Homotopy Analysis Method for an Epidemic Model

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Abstract

In this paper, Homotopy analysis method (HAM) is employed to solve a system of nonlinear equations. The model is the problem of the spread of a non-fatal disease in a population with a constant size, over the period of the epidemic. Mathematical modeling of the problem leads to a system of nonlinear ordinary differential equations. This system has been solved by HAM. Suitable values of the auxiliary parameter $\hbar$ are determined, using $\hbar$-curves. Approximate solutions are plotted and also presented in governing on the problem of the epidemic model.

Keywords: Homotopy analysis method, system of nonlinear ordinary differential equations, epidemic model

Introduction

The problem of spreading of a non-fatal disease over a population, which is assumed to have a constant size, over the period of the epidemic, is considered in [1]. Suppose that, at the time t, population consists of;

$x(t)$ Susceptible individuals; those who are uninfected so far and are liable to infection,
$y(t)$ Infected individuals; those who have the disease and are still at large,
$z(t)$ Isolated individuals, or those who have recovered and are therefore immune.

Assume that there is a steady constant rate between the susceptible and the infected, and that a constant proportion of these constantly results in transmission. Then, during the time interval δt, δx of the susceptible become infected, i.e.;

$$\delta x = -\beta xy\delta t, \tag{1}$$

where β is a positive constant. If $\gamma > 0$ is the rate at which current infected become isolated, then;

$$\delta y = \beta xy\delta t - \gamma y\delta t. \tag{2}$$

The number of new isolated, δz is given by;

$$\delta z = \gamma y\delta t. \tag{3}$$

Now, to determine the system governed on this phenomenon, let $\delta t \to 0$. Then, the following system determines the progress of the disease.

$$\begin{cases} \dfrac{dx}{dt} = -\beta xy, \\ \dfrac{dy}{dt} = \beta xy - \gamma y, \\ \dfrac{dz}{dt} = \gamma y. \end{cases} \tag{4}$$

subject to the following initial conditions.

$$x(0) = N_1, \; y(0) = N_2, \; z(0) = N_3. \tag{5}$$

This model is well-known as the SIR model [2]. The system (4) will be solved by Homotopy analysis method (HAM), and the results will be compared with those of ADM [3].

Homotopy analysis method

To show the basic idea of HAM, consider the following general nonlinear problem;

$$\mathcal{N}[u(t)] = 0, \tag{6}$$

where $\mathcal{N}$ is a nonlinear operator, t denotes the independent variable, and $u(t)$ is an unknown variable. HAM is based on the concept of Homotopy. However, instead of using the traditional Homotopy, a nonzero auxiliary parameter $\hbar \neq 0$ and a nonzero auxiliary function $H(t) \neq 0$ is introduced to construct such a new kind of Homotopy.

$$\mathcal{H}(\phi; q, \hbar, H) = (1-q)\,\mathcal{L}[\phi(t; q, \hbar, H) - u_0(t)] - q\,\hbar\,H(t)\,\mathcal{N}[\phi(t; q, \hbar, H)], \tag{7}$$

where $q \in [0,1]$ is an embedding parameter and $\phi(t; q, \hbar, H)$ is a function of $t, q, \hbar$ and $H(t)$. $u_0(t)$ denotes an initial approximation of $u(t)$ and $\mathcal{L}$ denotes an auxiliary linear operator with the property $\mathcal{L}f = 0$ results $f = 0$. The auxiliary parameter $\hbar$, the auxiliary function $H(t)$ and the auxiliary linear operator $\mathcal{L}$, play important roles in this method. This method is more general than the traditional Homotopy. By means of generalizing the traditional Homotopy method, Liao constructs the so-called zero-order deformation equations from $\mathcal{H}(\phi(t;q)) = 0$ [4]. Thus,

$$(1-q)\,\mathcal{L}[\phi(t;q) - u_0(t)] = q\,\hbar\,H(t)\,\mathcal{N}[\phi(t;q)]. \tag{8}$$

Obviously, as q increases from 0 to 1, $\phi(t; q, \hbar, H)$ varies from initial approximation $u_0(t)$ to the exact solution $u(t)$ of the original nonlinear problem. Expanding $\phi(t;q)$ in Taylor series with respect to q leads to;

$$\phi(t;q) = u_0(t) + \sum_{i=1}^{+\infty} u_i(t)\, q^i, \tag{9}$$

where

$$u_i(t) = \frac{1}{i!} \left. \frac{\partial^i \phi(t;q)}{\partial q^i} \right|_{q=0}. \tag{10}$$

