

$$u_1(z) = \frac{1}{48} M \Pr z^4 - \frac{1}{48} M \Pr z, \tag{47}$$

$$u_2(z) = \frac{-1}{960} M \Pr[\frac{1}{42}(20M^2 \Pr + 2M^2) z^7 + \frac{1}{12}(-5M^2 \Pr - 5M^2) z^4] + (\frac{5}{80640} M^3 \Pr^2 - \frac{31}{80640} M^3 \Pr) z, \tag{48}$$

$$u_3(z) = \frac{1}{322560} M \Pr[\frac{1}{90}(280M^4 \Pr^2 + 84M^4 \Pr + 11M^4) z^{10} + \frac{1}{42}(-140M^4 \Pr^2 - 294M^4 \Pr - 28M^4) z^7 + \frac{1}{12}(-5M^4 \Pr^2 + 66M^4 \Pr + 26M^4) z^4] + (\frac{115}{58060800} M^5 \Pr^3 + \frac{102}{58060800} M^5 \Pr^2 - \frac{292}{58060800} M^5 \Pr) z, \tag{49}$$

$\vdots$

According to Eq. (31) and the assumption p = 1;

$$g = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots, \quad h = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \cdots. \tag{50}$$

With choice M = 5, Pr = 1;

$$
\begin{aligned}
g(z) &= 0.8310701384z^{2} - 0.2959136461z^{5} \\
&+0.2146153977z^{8} + 0.001357657484z^{20} \\
&+0.05722180311z^{14} - 0.01350962363z^{17} \\
&-0.1361900018z^{11},
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
h(z) &= 0.2 - 0.33242805z + 0.29591364z^{4} \\
&-0.3433846364z^{7} + 0.2996180039z^{10} \\
&-0.1602210487z^{13} + 0.0459327204z^{16} \\
&-0.0054306299z^{19}.
\end{aligned}
\tag{52}
$$

Now, under the transformation $g(z) = \dfrac{f(\eta)}{5}$, $h(z) = \dfrac{\theta(\eta)}{5}$, the above equations are transformed to;

$$
\begin{aligned}
f(\eta) &= 0.166214027\eta^{2} - 4.734618338\,10^{-4}\eta^{5} \\
&+2.747077090\,10^{-6}\eta^{8} - 1.394585618\,10^{-8}\eta^{11} \\
&+4.687610111\,10^{-11}\eta^{14} - 8.853666940\,10^{-14}\eta^{17} \\
&+7.118035270\,10^{-17}\eta^{20},
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\theta(\eta) &= 1 - 0.3324280554\eta + 0.0023673092\eta^{4} \\
&-0.0000219766\eta^{7} + 1.5340441800\,10^{-7}\eta^{10} \\
&-6.56265415\,10^{-10}\eta^{13} + 1.50512338\,10^{-12}\eta^{16} \\
&-1.4236070530\,10^{-15}\eta^{19}.
\end{aligned}
\tag{54}
$$

Results and discussion

In this paper, the HPM, such as analytical technique, is employed for a nonlinear Blasius equation. **Figures 1 - 3** show the profiles of f (η), f ′(η) and $\theta(\eta)$ obtained by the HPM for different values of η in comparison with the numerical solutions. Good agreement can be seen between the present HPM and the numerical results. So, the solutions obtained with the present HPM are more accurate than [20,21]. Numerical comparison between the present HPM with other different approximate solutions is tabulated in **Tables 1 - 3**. It is that present HPM which is close to the numerical results in comparison with [20,21]. The approximations of the f ″(0) obtained by HPM and their relative error with respect to the Howarth number [15] results are listed in **Table 4**. **Figures 4 - 6** show the absolute error of f(η), f ′(η) and $\theta(\eta)$ related to their numerical solution at different values of η. However, the present results are more acceptable than the results obtained by [20,21].

