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An Operational Matrix of Fractional Derivatives of Laguerre Polynomials

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Abstract

In this paper, we derive the Laguerre operational matrix (LOM) of fractional derivatives, which is applied together with the spectral tau method for numerical solution of general linear multi-term fractional differential equations (FDEs) on the half line. A new approach implementing Laguerre operational matrix in combination with the Laguerre collocation technique is introduced for solving nonlinear multi-term FDEs. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations, greatly simplifying the problem. The proposed methods are applied for solving linear and nonlinear multi-term FDEs, subject to initial conditions, and the exact solutions are obtained for some tested problems.

Keywords: Multi-term fractional differential equations, nonlinear fractional differential equations, operational matrix, Laguerre polynomials, Tau method, collocation method, Caputo derivative

Introduction

The applications of fractional calculus, used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, and signal processing, can be successfully modelled by linear or nonlinear fractional differential equations (FDEs). So far, there have been several fundamental works on fractional derivative and fractional differential equations (see [1-4]). These works are an introduction to the theory of fractional derivative and FDEs, and provide a systematic understanding of fractional calculus, such as their existence and their uniqueness.

Finding the approximate or exact solutions of FDEs is an important task. Save in a limited number, there is difficulty in finding the analytical solutions for these equations. Therefore, there have been attempts to develop new methods for obtaining analytical solutions which reasonably approximate the exact solutions. Several such techniques have drawn special attention, such as Adomians decomposition methods [5], Homotopy analysis method [6,7] and Variational iteration method [8].

Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve certain differential equations. The main idea is to write the solution of the differential equation as a sum of certain orthogonal polynomials, and then obtain the coefficients in the sum in orderto satisfy the differential equation as well as possible. Due to their high order of accuracy, spectral methods have been increasing in popularity for several decades, especially in the field of computational fluid dynamics (see, e.g., [9,10] and the references therein). There are 4 versions of spectral methods, namely the Galerkin-type [11,12], Petrov-Galerkin [13-15], tau [15,16] and collocation methods [17,18].

Many researchers have paid attention to existence result of solution of the initial value problem for fractional differential equations, among them [19-21]. Recently, Doha *et al*. [22] introduced a shifted Chebyshev operational matrix and applied it with spectral methods for solving multi-term linear and nonlinear FDEs subject to initial and boundary conditions. Doha *et al*. [16] derived a new formula expressing explicitly any fractional-order derivatives of shifted Chebyshev polynomials of any degree in terms of shifted Chebyshev polynomials themselves, and used it, together with tau and collocation spectral methods, to find an approximate solutions for multi-term linear and nonlinear FDEs. Doha *et al*. [22] introduced a shifted Chebyshev operational matrix and applied it with spectral methods for solving multi-term linear and nonlinear FDEs subject to initial and boundary conditions. Moreover, Bhrawy and Alshomrani [23] introduced the shifted Legendre operational matrix for fractional derivatives and applied it with spectral methods for numerical solution of multi-term linear and nonlinear FDEs subject to multi-point boundary conditions.

Furthermore, Bhrawy *et al*. [24] proposed a suitable way to approximate the multi-term FDEs with variable coefficients subject to initial conditions, using a quadrature shifted Legendre tau approximation; this approach extended the tau method for variable coefficients FDEs by approximating the weighted inner products in the tau method by using the shifted Legendre-Gauss-Lobatto quadrature. The authors in [25-27] presented the spectral tau method for numerical solution of some FDEs, and in [28] Pedas and Tamme developed the spline collocation methods for solving FDEs. Recently, Esmaeili and Shamsi [29] introduced a direct solution technique for obtaining the spectral solution of a special family of fractional initial value problems using a pseudo-spectral method; moreover, Esmaeili *et al*. [30] presented a computational technique based on the collocation method and Müntz polynomials for the solution of FDEs. The algorithms in the present work are somewhat related to the ideas used by Doha *et al*. [16,22] and Bhrawy *et al*. [24,31,32] in developing accurate algorithms for various purposes. More recently, the authors in [33,34] constracted the operational matrix of fractional integration of Laguerre polynomials and modified generalized Laguerre polynomials to solve liear fractional differential equations on semi-infinite intervals.