If the initial approximation $u_0(t)$, auxiliary parameter $\hbar$, auxiliary function $H(t)$, and auxiliary linear operator $\mathcal{L}$ are so properly chosen that the series (9) converges at $q=1$, then the following series solution will be obtained.

$$u(t) = u_0(t) + \sum_{i=1}^{+\infty} u_i(t), \tag{11}$$

where $u_i(t)$, for $i = 1, 2, \ldots$ can be determined by the so-called high-order deformation equations which are introduced in what follows.
Define the vector

$$\vec{u}_m = \{u_o(t), u_1(t), \ldots, u_{m-1}(t)\}. \tag{12}$$

Differentiating Eq. (9), m times with respect to q, and then setting $q=0$, and finally dividing them by $m!$ results in the so-called mth-order deformation equation;

$$\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t) R_m(\vec{u}_{m-1}), \tag{13}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{m!} \frac{\partial^{m-1} \mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{14}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1. & m > 1 \end{cases} \tag{15}$$

Notice that $R_m(\vec{u}_{m-1})$ only depends upon $u_o(t), u_1(t), \ldots, u_{m-1}(t)$ which are known when solving the mth-order deformation equations. Therefore, the term $u_m(t)$ can be easily determined by the linear Eq. (13). Obviously, the solution $u(t)$ is dependent upon independent variable t, auxiliary parameter $\hbar$, and auxiliary function $H(t)$. Thus, the convergence region and rate of convergence of solution series given by the above approach might not be uniquely determined. In choosing $u_0(t)$, auxiliary parameter $\hbar$, auxiliary function $H(t)$ and auxiliary operator $\mathcal{L}$, refer to [4,5]. HAM has been used to solve many functional equations so far [6-12].

HAM for epidemic model

To apply HAM to the system of nonlinear Eq. (4), the following series form solutions are considered.

$$\begin{cases} \phi_1(t;q) = x_0(t) + \sum_{i=1}^{+\infty} x_i(t)\, q^i, \\ \phi_2(t;q) = y_0(t) + \sum_{i=1}^{+\infty} y_i(t)\, q^i, \\ \phi_3(t;q) = z_0(t) + \sum_{i=1}^{+\infty} z_i(t)\, q^i, \end{cases} \tag{16}$$

and the following nonlinear operators;

$$\begin{cases} \mathcal{N}_1[\phi_1(t;q),\phi_2(t;q),\phi_3(t;q)] = \dfrac{\partial \phi_1(t;q)}{\partial t} + \beta\,\phi_1(t;q)\,\phi_2(t;q), \\ \mathcal{N}_2[\phi_1(t;q),\phi_2(t;q),\phi_3(t;q)] = \dfrac{\partial \phi_2(t;q)}{\partial t} - \beta\,\phi_1(t;q)\,\phi_2(t;q) + \gamma\,\phi_2(t;q), \\ \mathcal{N}_3[\phi_1(t;q),\phi_2(t;q),\phi_3(t;q)] = \dfrac{\partial \phi_3(t;q)}{\partial t} - \gamma\,\phi_2(t;q). \end{cases} \tag{17}$$

According to the system of Eq. (4) and the initial condition (5) the solutions can be expressed by a set of base functions;

$$\left\{ t^n \middle| n = 0,1,2,... \right\}, \tag{18}$$

such as

$$\begin{cases} x(t) = \sum_{i=1}^{+\infty} \alpha_{1i}\, t^i, \\ y(t) = \sum_{i=1}^{+\infty} \alpha_{2i}\, t^i, \\ z(t) = \sum_{i=1}^{+\infty} \alpha_{3i}\, t^i. \end{cases} \tag{19}$$

α_{1i}, α_{2i}, and α_{3i} are coefficients to be determined. The initial approximations, the auxiliary function $H(t)$, and the auxiliary linear operator $\mathcal{L}$ should be chosen so that the solutions $x(t)$, $y(t)$ and $z(t)$ be of the form (19). This rule is so-called the rule of solution expression [4]. The following linear operator is considered.

