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A New Mathematical Model for Nonlinear Wave in a Hyperelastic Rod and Its Analytic Approximate Solution

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Abstract

This paper proposes a fractional model for nonlinear waves in hyperelastic rods, which describes far-field, finite length, finite amplitude radial deformation waves in cylindrical compressible hyperelastic rods. In this model, fractional derivatives are described in the Caputo sense. The error analysis shows that our approximate solution converges very rapidly to the exact solution and the numerical solution is compared with the known analytical solution which is nearly identical with the exact solution. The method introduces a promising tool for solving time the fractional hyperelastic rod equation.

Keywords: Hyperelastic rod, fractional derivative, analytic approximate solution, homotopy perturbation method

Introduction

The hyperelastic rod equation was first derived in 1998 by Dai and Hau [1,2] as a one-dimensional model for finite-length and small-amplitude axial deformation waves in thin cylindrical rods composed of a compressible Mooney-Rivlin material. The derivation relied upon a reductive perturbation technique, and took into account the nonlinear dispersion of pulses propagating along a rod. It was assumed that each cross section of the rod is subject to a stretching and rotation in space. This equation models propagation of nonlinear waves in cylindrical axially symmetric which is defined as;

$$\frac{\partial u}{\partial t}+3u\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial^2 x\partial t}-\gamma\left(\frac{2\partial u}{\partial x}\frac{\partial^2 u}{\partial^2 x}+u\frac{\partial^3 u}{\partial x^3}\right)=0 \tag{1}$$

with the initial condition $u(x,0)=f(x)$. The solution $u(x,t)$ to the hyperelastic rod equation represents the radial stretch relative to a pre-stressed state, while the material parameter γ is a fixed constant depending upon the pre-stress and the material used in the rod. In the physical derivation of (1), the material parameter γ is fixed and ranges from −29.47 to 3.41. For example, with $\gamma=1$ we obtain the Camassa- Holm equation [3,4], a model for the propagation of shallow water waves, with $u(x,t)$ standing for the water's free surface over a flat bed. If $\gamma=0$, Eq. (1) is the BBM equation which was proposed by Benjamin, Bona and Mahony [5] as a model for the unidirectional evolution of long waves. The solitary wave solution of this equation is global and orbitally stable [6,7]. But throughout this paper we allow for the real value $\gamma=2$. This hyperelastic rod wave equation has an exact solution;

$$u(x,t)=\frac{c}{2}\left(1-\frac{1}{\gamma}\right)+\frac{c}{2}\left(\frac{3}{\gamma}-1\right)e^{-\frac{1}{\sqrt{\gamma}}|x-ct-\varsigma|}. \tag{2}$$

Recently, many experts have paid great attention to the construction of solutions of the hyperelastic rod equation by different methods as H^1-perturbations of smooth solution for a weakly dissipative hyperelastic- rod wave equation by Bendahamane [8], exact travelling wave solutions in a nonlinear elastic rod equation by Li and Zang [9], novel phenomenon in a compressible hyperelastic rod by Zhengrong and Bengong [10], global conservative solutions of the generalized hyperelastic rod equation by Holden and Raynaud [11], global weak solutions to a generalized hyperelastic rod wave equation by Coclite *et al*. [12], blow up of solutions to a periodic nonlinear dispersive rod equation by Jin *et al*. [13], on the blow up of solutions to a nonlinear dispersive rod equation by Wahlen [14].

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [15]. In past years, it has turned out that differential equations involving derivatives of non-integer order can be adequate models for various physical phenomena [16]. Many important phenomena are well described by fractional differential equations in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science. This is because a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. The book by Oldham and Spanier [17] has played a key role in the development of the subject. Some fundamental results related to solving fractional differential equations may be found in Miller and Rose [18], Kilbas and Srivastava [19], Diethelm and Ford [20], Diethelm [21], Samko [22].

