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A Legendre Computational Matrix Method for Solving High-Order Fractional Differential Equations

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Abstract

In this paper, a matrix method for the approximate solution of high order fractional differential equations (FDEs) in terms of a truncated Legendre series is presented. The FDEs and its initial or boundary conditions are transformed to matrix equations, which correspond to a system of algebraic equations with unknown Legendre coefficients. The solution of this system yields the Legendre coefficients of the solution formula. Several numerical examples, such as Cauchy and Bagley-Torvik fractional differential equations, are provided to confirm the accuracy and the effectiveness of the proposed method.

Keywords: Ordinary fractional differential equations, shifted Legendre polynomials, Caputo derivatives, computational matrix method, cauchy equations, Bagley-Torvik equations

Introduction

FDEs have been the focus of many studies, due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering [1]. Consequently, considerable attention has been given to the solutions of fractional differential equations and integral equations of physical interest. Most FDEs do not have exact analytic solutions, so approximate and numerical techniques [2-13] must be used.

Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory, and forms the basis of solutions of differential equations [14,15]. In [16] Khader introduced an efficient numerical method for solving fractional diffusion equations using shifted Chebyshev polynomials, and also introduced in [17] an operational matrix method for solving nonlinear multi-order fractional differential equations. In [18] a spectral method for solving multi-term fractional orders differential equations was introduced. In [19] the Chebyshev collocation method was used to solve high order nonlinear ordinary differential equations. In [20], Bharawy *et al*. introduced a quadrature Tau method for solving fractional differential equations with variable coefficients.Collocation methods have become increasingly popular for solving differential equations. They are also very useful in providing highly accurate solutions to nonlinear differential equations [21-25]. A nonlinear fractional Langevin equation with three-point boundary conditions was solved in [26] using the Jacobi-Gauss-Lobatto collocation method. In [27], Doha *et al*. derived the Jacobi operational matrix of fractional derivatives, which was applied together with spectral tau and collocation methods to obtain the numerical solutions of general linear and nonlinear multi-term FDEs, respectively, which may be considered a generalization of [28,29]. More recently, a shifted Legendre tau method was introduced to get a direct solution technique for solving multi-order FDEs subject to multi-point boundary conditions in [30]. Yuzbasi [31] proposed a new collocation method based on Bessel functions of the first kind to introduce an approximate solution of the Bagley-Torvik equation, which is a class of FDEs. Recent papers [32,33] are referred to, where the Haar wavelet operational matrix of fractional integration, Chebyshev wavelets, the generalized block pulse operational matrix of fractional integration, and the operational Chebyshev matrix of fractional integration, were developed for solving linear and nonlinear FDEs.

In this study, a computational matrix method is presented to find the approximate solutions of high order FDEs with variable coefficients in terms of shifted Legendre polynomials, via Legendre collocation points in the interval $[0, L]$. The main characteristic of this new technique is that it gives a straightforward algorithm in converting FDEs to a system of algebraic equations. This algorithm has several advantages, such as being non-differentiable, non-integral, and easily implemented on a computer, because its structure is dependent on matrix operations only.

The aim of the present paper is concerned with the application of this approach to obtain the approximate solution of FDEs of the following linear form;

$$D^{\nu}u(x) + \sum_{i=1}^{k-1} R_i(x)D^{\beta_i}u(x) + R_k(x)u(x) = f(x), \qquad 0 < x < L, \tag{1}$$

and the more complex non-linearity form;

$$D^{\nu}u(x) + \sum_{i=0}^{k} [R_i(x)(u^i(x)D^{\beta_i}u(x)) + S_i(x)(D^{\gamma_i}u(x)D^{\beta_i}u(x))] = g(x), \tag{2}$$

subject to the multi-point boundary conditions;

$$\sum_{i=0}^{m-1} \varepsilon_{ij}u^{(i)}(\tau_j) = \lambda_j, \qquad j = 1,2,\dots,m, \qquad \tau_j \in [0, L], \tag{3}$$

where $u(x)$ is an unknown function from $C^{m}[0, L]$, the known functions $R_i(x),\ S_i(x)$ are defined on the interval $[0, L]$, $0 < \beta_0 < \beta_1 < \dots < \beta_{k-1} < \beta_k < \nu$ and $0 < \gamma_0 < \gamma_1 < \dots < \gamma_{k-1} < \gamma_k < \nu$, $m - 1 < \nu \leq m$, the parameter ν refers to the fractional order of spatial derivative, $f(x)$ and $g(x)$ are the source functions and ε_{ij}, λ_j are constants. The existence and uniqueness of the solutions of FDEs (1) - (2) have been studied in [34].

The structure of this paper is arranged in the following way. In section 2, some basic definitions about Caputo fractional derivatives and properties of the shifted Legendre polynomials are introduced. In section 3, the fundamental relations for the new operational matrix method are introduced. In section 4, the procedure of solution for FDEs of linear form is clarified. In section 5, the procedure of solution for FDEs of non-linear form is clarified. In section 6, numerical examples are given to show the accuracy of the presented method. Finally, in section 7, the report ends with a brief conclusion and some remarks.

Preliminaries and notations

The fractional derivative in the Caputo sense

In this subsection, some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper are presented.