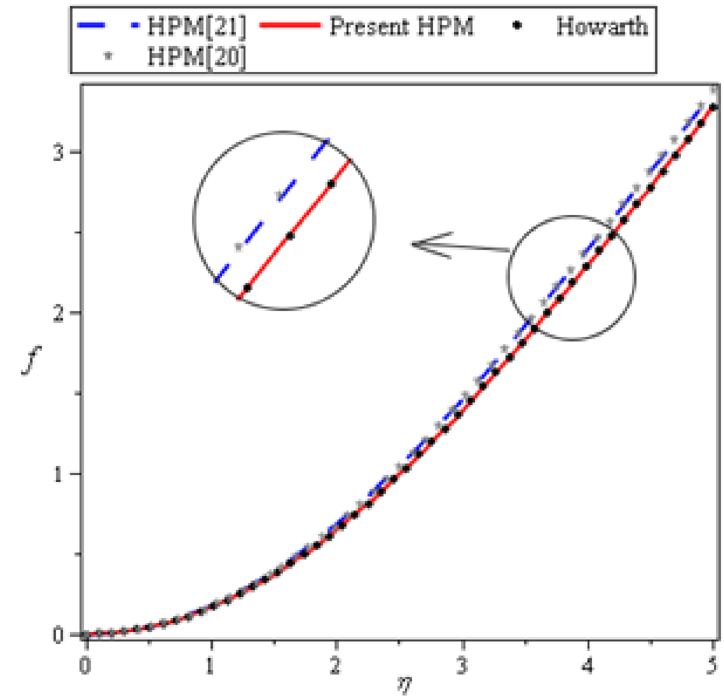

Figure 1 The comparison of answers obtained by HPM and numerical solution for $f(\eta)$.

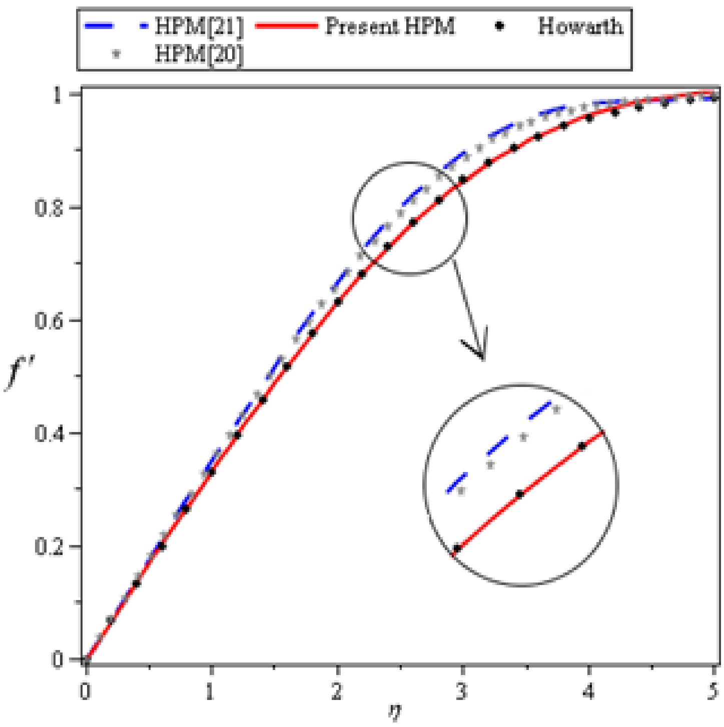

Figure 2 The comparison of answers obtained by HPM and numerical solution for $f'(\eta)$.