The main aim of this article is to present a mathematical model of the hyperelastic rod wave equation with fractional time derivative α ($0<\alpha\leq 1$) in the form of a rapidly convergent series with easily computable components. The Homotopy perturbation method (HPM) is used to solve the time fractional hyperelastic rod wave equation. Using the initial condition, the approximate analytical expressions of $u(x,t)$ for different fractional Brownian motions and also for standard motion are obtained. The approximate solution is obtained numerically, and is depicted graphically. The HPM was proposed first by He in 1998 and was further developed and improved by him [23-26] and was successfully applied to solve many linear and nonlinear fractional differential equation [27-33]. Some authors [34-38] have solved many differential equations by different new techniques. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solution of the problem.

Basic definitions of fractional calculus

In this section, we first give the definitions of fractional order integration and fractional order differentiation. For the concept the fractional derivative, we will adopt Caputo's definition which is a modification of the Riemann Liouville definition and has the advantage of the dealing properly with the initial value problem.

Definition 1 A real function $f(t),\ t>0$ is said to be in the space $C_\mu, \mu\in R$ if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$ where $f_1(t)\in C(0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)}\in C_\mu, n\in N.$

Definition 2 The Riemann-Liouville fractional integral $\left(J_t^\alpha\right)$ operator of order α of a function $f\in C_\mu, \mu\geq -1$ is defined as;

$$J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ (\alpha > 0, \ t > 0),$$
$$J_t^{\alpha} f(t) = f(t). \tag{3}$$

where $\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator (j_t^{α}), can be found in [16-21], we mention only the following. For $f \in C_{\mu}$, $\mu \ge -1$, α, $\beta \ge 0$ and $\gamma \ge -1$:

(1) $$J_t^{\alpha} J_t^{\beta} f(t) = J_t^{\alpha+\beta} f(t), \tag{4}$$

(2) $$\left(J_t^{\alpha} J_t^{\beta}\right) f(t) = \left(J_t^{\beta} J_t^{\alpha}\right) f(t), \tag{5}$$

(3) $$J_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}. \tag{6}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator $\left(D_t^{\alpha}\right)$ proposed by Caputo in his work in the theory of viscoelasticity [39].

Definition 3 The fractional derivative D_t^{α} of $f(t)$ in a Caputo sense is defined as;

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{7}$$

where $m-1 < \alpha \le m, \ m \in N, \ t > 0, \ f \in C_{-1}^{m}$.
The following are 2 basic properties of the Caputo's fractional derivative;

Lemma 1 If $m-1 < \alpha \le m, \ m \in N$ and $f \in C_{\mu}^{n}, \ \mu \ge -1,$ then;

$$\begin{cases} (D_t^{\alpha} J_t^{\alpha}) f(t) = f(t), \\ (J_t^{\alpha} D_t^{\alpha}) f(t) = f(t) - \sum_{i=0}^{m-1} f^{i}(0^{+}) \dfrac{t^{i}}{i!}, \end{cases} \tag{8}$$

Solution of the given problem

In this section the application of the HPM for time fractional hyperelastic rod-equation with an initial condition is discussed. We first consider the following time fractional hyperelastic rod-equation as;

$$D_t^{\alpha} u - D_{xxt} u + 3u D_x u - 4 D_x u D_{xx} u - 2u D_{xxx} u = 0, \qquad 0 < \alpha \le 1, \tag{9}$$

with initial conditions $u(x,0) = \frac{c}{4}\left(1 + e^{-\frac{x}{\sqrt{2}}}\right)$. Eq. (9) has the exact solution $u(x,t) = \frac{c}{4}\left(1 + e^{-\frac{1}{\sqrt{2}}(x-ct)}\right)$ for $\alpha = 1$.

To solve (9) by HPM, we consider the following convex homotopy;

$$(1-p)D_t^{\alpha}u+p(D_t^{\alpha}u-D_{xxt}u+3uD_xu-4D_xuD_{xx}u-2uD_{xxx}u=0,\quad 0<\alpha\le 1. \tag{10}$$