$$\mathcal{L}(\phi_i(t;q)) = \frac{\partial \phi_i(t;q)}{\partial t}, \quad i = 1,2,3 \tag{20}$$

with the property $\mathcal{L}(c) = 0,$ where c is constant.
From Eqs. (4), (16) and (17), the following three zero-order deformation equations can be constructed.

$$\begin{cases} (1-q)\,\mathcal{L}[\phi_1(t;q) - x_0(t)] = q\,\hbar\,H(t)\,\mathcal{N}_1[\phi_1(t;q)], \\ (1-q)\,\mathcal{L}[\phi_2(t;q) - y_0(t)] = q\,\hbar\,H(t)\,\mathcal{N}_2[\phi_2(t;q)], \\ (1-q)\,\mathcal{L}[\phi_3(t;q) - z_0(t)] = q\,\hbar\,H(t)\,\mathcal{N}_3[\phi_3(t;q)]. \end{cases} \tag{21}$$

Obviously, for $q = 0,$ and $q = 1,$

$$\begin{cases} \phi_1(t;0) = x_0(t), \\ \phi_2(t;0) = y_0(t), \\ \phi_3(t;0) = z_0(t), \end{cases} \tag{22}$$

$$\begin{cases} \phi_1(t;1) = x(t), \\ \phi_2(t;1) = y(t), \\ \phi_3(t;1) = z(t). \end{cases} \tag{23}$$

Thus, from these zero-order deformation equations, we have 3 high-order deformation equations as follows.

$$\begin{cases} \mathcal{L}[x_m(t) - \chi_m x_{m-1}(t)] = \hbar\, H(t)\, R_m^1(\vec{x}_{m-1}), \\ \mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar\, H(t)\, R_m^2(\vec{y}_{m-1}), \\ \mathcal{L}[z_m(t) - \chi_m z_{m-1}(t)] = \hbar\, H(t)\, R_m^3(\vec{z}_{m-1}), \end{cases} \tag{24}$$

subject to the initial conditions;

$$x_m(0) = \begin{cases} N_1, & m = 0 \\ 0. & m > 0 \end{cases} \qquad y_m(0) = \begin{cases} N_2, & m = 0 \\ 0. & m > 0 \end{cases} \qquad z_m(0) = \begin{cases} N_3, & m = 0 \\ 0. & m > 0 \end{cases} \tag{25}$$

where

$$\begin{cases} R_m^1(\vec{x}_{m-1}) = \dfrac{\partial\, x_{m-1}(t)}{\partial t} + \beta \sum_{i=0}^{m-1} x_i(t)\, y_{m-1-i}(t), \\ R_m^2(\vec{y}_{m-1}) = \dfrac{\partial\, y_{m-1}(t)}{\partial t} - \beta \left[\sum_{i=0}^{m-1} x_i(t)\, y_{m-1-i}(t) \right] + \gamma\, y_{m-1}(t), \\ R_m^3(\vec{z}_{m-1}) = \dfrac{\partial\, z_{m-1}(t)}{\partial t} - \gamma\, y_{m-1}(t). \end{cases} \tag{26}$$

From (24), (25), and (26), the following solutions will result.

$$\begin{cases} x_m(t) = \chi_m x_{m-1}(t) + \hbar \int_0^\tau H(t)\, R_m^1(\vec{x}_{m-1})\, d\tau + c_1, \\ y_m(t) = \chi_m y_{m-1}(t) + \hbar \int_0^\tau H(t)\, R_m^2(\vec{y}_{m-1})\, d\tau + c_2, \\ z_m(t) = \chi_m z_{m-1}(t) + \hbar \int_0^\tau H(t)\, R_m^3(\vec{z}_{m-1})\, d\tau + c_3. \end{cases} \tag{27}$$