The aim of this paper is to introduce the Laguerre operational matrix (LOM) of fractional derivative, which is based on the Laguerre tau method, for solving numerically linear multi-order FDEs with initial conditions. Also, we introduce a suitable way to approximate the nonlinear multi-order fractional initial value problems on the interval $\Lambda = (0, \infty)$, by the spectral Laguerre collocation method based on the LOM to find the solution $u_N(x)$. The nonlinear FDE is collocated at $(N-m+1)$ points. For suitable collocation points, we use the $(N-m+1)$ nodes of the Laguerre-Gauss interpolation on Λ. These equations, together with m initial conditions, generate $(N+1)$ nonlinear algebraic equations, which can then be solved using Newton's iterative method. Another considerable advantage of the proposed method is that our *N*-th order approximation gives the exact solution when the solution is polynomial of a degree equal to or less than *N*. If the solution is not polynomial, Laguerre approximation converges to the exact solution as *N* increases. Finally, the accuracy of the proposed algorithms is demonstrated by test problems.

The article is organized as follows: We begin by reviewing certain basic facts about fractional calculus theory and Laguerre polynomials, which are required for establishing our results in Section 2. In Section 3 the LOM of fractional derivative is obtained. Section 4 is devoted to applying the LOM of fractional derivative for solving linear and nonlinear multi-order FDEs. Finally, in Section 5 the proposed method is applied to several examples.

Preliminary

The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator. We give some definitions and properties of fractional derivatives and Laguerre polynomials.

Definition 1 The Riemann-Liouville fractional integral operator of order $\nu(\nu > 0)$ is defined as;

$$\begin{aligned} J^{\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\,dt, \quad \nu > 0, \quad x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \tag{1}$$

Definition 2 The Caputo fractional derivatives of order ν is defined as;

$$D^{\nu} f(x) = J^{m-\nu} D^{m} f(x) = \frac{1}{\Gamma(m-\nu)} \int_{0}^{x} (x-t)^{m-\nu-1} \frac{d^{m}}{dt^{m}} f(t) dt, \quad m-1 < \nu < m, \quad x > 0, \tag{2}$$

where D^{m} is the classical differential operator of order m.

For the Caputo derivative we have;

$$D^{\nu} C = 0, \qquad (C \text{ is a constant}), \tag{3}$$

$$D^{\nu} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \text{for } \beta \in N_0 \text{ and} \beta \geq \lceil \nu \rceil \text{ or } \beta \notin N \text{and } \beta > \lfloor \nu \rfloor. \end{cases} \tag{4}$$

where $\lceil \nu \rceil$ and $\lfloor \nu \rfloor$ are the ceiling and floor functions respectively, while $N = \{1,2,\ldots\}$ and $N_0 = \{0,1,2,\ldots\}$.

The Caputo's fractional differentiation is a linear operation, similar to the integer-order differentiation

$$D^{\nu} (\lambda f(x) + \mu g(x)) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x), \tag{5}$$

where λ and μ are constants. For more details on the geometric and physical interpretation for fractional derivatives for both the Riemann-Liouville and Caputo types, see [4].