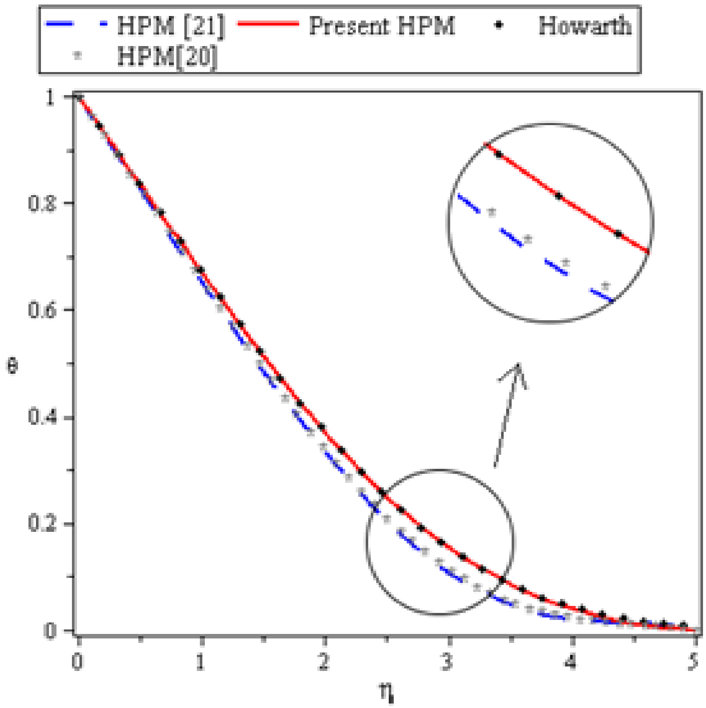

Figure 3 The comparison of answers obtained by HPM and numerical solution for $\theta(\eta)$.

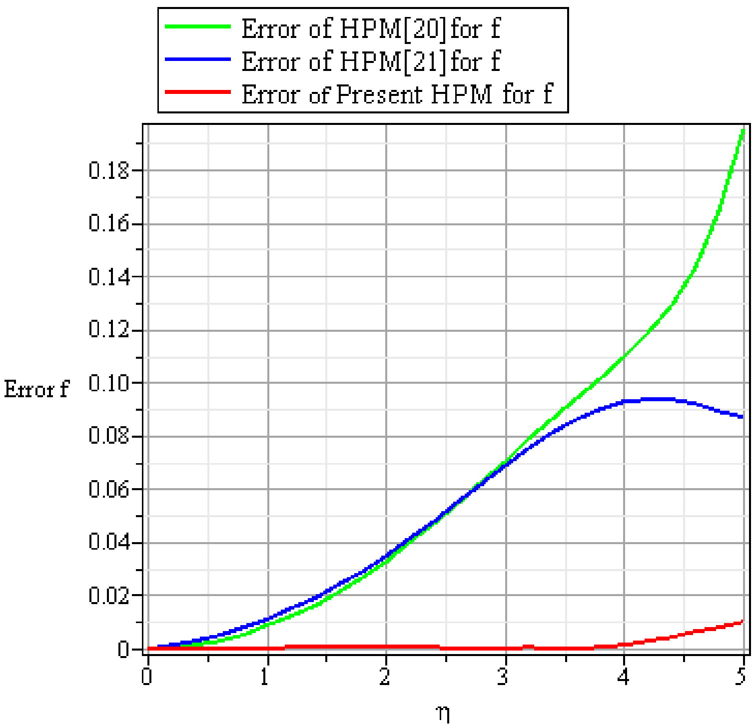

Figure 4 The comparison of the absolute errors for $f(\eta)$ with respect to the Howarth number [15] results.

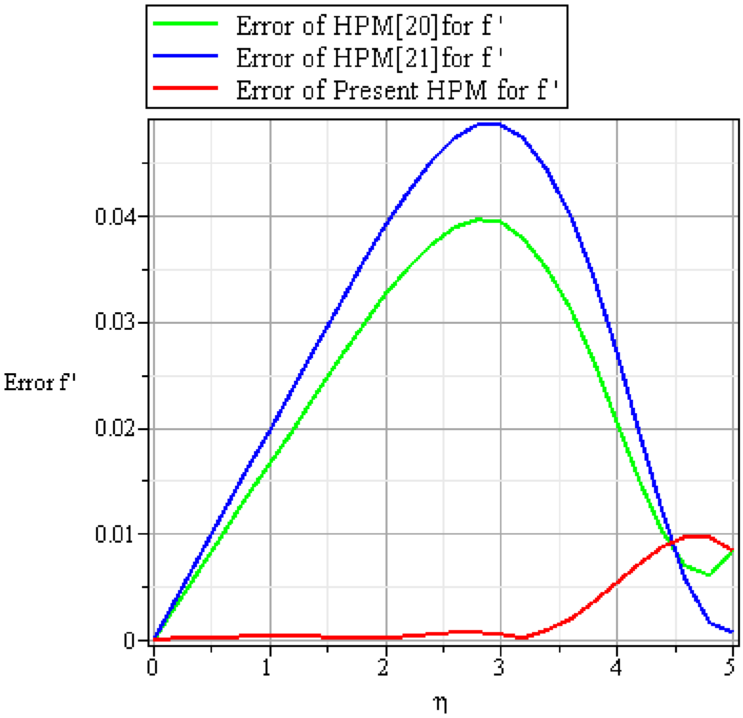

Figure 5 The comparison of the absolute errors for $f'(\eta)$ with respect to the Howarth number [15] results.