We seek the solution of (9) in the following form;

$$u(x,t)=\lim_{N\to\infty}\sum_{i=0}^{N}p^iu_i(x,t). \tag{11}$$

where $u_i(x,t),\ i=0,1,2,3,...,$ is the function to be determined. We use the following iterative scheme to evaluate $u_i(x,t)$. Substituting Eq. (11) into (12) and equating the coefficient of p with the same power, one gets;

$$p^0:\ D_t^{\alpha}u_0=0, \tag{12}$$

$$p^1:\ D_t^{\alpha}u_1=D_{xxt}u_0-3u_0D_xu_0+4D_xu_0D_{xx}u_0+2u_0D_{xxx}u_0, \tag{13}$$

$$p^2:\ D_t^{\alpha}u_2=D_{xxt}\,u_1-3(u_0D_xu_1+u_1D_xu_0)+4(D_xu_0D_{xx}u_1+D_xu_1D_{xx}u_0)+2(u_0D_{xxx}u_1+u_1D_{xxx}u_0), \tag{14}$$

$$\begin{aligned}p^3:\ D_t^{\alpha}u_3&=D_{xxt}\,u_2-3(u_0D_xu_2+u_1D_xu_1+u_2D_xu_0)+4(D_xu_0D_{xx}u_2+D_xu_1D_{xx}u_1+D_xu_2D_{xx}u_0)\\&+2(u_0D_{xxx}u_2+u_1D_{xxx}u_1+u_2D_{xxx}u_0),\end{aligned} \tag{15}$$

$$\begin{aligned}p^4:\ D_t^{\alpha}u_4&=D_{xxt}\,u_3-3(u_0D_xu_3+u_1D_xu_2+u_2D_xu_1+u_3D_xu_0)+4(D_xu_0D_{xx}u_3+D_xu_1D_{xx}u_2+D_xu_2D_{xx}u_1+D_xu_3D_{xx}u_0)\\&+2(u_0D_{xxx}u_3+u_1D_{xxx}u_2+u_2D_{xxx}u_1+u_3D_{xxx}u_0),\end{aligned} \tag{16}$$

and so on.

The above system of nonlinear equations can be easily solved by applying the operator J_t^{α} to obtain the various components $u_n(x,t)$, thus enabling the series solution to be entirely determined. The first few components of the homotopy perturbation solutions for the Eq. (3) are given as follows;

$$u_0(x,t)=u(x,0)=\frac{c}{4}\left(1+e^{-\frac{x}{\sqrt{2}}}\right), \tag{17}$$

$$u_1(x,t)=-\frac{c^2e^{-\frac{x}{\sqrt{2}}}}{8\sqrt{2}}\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{18}$$

$$u_2(x,t)=-\frac{c^3e^{-\frac{x}{\sqrt{2}}}}{32}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{c^2e^{-\frac{x}{\sqrt{2}}}\alpha\,\Gamma(\alpha)}{16\sqrt{2}\Gamma(1+\alpha)}\frac{t^{2\alpha}}{\Gamma(2\alpha)}, \tag{19}$$

$$u_3(x,t) = \frac{c^2 e^{-\frac{x}{\sqrt{2}}}\alpha(2\alpha+1)\Gamma(\alpha)\Gamma(2\alpha-1)}{32\sqrt{2}\Gamma(2\alpha)(\alpha+1)}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{c^3 e^{-\frac{x}{\sqrt{2}}}\alpha}{32}\left(\frac{\Gamma(\alpha)}{2\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha)}{\Gamma(2\alpha+1)}\right)\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - \frac{c^4 e^{-\frac{x}{\sqrt{2}}}}{64\sqrt{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \quad (20)$$

In this manner the rest of components of the homotopy perturbation solution can be obtained. Thus the solution $u(x,t)$ of the Eq. (3) is as follows;

$$\tilde{u}(x,t) = \sum_{n=0}^{8} u_n(x,t). \quad (21)$$

The series solution converges very rapidly. The rapid convergence means only a few terms are required to get an analytic function.

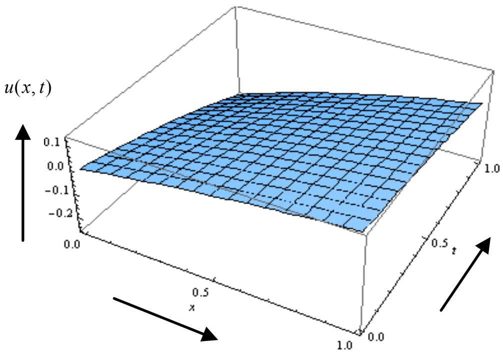

Figure 1 The exact solution $u(x,t)$.