Now, let $\Lambda = (0,\infty)$ and $w(x) = e^{-x}$ be a weight function on Λ in the usual sense. Define $L_w^2(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \| v \|_w < \infty\}$, equipped with the following inner product and norm $(u,v)_w = \int_{\Lambda} u(x)\, v(x)\, w(x)\, dx, \qquad \| v \|_w = (u,v)_w^{\frac{1}{2}}.$

Next, let the Laguerre polynomial of degree ℓ be defined by;

$$L_{\ell}(x) = \frac{1}{\ell!} e^{x} \partial_x^{\ell} (x^{\ell} e^{-x}), \quad \ell = 0,1,\cdots. \tag{6}$$

They satisfy the equations $\partial_x (x e^{-x} \partial_x L_{\ell}(x)) + \ell e^{-x} L_{\ell}(x) = 0 \quad x \in \Lambda,$ and $L_{\ell}(x) = \partial_x L_{\ell}(x) - \partial_x L_{\ell+1}(x), \quad \ell \geq 0.$

The set of Laguerre polynomials is the $L_w^2(\Lambda)$ -orthogonal system, namely;

$$\int_{\Lambda} L_j(x) L_k(x) w(x) dx = \delta_{jk}, \quad \forall \quad i,j \geq 0, \tag{7}$$

where δ_{jk} is the Kronecher function.

The analytical form of Laguerre polynomials of degree i on the interval $\Lambda \equiv (0,\infty)$ is given by;

$$L_i(x)=\sum_{k=0}^{i}(-1)^k\,\frac{i!}{(i-k)!\,(k!)^2}\,x^k,\quad i=0,1,\cdots. \tag{8}$$

The special value

$$D^q L_i(0)=(-1)^q\sum_{j=0}^{i-q}\frac{(i-j-1)!}{(q-1)!(i-j-q)!}, \tag{9}$$

where q is a positive integer, will be of important use later.

Laguerre operational matrix of fractional derivative

Let $u(x)\in L_w^2(\Lambda)$; then $u(x)$ may be expressed in terms of Laguerre polynomials as;

$$u(x)=\sum_{j=0}^{\infty}a_jL_j(x),\qquad a_j=\int_0^{\infty}u(x)L_j(x)w(x)dx,\quad j=0,1,2,\cdots. \tag{10}$$

In practice, only the first $(N+1)$-terms Laguerre polynomials are considered. Then we have;

$$u_N(x)=\sum_{j=0}^{N}a_jL_j(x)=C^T\phi(x). \tag{11}$$

where the Laguerre coefficient vector C and the Laguerre vector $\phi(x)$ are given by;

$$C^T=[c_0,c_1,.....,c_N],\ \phi(x)=[L_0(x),L_1(x),.......,L_N(x)]^T. \tag{12}$$

The derivative of the vector $\phi(x)$ can be expressed by;

$$\frac{d\phi(x)}{dx}=D^{(1)}\phi(x), \tag{13}$$

where $D^{(1)}$ is the $(N+1)\times(N+1)$ operational matrix of derivative given by;

$$D^{(1)} = -\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

By using Eq. (13), it is clear that;

$$\frac{d^n\phi(x)}{dx^n} = (D^{(1)})^n \phi(x), \tag{14}$$

where $n \in N$ and the superscript in $D^{(1)}$, denotes matrix powers. Thus;

$$D^{(n)} = (D^{(1)})^n, \qquad n = 1,2,..... \tag{15}$$

Lemma 1 Let $L_i(x)$ be a Laguerre polynomial; then,

$$D^{\nu} L_i(x) = 0, \quad i = 0,1,\cdots,\lceil \nu \rceil - 1, \quad \nu > 0. \tag{16}$$

Proof. Immediately, if we use Eqs. (4) - (5) in Eq. (8), the lemma can be proved.

In the following theorem, we generalize the operational matrix of derivatives of Laguerre polynomials given in Eq. (13) for fractional derivatives.