Under the rule of solution expression denoted by (19), the auxiliary function $H(t)$ can be chosen in the form;

$$H(t) = t^{2\kappa}. \tag{28}$$

In order to obey both of the first rule of solution expression and the rule of coefficient ergodicity [4], $\kappa = 0$. Thus, one can obtain uniquely the corresponding auxiliary function $H(t) = 1$. Now,

$$\begin{cases} x_1(t) = \hbar\beta N_1 N_2 t \\ y_1(t) = \hbar N_2(\gamma - N_1\beta)t \\ z_1(t) = -\hbar N_2\gamma t \end{cases}$$

$$\begin{cases} x_2(t) = \frac{1}{2}\hbar^2\beta N_1 N_2 t^2(-N_1\beta + \beta N_2 + \gamma) + (\hbar^2\beta N_1 N_2 + \hbar\beta N_1 N_2)t \\ y_2(t) = \frac{1}{2}\hbar^2 t^2 N_2(\gamma^2 - 2\beta N_1\gamma - N_2\beta^2 N_1 + \beta^2 N_1{}^2) + (\hbar N_2(\gamma - N_1\beta) + \hbar^2 N_2(\gamma - N_1\beta))t \\ z_2(t) = \frac{1}{2}\hbar^2\gamma N_2(-N_1\beta + \gamma)t^2 + (-\hbar\gamma N_2 - \hbar^2\gamma N_2)t \end{cases} \tag{29}$$

$$\vdots$$

are successively obtained, along with other terms of the series.

All of linear Eq. (27) can be easily solved. There are truncated series;

$$\begin{cases} x(t) = x_0(t) + \sum_{i=1}^{n} x_i(t), \\ y(t) = y_0(t) + \sum_{i=1}^{n} y_i(t), \\ z(t) = z_0(t) + \sum_{i=1}^{n} z_i(t), \end{cases} \tag{30}$$

as an approximation of the solution of the system (4). It has been proved that, as long as a series solution given HAM converges, the limit must be an exact solution. So, it is important to ensure that the solution series (16) is convergent at $q = 1$. Notice that the solutions are dependent upon auxiliary parameter $\hbar$. Thus, there is still the freedom to choose the auxiliary parameter $\hbar$. With comparison between the above solutions and the solutions obtained by ADM [3], it is clear that for $\hbar = -1$, solutions of these two methods are the same. But $\hbar$ can be chosen such that the solution series converges in a larger region. To investigate the influence of $\hbar$ on the solution series, one can consider the convergence of some related series such as $x'(0.1)$, $y'(0.1)$ and $z'(0.1)$. This is done by plotting the so-called $\hbar$-curves to ensure solution series converge, as suggested by Liao [4]. Then, a valid region of $\hbar$ from the horizontal line segment of curves is chosen. $\hbar$-curves of $x'(0.1)$, $y'(0.1)$ and $z'(0.1)$.

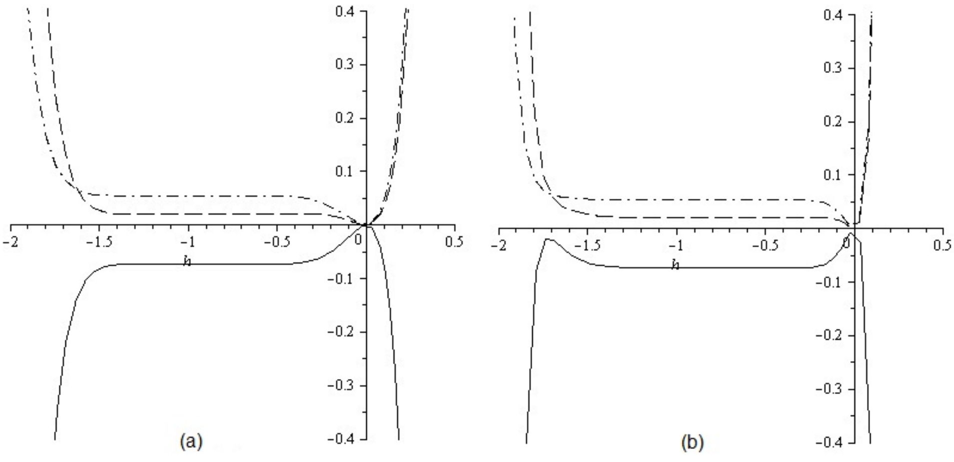

Figure 1 The $\hbar$-curves of $x'(0.1)$, $y'(0.1)$ and $z'(0.1)$ versus $\hbar$ that are respectively denoted by solid, dash and dash-dotted lines. (a) and (b) are respectively for 10-order and 20-order approximations.

Results and analysis

Consider the following values for parameters of system of Eq. (4):

$N_1 = 20$: Initial population of $x(t)$, who are susceptible.

$N_2 = 15$: Initial population of $y(t)$, who are infective.