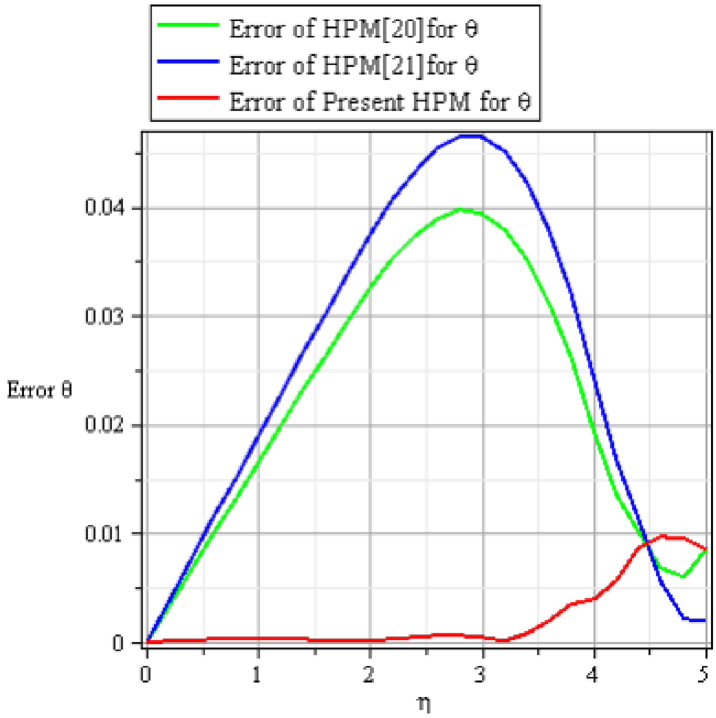

Figure 6 The comparison of the absolute errors for $\theta(\eta)$ with respect to the numerical solution.

Table 1 Obtained results, in comparison with HPM [20,21] and numerical method (Howarth number [15]) for *f*(η).

η	Howarth [15]	Hpm [20]	Hpm [21]	Present Hpm	Error of Hpm [20]	Error of Hpm [21]	Error of present
0	0	0	0	0	0	0	0
0.2	0.0066412	0.0069699	0.0077932	0.0066484	0.0003287	0.0011520	0.0000072
0.4	0.0266762	0.0278758	0.0293860	0.0265894	0.0011996	0.0027098	0.0000868
0.6	0.0597215	0.0626959	0.0647567	0.0598003	0.0029744	0.0050352	0.0000788
0.8	0.1061082	0.1113738	0.1138493	0.1062224	0.0052656	0.0077411	0.0001142
1.0	0.1655717	0.1738016	0.1765564	0.1657432	0.0082299	0.0109847	0.0001715
1.2	0.2379487	0.2498038	0.2527029	0.2381818	0.0118551	0.0147542	0.0002331
1.4	0.3229815	0.3391217	0.3420312	0.3232729	0.0161402	0.0190497	0.0002914
1.6	0.4203207	0.4414008	0.4441877	0.4206587	0.0210801	0.0238670	0.0003380
1.8	0.5295180	0.5561797	0.5587117	0.5298809	0.0266617	0.0291937	0.0003629
2.0	0.6500243	0.6828833	0.6850282	0.6503809	0.0328590	0.0350039	0.0003566
2.2	0.7811933	0.8208206	0.8224437	0.7815042	0.0396273	0.0412504	0.0003109
2.4	0.9222901	0.9691873	0.9701481	0.9225139	0.0468972	0.0478580	0.0002238
2.6	1.0725059	1.1270772	1.1272213	1.0726070	0.0545713	0.0547154	0.0001011
2.8	1.2309773	1.2935005	1.2926472	1.2309370	0.0625232	0.0616699	0.0000403
3.0	1.3968082	1.4674133	1.4653338	1.3966410	0.0706051	0.0685256	0.0001672
3.2	1.5690949	1.6477584	1.6441417	1.5688630	0.0786635	0.0750468	0.0002319
3.4	1.7469501	1.8335195	1.8279185	1.7467840	0.0865694	0.0809684	0.0001661
3.6	1.9295251	2.0237911	2.0155409	1.9296270	0.0942660	0.0860158	0.0001019
3.8	2.1160298	2.2178650	2.2059613	2.1166770	0.1018352	0.0899315	0.0006472
4.0	2.3057464	2.4153361	2.3982576	2.3072780	0.1095897	0.0925112	0.0015316
4.2	2.4980396	2.6162294	2.5916832	2.5008250	0.1181898	0.0936436	0.0027854
4.4	2.6923609	2.8211494	2.7857122	2.6967390	0.1287885	0.0933513	0.0043781
4.6	2.8882480	3.0314545	2.9800744	2.8944690	0.1432065	0.0918264	0.0062210
4.8	3.0853206	3.2494582	3.1747721	3.0934800	0.1641376	0.0894515	0.0081594
5.0	3.2832736	3.4786579	3.3700690	3.2932580	0.1953843	0.0867954	0.0099844