Theorem 2 Let $\phi(x)$ be Laguerre vector defined in Eq. (12), and also suppose $\nu > 0$;then,

$$D^{\nu}\phi(x) = D^{(\nu)}\phi(x), \tag{17}$$

where $D^{(\nu)}$ is the $(N+1)\times(N+1)$ operational matrix of derivatives of order ν in the Caputo sense, and is defined as follows;

$$D^{(\nu)}=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_\nu(\lceil\nu\rceil,0) & S_\nu(\lceil\nu\rceil,1) & S_\nu(\lceil\nu\rceil,2) & \cdots & S_\nu(\lceil\nu\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\nu(i,0) & S_\nu(i,1) & S_\nu(i,2) & \cdots & S_\nu(i,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\nu(N,0) & S_\nu(N,1) & S_\nu(N,2) & \cdots & S_\nu(N,N) \end{pmatrix} \tag{18}$$

where $S_\nu(i,j)=\sum_{k=\lceil\nu\rceil}^{i}\frac{(-1)^k i!\,\Gamma(j-k+\nu)}{j!\,(i-k)!\,k!\,\Gamma(-k+\nu)}$.

Note that in $D^{(\nu)}$, the first $\lceil\nu\rceil$ rows are all zero.

Proof. The analytic form of the Laguerre polynomials $L_i(x)$ of degree i is given by (8). Using Eqs. (4), (5) and (8), we have;

$$\begin{aligned} D^\nu L_i(x) &= \sum_{k=0}^{i}(-1)^k\frac{i!}{(i-k)!\,(k!)^2}D^\nu x^k \\ &= \sum_{k=\lceil\nu\rceil}^{i}(-1)^k\frac{i!}{(i-k)!\,\Gamma(k-\nu+1)\,k!}x^{k-\nu}, \quad i=\lceil\nu\rceil,\cdots,N. \end{aligned} \tag{19}$$

Now, approximating $x^{k-\nu}$ by $N+1$ terms of Laguerre series, we have;

$$x^{k-\nu}=\sum_{j=0}^{N}b_j L_j(x), \tag{20}$$

where b_j is given from (10) with $u(x)=x^{k-\nu}$, and

$$b_j=\sum_{\ell=0}^{j}(-1)^\ell\frac{\Gamma(k-\nu+\ell+1)\,j!}{(j-\ell)!\,(\ell!)^2}, \tag{21}$$

Employing Eqs. (19) - (21), we get;

$$D^\nu L_i(x)=\sum_{j=0}^{N}S_\nu(i,j)L_j(x), \quad i=\lceil\nu\rceil,\cdots,N, \tag{22}$$

where $S_\nu(i,j)=\sum_{k=\lceil\nu\rceil}^{i}\sum_{\ell=0}^{j}\frac{(-1)^{k+\ell}i!\,j!\,\Gamma(k-\nu+\ell+1)}{(i-k)!\,\Gamma(k-\nu+1)\,k!(j-\ell)!\,(\ell!)^2}$.

Accordingly, Eq. (22) can be written in a vector form as follows;

$$D^{\nu} L_i(x) = [S_{\nu}(i,0), S_{\nu}(i,1), S_{\nu}(i,2), \cdots, S_{\nu}(i,N)]\phi(x), \qquad i = \lceil \nu \rceil, \cdots, N. \tag{23}$$

Also, according to Lemma 1, we can write;

$$D^{\nu} L_i(x) = [0,0,0,\cdots,0]\phi(x), \qquad i = 0,1,\cdots,\lceil \nu \rceil - 1. \tag{24}$$

A combination of Eqs. (23) and (24) leads to the desired result.
Remark. If $\nu = n \in N$, then Theorem 2 gives the same result as Eq. (14).

Applications of the Laguerre operational matrix (LOM) for FDEs

The main aim of this section is to propose a suitable way to approximate linear multi-term FDEs with constant coefficients using the Laguerre tau method based on the LOM, such that it can be implemented efficiently.

Linear multi-order initial FDEs

Consider the linear FDE;

$$D^{\nu} u(x) = \sum_{j=1}^{k} \gamma_j D^{\beta_j} u(x) + \gamma_{k+1} u(x) + g(x), \quad \text{in } \Lambda = (0, \infty), \tag{25}$$

with initial conditions;

$$u^{(i)}(0) = d_i, \qquad i = 0, \cdots, m-1, \tag{26}$$

where γ_j for $j = 1, \cdots, k+1$ are real constant coefficients and also $m-1 < \nu \le m,\ 0 < \beta_1 < \beta_2 < \cdots < \beta_k < \nu$. Moreover $D^{\nu} u(x) \equiv u^{(\nu)}(x)$ denotes the Caputo fractional derivative of order ν for $u(x)$, the values of $d_i\ (i = 0, \cdots, m-1)$ describe the initial state of $u(x)$, and $g(x)$ is a given source function.