$N_3 = 10$: Initial population of $z(t)$, who are immune.

$\beta_1 = 0.01$: Rate of change of susceptible to the infective population.

$\gamma = 0.02$: Rate of change of infective to the immune population.

By substituting the above parameters in (29), solutions are obtained that are dependent upon auxiliary parameter $\hbar$. The $\hbar$-curves are shown in **Figure 1**. As shown in **Figure 1**, it is easy to discover the valid region of $\hbar$. Numerical solutions for the 5-order, 10-order, and 20-order approximations by various $\hbar$ are shown in **Figures 2 - 4**. It is clear that when $\hbar = -1$, (equivalent to the solution of traditional Homotopy method and ADM [1]), convergence interval is smaller than $\hbar = -0.6$, and $\hbar = -0.4$. **Figures 2 - 4** show that with various $\hbar$, there are various convergence regions and convergence rates. Thus, the convergence region and convergence rate of a series solution can be freely enhanced via an appropriate choice of the auxiliary parameter $\hbar$. As shown in **Figures 2 - 4**, in this epidemic model it is seems that $\hbar = -0.4$ has a larger convergence region than $\hbar = -0.6$ and $\hbar = -1$.

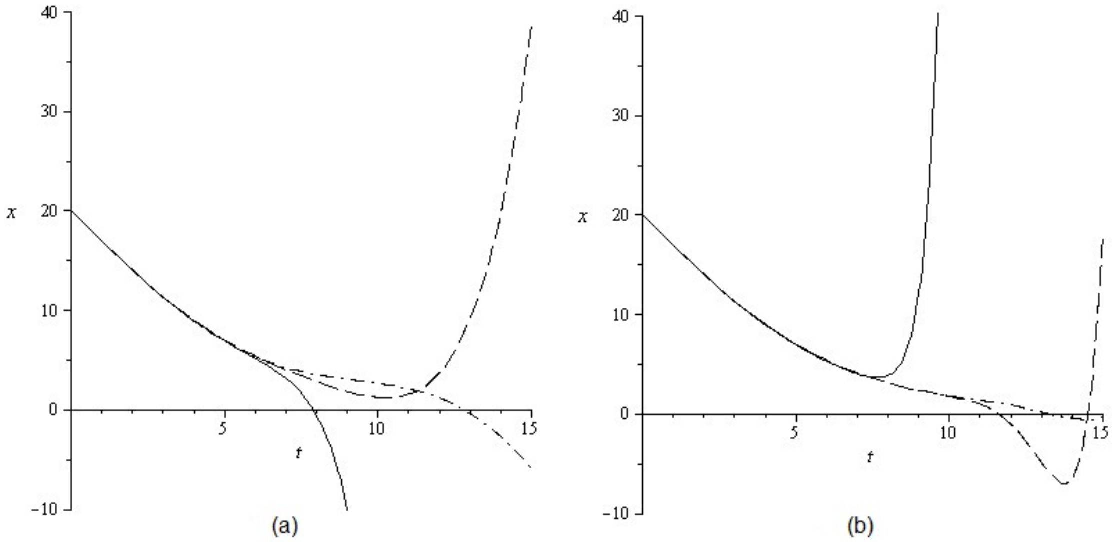

Figure 2 Plots of (a): 10-order and (b): 20-order approximations for $x(t)$ for various auxiliary parameter $\hbar$, solid line: $\hbar=-1$; dashed line: $\hbar=-0.6$; dash-dotted line: $\hbar=-0.4$.