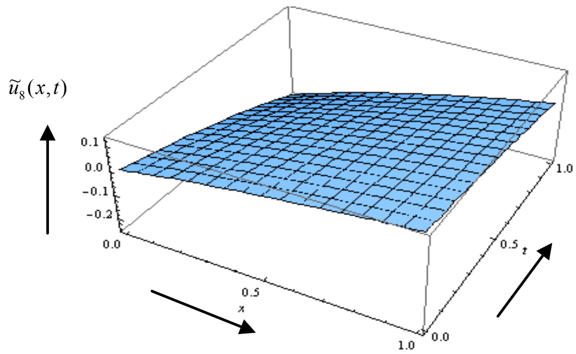

Figure 2 The approximate solution $\tilde{u}(x,t)$ for $\alpha = 1$.

The observations are depicted in **Figures 1 - 5**. **Figures 1 - 2** shows the comparison between the exact solution and approximate solution (by HPM) for the standard hyperelastic rod-equation i.e. for $\alpha = 1$. It can be seen from **Figures 1 - 2** that the approximate solution $\tilde{u}(x,t)$ obtained by the present method is nearly identical to the exact solution $u(x,t)$.

Numerical result and discussion

The simplicity and accuracy of the proposed method is illustrated by computing the absolute error $E_{u_8}(x,t) = |u(\mathrm{x,t}) - \tilde{u}(\mathrm{x,t})|$, where $u(x,t)$ is the exact solution and $\tilde{u}(x,t)$ is approximate solutions of (9) obtained by truncating the respective solution series (11) at level $N = 8$. **Figure 3** represents the absolute error which shows our approximate solution $\tilde{u}(x,t)$ converges to the exact solution very rapidly.

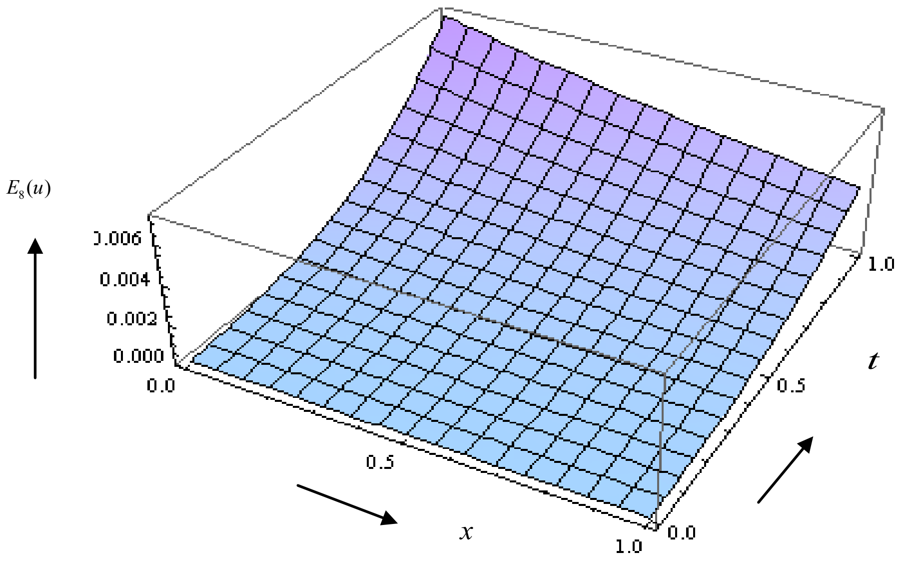

Figure 3 The absolute error $E_8(u)$ between the exact solution and approximate solution.