To solve problem Eqs. (25) with conditions (26), we approximate $u(x)$ and $g(x)$ by the Laguerre polynomials as;

$$u(x) = \sum_{i=0}^{N} c_i L_i(x) = C^T \phi(x), \tag{27}$$

$$g(x) = \sum_{i=0}^{N} g_i L_i(x) = G^T \phi(x), \tag{28}$$

where vector $G = [g_0, \cdots, g_N]^T$ is known but $C = [c_0, \cdots, c_N]^T$ is an unknown vector.

By using Theorem 2 (relation Eqs. (17) and (27)) we have;

$$D^{\nu}u(x)=C^{T}D^{\nu}\phi(x);C^{T}D^{(\nu)}\phi(x), \tag{29}$$

$$D^{\beta_j}u(x)=C^{T}D^{\beta_j}\phi(x)=C^{T}D^{(\beta_j)}\phi(x), \quad j=1,\cdots,k. \tag{30}$$

Employing Eqs. (27) - (30), the residual $R_N(x)$ for Eq. (25) can be written as;

$$R_N(x)=(C^{T}D^{(\nu)}-C^{T}\sum_{j=1}^{k}\gamma_j D^{(\beta_j)}-\gamma_{k+1}C^{T}-G^{T})\phi(x). \tag{31}$$

As in a typical tau method, see [9], we generate $N-m+1$ linear equations by applying;

$$\langle R_N(x),L_j(x)\rangle=\int_0^{\infty}w(x)R_N(x)L_j(x)dx=0 \qquad j=0,1,\cdots,N-m. \tag{32}$$

Also by substituting Eqs. (15) and (27) in Eq. (26), we get;

$$u^{(i)}(0)=C^{T}D^{(i)}\phi(0)=d_i, \qquad i=0,1,\cdots,m-1, \tag{33}$$

Eqs. (32) and (33) generate $(N-m+1)$ and m set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $u(x)$ given in Eq. (27) can be calculated, which give the solution of the initial value problem in Eqs. (25) and (26).

Nonlinear multi-order initial FDEs

Consider the nonlinear FDE;

$$D^{\nu}u(x)=F(x,u(x),D^{\beta_1}u(x),\cdots,D^{\beta_k}u(x)), \quad \text{in } \Lambda=(0,\infty), \tag{34}$$

with initial conditions;

$$u^{(i)}(0)=d_i, \qquad i=0,\cdots,m-1, \tag{35}$$

with initial conditions (26), where F can be nonlinear in general.

In order to use Laguerre polynomials for this problem, we first approximate $u(x)$, $D^{\nu}u(x)$ and $D^{\beta_j}u(x)$, for $j=1,\cdots,k$ as Eqs. (27), (29) and (30) respectively; by substituting these equations in Eq. (34), we get;

$$C^{T}D^{(\nu)}\phi(x)=F(x,C^{T}\phi(x),C^{T}D^{(\beta_1)},\cdots,C^{T}D^{(\beta_k)}). \tag{36}$$

Also, by substituting Eqs. (14) and (27) in Eq. (35), we obtain;

$$u^{(i)}(0)=C^{T}D^{(i)}\phi(0)=d_i, \qquad i=0,1,\cdots,m-1, \tag{37}$$

To find the solution $u(x)$, we first collocate Eq. (36) at $(N-m+1)$ points, For suitable collocation points, we use the first $(N-m+1)$ Laguerre roots of $L_i(x)$. These equations, together with Eq. (37), generate $(N+1)$ nonlinear equations, which can be solved using Newton's iterative method. Consequently, the approximate solution $u(x)$ can be obtained.