Table 2 Obtained results, in comparison with HPM [20,21] and numerical method (Howarth number [15]) for f′ (η).

η	Howarth [15]	Hpm [20]	Hpm [21]	Present hpm	Error of HPM [20]	Error of HPM [21]	Error of present
0	0	0	0	0	0	0	0
0.2	0.0664077	0.0696975	0.0703281	0.0664818	0.0032898	0.0039204	0.0000741
0.4	0.1327641	0.1393444	0.1406060	0.1329106	0.0065803	0.0078419	0.0001465
0.6	0.1989372	0.2088105	0.2107046	0.1991507	0.0098733	0.0117674	0.0002135
0.8	0.2647094	0.2778800	0.2804100	0.2649775	0.0131706	0.0157006	0.0002681
1.0	0.3297800	0.3462538	0.3494253	0.3300826	0.0164738	0.0196453	0.0003026
1.2	0.3937761	0.4135539	0.4173749	0.3940826	0.0197778	0.0235988	0.0003065
1.4	0.4562617	0.4793309	0.4838112	0.4565324	0.0230692	0.0275495	0.0002707
1.6	0.5167567	0.5430747	0.5482248	0.5169439	0.0263180	0.0314681	0.0001872
1.8	0.5747581	0.6042289	0.6100571	0.5748116	0.0294708	0.0352990	0.0000535
2.0	0.6297657	0.6622097	0.6687189	0.6296405	0.0324440	0.0389532	0.0001252
2.2	0.6813103	0.7164291	0.7236108	0.6809787	0.0351188	0.0423005	0.0003316
2.4	0.7289819	0.7663226	0.7741498	0.7284483	0.0373407	0.0451679	0.0005336
2.6	0.7724550	0.8113803	0.8197988	0.7717748	0.0389253	0.0473438	0.0006802
2.8	0.8115096	0.8511819	0.8600992	0.8108053	0.0396723	0.0485896	0.0007043
3.0	0.8460444	0.8854328	0.8947068	0.8455199	0.0393884	0.0486624	0.0005245
3.2	0.8760814	0.9140010	0.9234279	0.8760235	0.0379196	0.0473465	0.0000579
3.4	0.9017612	0.9369507	0.9462547	0.9025271	0.0351895	0.0444935	0.0007659
3.6	0.9233296	0.9545718	0.9633968	0.9253044	0.0312422	0.0400672	0.0019748
3.8	0.9411181	0.9673977	0.9753066	0.9446574	0.0262796	0.0341885	0.0035393
4.0	0.9555182	0.9762106	0.9826929	0.9608570	0.0206924	0.0271747	0.0053388
4.2	0.9669570	0.9820237	0.9865191	0.9741165	0.0150667	0.0195621	0.0071595
4.4	0.9758708	0.9860369	0.9879789	0.9845695	0.0101661	0.0121081	0.0086987
4.6	0.9826835	0.9895542	0.9884434	0.9922967	0.0068707	0.0057599	0.0096132
4.8	0.9877895	0.9938540	0.9893700	0.9973820	0.0060645	0.0015805	0.0095925
5.0	0.9915419	0.9999999	0.9921642	0.9999982	0.0084580	0.0006223	0.0084563

Table 3 Obtained results, in comparison with HPM [20,21] and numerical method for θ (η).