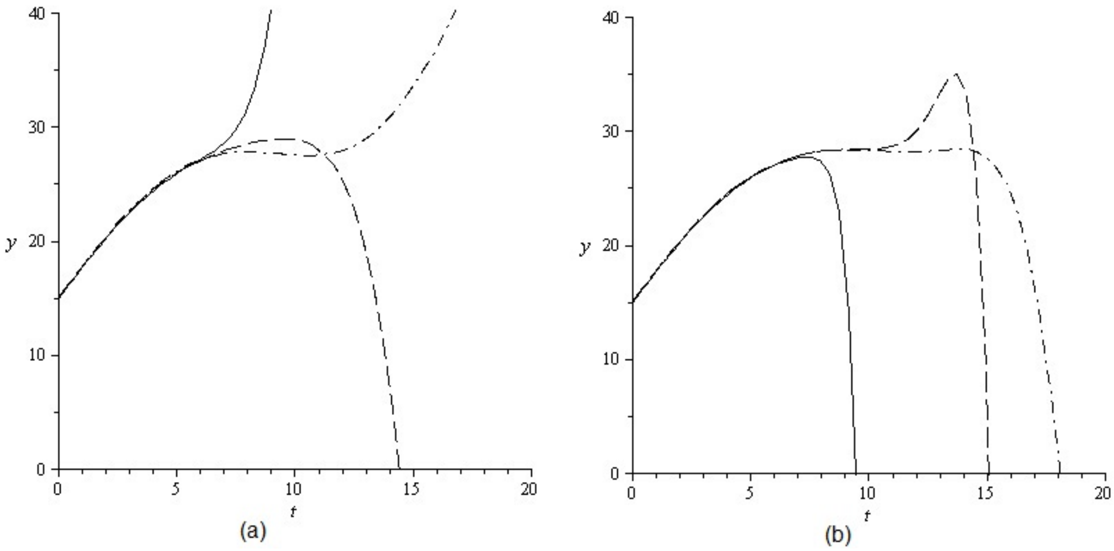

Figure 3 Plots of (a): 10-order and (b): 20-order approximations for $y(t)$ for various auxiliary parameter $\hbar$, solid line: $\hbar=-1$; dashed line: $\hbar=-0.6$; dash-dotted line: $\hbar=-0.4$.

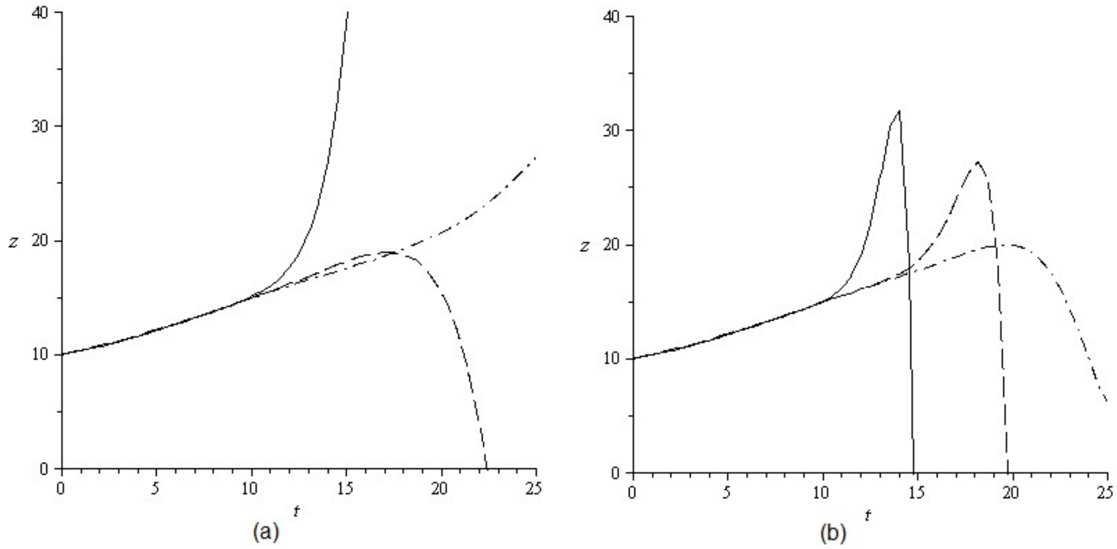

Figure 4 Plots of (a): 10-order and (b): 20-order approximations for $z(t)$ for various auxiliary parameter $\hbar$, solid line: $\hbar = -1$; dashed line: $\hbar = -0.6$; dash-dotted line: $\hbar = -0.4$.

Conclusions

HAM has been applied to solve the epidemic model successfully. The results have been shown by some plots to determine the convergence region of the results for various auxiliary parameters $\hbar$. The interval of valid values for $\hbar$ has been obtained by $\hbar$-curves. The results show that for $\hbar = -0.4$, there is a larger convergence region than for others. In addition, it has been shown that the results by ADM [3] are the same as the results by HAM for $\hbar = -1$, which has a smaller convergence region. This study shows the flexibility of HAM to solve nonlinear system of equations.

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