Numerical results

To illustrate the effectiveness of the proposed methods in the present paper, several test examples are carried out in this section.

Example 1 As the first example, we consider the following initial value problem in the case of the inhomogeneous Bagely-Torvik equation;

$$D^2u(x)+D^{\frac{3}{2}}u(x)+u(x)=1+x, \qquad u(0)=1, \quad u'(0)=1, \qquad x\in\Lambda. \tag{38}$$

The exact solution of this problem is $u(x)=1+x$.
By applying the technique described in Section 4.1 with $N=2$, we approximate the solution as;

$$u(x)=c_0L_0(x)+c_1L_0(x)+c_2L_2(x)=C^T\phi(x). \tag{39}$$

Here, we have;

$$D^{(1)}=-\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\end{pmatrix}, D^{(2)}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, D^{(\frac{3}{2})}=\begin{pmatrix}0&0&0\\0&0&0\\1&-\frac{1}{2}&-\frac{1}{8}\end{pmatrix}, G=\begin{pmatrix}2\\-1\\0\end{pmatrix}.$$

Therefore using Eq. (32), we obtain;

$$c_0+2c_2-2=0. \tag{40}$$

Now, by applying Eq. (33), we have;

$$-c_1-2c_2-1=0, \tag{41}$$

$$c_0+c_1+c_2=1, \tag{42}$$

Finally, by solving Eqs. (40) - (42), we get;

$$c_0=2, \qquad c_1=-1, \qquad c_2=0. \tag{43}$$

Thus, we can write;

$$u(x) = \begin{pmatrix} 2, & -1, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1-x \\ \frac{1}{2}(x^2-4x+2) \end{pmatrix} = 1 + x. \tag{44}$$

which is the exact solution.

Example 2 Consider the equation;

$$D^2 u(x) + D^{\frac{1}{2}} u(x) + u(x) = x^2 + 2 + \frac{2.6666666667}{\Gamma(0.5)} x^{1.5}, \qquad u(0) = 0, \quad u'(0) = 0, \qquad x \in \Lambda, \tag{45}$$

whose exact solution is given by $u(x) = x^2$.

By applying the technique described in Section 4.1 with $N = 2$, we approximate the solution as;

$$u(x) = c_0 L_0(x) + c_1 L_0(x) + c_2 L_2(x) = C^T \phi(x). \tag{46}$$

Here, we have;

$$D^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, D^{(\frac{1}{2})} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{8} \\ -1 & -\frac{1}{2} & \frac{5}{8} \end{pmatrix}, G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}. \tag{47}$$

Therefore, using Eq. (32), we obtain;

$$c_0 - c_1 = 6. \tag{48}$$

Now, by applying Eq. (33), we have;

$$c_0 + c_1 + c_2 = 0, \tag{49}$$

$$-c_1 - 2c_2 = 0, \tag{50}$$

Finally, by solving Eqs. (48) - (50), we get;

$$c_0 = 2, \qquad c_1 = -4, \qquad c_2 = 2. \tag{51}$$

Thus, we can write;

$$u(x) = \begin{pmatrix} 2, & -4, & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1-x \\ \frac{1}{2}(x^2 - 4x + 2) \end{pmatrix} = x^2. \tag{52}$$

which is the exact solution.

Example 3 Consider the equation;

$$D^2u(x) - 2Du(x) + D^{\frac{1}{2}}u(x) + u(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}}x^{2.5}, \quad u(0) = 0, u'(0) = 0, \qquad x \in \Lambda, \tag{53}$$

whose exact solution is given by $u(x) = x^3$.