η	Numerical	Hpm [20]	Hpm [21]	Present HPM	Error of HPM [20]	Error of HPM [21]	Error of present
0	1	1	1	1	0	0	0
0.2	0.9335922	0.9303024	0.9298261	0.9335182	0.0032898	0.0037661	0.0000740
0.4	0.8672358	0.8606555	0.8597026	0.8670894	0.0065803	0.0075332	0.0001464
0.6	0.8010627	0.7911894	0.7897585	0.8008493	0.0098733	0.0113042	0.0002134
0.8	0.7352908	0.7221199	0.7202081	0.7350225	0.0131709	0.0150827	0.0002683
1.0	0.6702199	0.6537461	0.6513486	0.6699174	0.0164738	0.0188713	0.0003025
1.2	0.6062238	0.5864460	0.5835559	0.6059174	0.0197778	0.0226679	0.0003064
1.4	0.5437381	0.5206690	0.5172777	0.5434676	0.0230691	0.0264604	0.0002705
1.6	0.4832432	0.4569252	0.4530234	0.4830561	0.0263180	0.0302198	0.0001871
1.8	0.4252418	0.3957710	0.3913508	0.4251884	0.0294708	0.0338910	0.0000534
2.0	0.3702342	0.3377909	0.3328480	0.3703595	0.0324433	0.0373862	0.0001253
2.2	0.3186896	0.2835708	0.2781117	0.3190213	0.0351188	0.0405779	0.0003317
2.4	0.2710180	0.2336773	0.2277213	0.2715516	0.0373407	0.0432967	0.0005336
2.6	0.2275449	0.1886196	0.1822082	0.2282253	0.0389253	0.0453367	0.0006804
2.8	0.1884903	0.1488180	0.1420233	0.1891947	0.0396723	0.0464670	0.0007044
3.0	0.1539554	0.1145671	0.1075010	0.1544800	0.0393883	0.0464544	0.0005246
3.2	0.1239183	0.0859989	0.0788228	0.1239763	0.0379194	0.0450955	0.0000580
3.4	0.0982386	0.0630492	0.0559820	0.0974720	0.0351894	0.0422566	0.0007666
3.6	0.0766702	0.0454281	0.0387528	0.0746950	0.0312421	0.0379174	0.0019752
3.8	0.0588819	0.0326022	0.0266662	0.0553421	0.0262797	0.0322157	0.0035398
4.0	0.0430429	0.0237893	0.0189977	0.0391421	0.0192536	0.0240452	0.0039008
4.2	0.0314817	1.80E-02	1.48E-02	2.59E-02	0.0135055	0.0167097	0.0055981
4.4	0.0241292	0.0139630	0.0127911	0.0154301	0.0101662	0.0113381	0.0086991
4.6	0.0173165	0.0104457	0.0116918	0.0077040	0.0068708	0.0056247	0.0096125
4.8	0.0122105	0.0061459	0.0100447	0.0026180	0.0060646	0.0021658	0.0095925
5.0	0.0084581	3.36E-10	6.50E-03	2.00E-06	0.0084581	0.0019576	0.0084561

Table 4 Obtained results, in comparison with HPM [18-21] and numerical method for f ″ (0). (Howarth number [15]: f ″(0) = 0.332057).

Method	f ″(0)	Relative error
HPM [Present method]	0.332428	0.1 %
HPM [20]	0.348505	4.9 %
HPM [21]	0.348	4.8 %
HPM [18]	0.3095	6.8 %
HAM [19]	0.3296	0.7 %

Conclusions

In this paper, the HPM has been successfully applied for nonlinear equations of momentum and energy. Since a special technique was used, the obtained results have excellent accuracy for $\eta \le 5$. It is also shown that, the present work result for these values of η provide more accuracy than [20,21] and are in acceptable agreement with the ones derived by the numerical method.

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