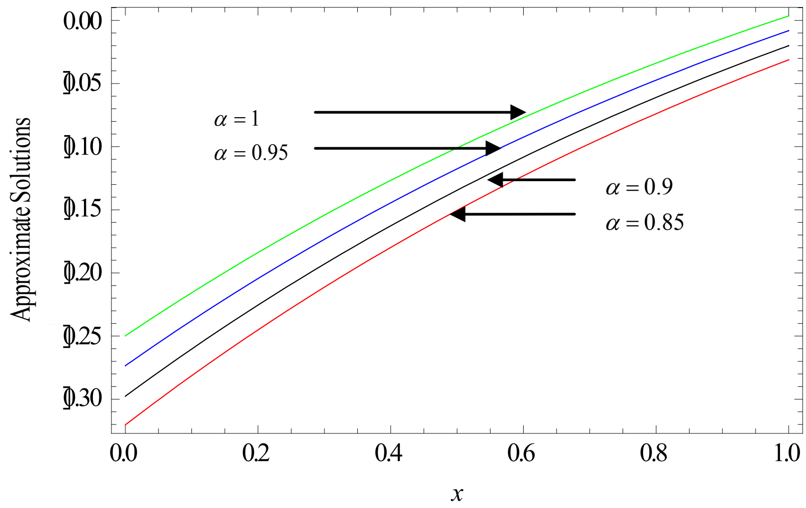

Figure 4 The approximate solution $\tilde{u}(x,t)$ at $c = t = 1$ and for different values of α.

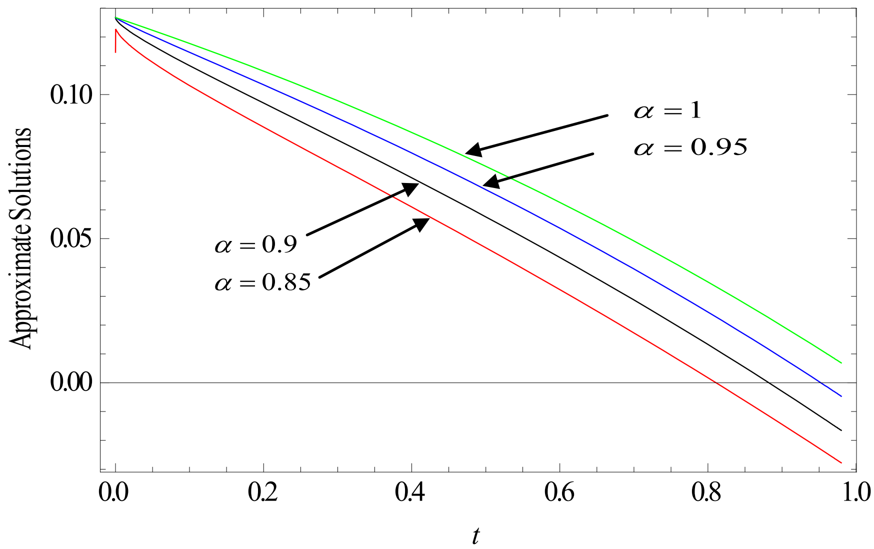

Figure 5 The approximate solution $\tilde{u}(x,t)$ at $c = x = 1$ and for different value of α.

Figures 4 - 5 show the evolution results for the approximate solution $\tilde{u}(x,t)$ Eq. (9) obtained for different values α by using the HPM. It is seen from **Figure 4** that the approximate solution $\tilde{u}(x,t)$ increases with increases in x for different values of $\alpha = 0.85,\ 0.90,\ 0.95$ and for the standard hyperelastic rod equation at $c = t = 1$. Similarly from **Figure 5**, it can be seen that the approximate solution $\tilde{u}(x,t)$ decreases with increases in t for different values of $\alpha = 0.85,\ 0.90,\ 0.95$ and for the standard hyperelastic rod equation at $c = x = 1$. **Figures 4** and **5** show the behaviors of the approximate solution when one variable is constant and other is varied. It is to be noted that only eight terms of the homotopy perturbation series were used in evaluating the approximate solutions in all figures.

Conclusions

In this paper, the HPM is applied to obtain approximate solutions of the time fractional nonlinear partial differential equation such as the hyperelastic rod model, which represents the radial stretch relative to a pre-stressed state. We have demonstrated the accuracy and efficiency of this method by solving the considered time fractional nonlinear partial differential equation. The computations associated with the example in this paper are performed using Mathematica 7.

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