By applying the technique described in Section 1 with $N = 3$, we approximate the solution as;

$$u(x) = \sum_{i=0}^{3} c_i L_i(x) = C^T \phi(x). \tag{54}$$

Here, we have;

$$D^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}, \qquad D^{(1)} = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$D^{(\frac{1}{2})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{8} & \frac{1}{16} \\ -1 & -\frac{1}{2} & \frac{5}{8} & \frac{2}{16} \\ -1 & -\frac{1}{2} & -\frac{3}{8} & \frac{11}{16} \end{pmatrix}, G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}. \tag{55}$$

Therefore, using Eq. (32), we obtain;

$$c_0 + c_1 + 2c_2 + 3c_3 - g_0 = 0, \tag{56}$$

$$\frac{1}{2}c_1+\frac{1}{2}c_2+\frac{5}{2}c_3-g_1=0. \tag{57}$$

Now, by applying Eq. (33), we have;

$$\begin{aligned} C^T\phi(0)&=c_0+c_1+c_2+c_3=0,\\ C^T D^{(1)}\phi(0)&=-c_1-2c_2-3c_3=0. \end{aligned} \tag{58}$$

Finally, by solving Eqs. (56) - (58), we get;

$$c_0=6, \qquad c_1=-18, \qquad c_2=18, \qquad c_3=-6. \tag{59}$$

Thus, we can write;

$$u(x)=\begin{pmatrix} 6, & -18, & 18,-6 \end{pmatrix}\begin{pmatrix} L_0(x)\\ L_1(x)\\ L_2(x)\\ L_3(x) \end{pmatrix}=x^3. \tag{60}$$

Numerical results will not be presented, since the exact solution is obtained.

Example 4 We next consider the following nonlinear initial value problem;

$$D^2u(x)+D^\nu u(x)+u^2(x)=g(x), \qquad u(0)=1, u'(0)=0, \qquad x\in(0,20), \tag{61}$$

where

$$g(x)=cos^2(\gamma x)-\gamma^2 cos(\gamma x)+\frac{1}{\Gamma(-\nu)}\int_0^x (x-t)^{-\nu-1}u(t)dt \tag{62}$$

and the exact solution is given by $u(x)=cos(\gamma x)$.

The solution of this problem is obtained by applying the technique described in Section 2. The maximum absolute error for $\gamma=\frac{1}{30}$ and $\gamma=0.01$ with various choices of N and ν are shown in **Tables 1** and **2**.

Table 1 Maximum absolute error for $\gamma = \frac{1}{30}$ and different values of ν and N for Example 4.

N	ν	error	ν	error
20		$1.93.10^{-1}$		$2.67.10^{-1}$
30	$\frac{1}{2}$	$5.90.10^{-2}$	$\frac{9}{10}$	$4.72.10^{-2}$
40		$1.44.10^{-2}$		$2.85.10^{-2}$

Table 2 Maximum absolute error for $\gamma = 0.01$ and different values of ν and N for Example 4.

N	ν	error	ν	Error	ν	error
20		$6.20.10^{-1}$		$2.12.10^{-1}$		$3.14.10^{-1}$
30	$\frac{2}{10}$	$2.16.10^{-1}$	$\frac{1}{2}$	$4.32.10^{-2}$	$\frac{9}{10}$	$6.60.10^{-2}$
40		$5.51.10^{-2}$		$2.19.10^{-2}$		$1.21.10^{-2}$

Conclusions

In this paper, we have derived a general formulation for the Laguerre operational matrix of fractional derivatives, which is used to approximate the numerical solution of a class of fractional differential equations on the half-line. Our approach was based on the Laguerre tau and collocation methods. To the best of our knowledge, this is the first study concerning the spectral tau Laguerre method based on the Laguerre operational matrix of fractional derivatives for solving multi-term FDEs on the half-line.

An efficient and accurate numerical scheme based on the Laguerre collocation spectral method is proposed for solving the nonlinear FDEs on the half-line. The problem is reduced to the solution of nonlinear algebraic equations. The numerical results given in the previous section demonstrate the good accuracy of these algorithms. Moreover, only a small number of Laguerre polynomials is needed to obtain a satisfactory result.

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