Definition 1

The Caputo fractional derivative D^{ν} of order ν is defined in the following form;

$$D^{\nu}f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\nu-m+1}} d\xi, \qquad \nu > 0, \quad x > 0,$$

where $m - 1 < \nu \leq m,\ m \in \mathbb{N}$.

Similar to integer-order differentiation, the Caputo fractional derivative operator is a linear operation:

$$D^{\nu}(\lambda\, p(x) + \mu\, q(x)) = \lambda\, D^{\nu}\, p(x) + \mu\, D^{\nu}\, q(x), \tag{4}$$

where λ and μ are constants. The Caputo derivative is obtained as;

$$D^{\nu}\, C = 0, \qquad C \text{ is a constant}, \tag{5}$$

$$D^{\nu}\, x^{n} = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \nu \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \nu \rceil. \end{cases} \tag{6}$$

The ceiling function $\lceil \nu \rceil$ is used to denote the smallest integer greater than or equal to ν and $\mathbb{N}_0 = \{0,1,\ldots\}$. Recall that for $\nu \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of the integer order.

For more details on fractional derivatives definitions and its properties, see [35,36].

Some properties of the shifted Legendre polynomials

The well known Legendre polynomials $P_n(z)$, defined on the interval $[-1,1]$, have the following properties;

$$P_n(-z) = (-1)^n P_n(z), \quad P_n(-1) = (-1)^n, \quad P_n(1) = 1.$$

It is well known that the weight function is $\omega(z) = 1$, and the weighted space $L^2_\omega(-1,1)$ is equipped with the following inner product and norm;

$$(u,v) = \int_{-1}^{1} u(z)v(z)\omega(z)dz, \quad \| u \| = (u,u)^{\frac{1}{2}}.$$

The set of Legendre polynomials forms a complete orthogonal system $L^2(-1,1)$ and;

$$\| P_n(z) \|^2 = h_n = \frac{2}{2n+1},$$

is obtained. In order to use these polynomials on the interval $[0,L]$ the so-called shifted Legendre polynomials are defined by introducing the change of variable $z = \frac{2x}{L} - 1$.

The shifted Legendre polynomials are defined as;

$$P^*_n(x) = P_n\left(\frac{2x}{L} - 1\right) \text{ where } \quad P^*_n(0) = (-1)^n.$$

The analytic form of the shifted Legendre polynomial $P^*_n(x)$ of degree n is given by;

$$P^*_n(x) = \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k)!}{(n-k)!(k!)^2 L^k} x^k. \tag{7}$$

Let $\omega_L(x) = 1$, and the weighted space $L^2_{\omega_L}(0,L)$ is defined with the following inner product and norm;

$$(u,v)_{\omega_L} = \int_0^L u(x)v(x)\omega_L(x)dx, \quad \| u \|_{\omega_L} = (u,u)^{\frac{1}{2}}_{\omega_L}.$$

The set of the shifted Legendre polynomials forms a complete $L^2_{\omega_L}(0,L)$ orthogonal system and;

$$\| P^*_n(x) \|^2_{\omega_L} = \frac{L}{2} h_n = \frac{L}{2n+1}.$$

is obtained. The function $u(x)$ which is square integrable in $[0, L]$, may be expressed in terms of shifted Legendre polynomials as;

$$u(x) = \sum_{i=0}^{\infty} c_i \, P_i^*(x),$$

where the coefficients c_i are given by;

$$c_i = \frac{1}{\|P_i^*(x)\|_{\omega_L}^2} \int_0^L u(x) P_i^*(x) \omega_L(x) dx, \quad i = 0,1,2,\ldots. \tag{8}$$

Fundamental relations

It is suggested that the solution $u(x) \in C^{\,m}[0, L]$ can be approximated in terms of the first $(m+1)$-terms of shifted Legendre polynomials only as:

$$u_m(x) = \sum_{i=0}^{m} \; c_i \, P_i^*(x). \tag{9}$$

To express the fractional derivative of the function $u(x)$ in terms of shifted Legendre polynomials, the following theorem is introduced.

Theorem 1

Let $u(x)$ be approximated by shifted Legendre polynomials (9) and also $\nu > 0$; then, its Caputo fractional derivative can be written in the following form;

$$D^{\,\nu}(u_m(x)) \cong \sum_{j=0}^{m} \sum_{i=\lceil \nu \rceil}^{m} \; \sum_{k=\lceil \nu \rceil}^{i} \frac{(-1)^{i+k}\,(2j+1)\,(i+k)!\,(k-j-\nu+1)_j\, c_i}{L^{\nu}\,(i-k)!\,k!\,\Gamma(k-\nu+1)\,(k-\nu+1)_{j+1}} \; P_j^*(x), \tag{10}$$

where $i = \lceil \nu \rceil, \lceil \nu \rceil + 1, \ldots, m$ and $(a)_n$ is the pochhammer symbol.

Proof. Since the Caputo's fractional differentiation is a linear operation;

$$D^{\nu}(u_m(x)) = \sum_{i=0}^{m} c_i \, D^{\nu} P_i^*(x) \tag{11}$$

is obtained. Employing Eqs. (4) - (6) in Eq. (7);

$$D^{\nu} \, P_i^*(x) = 0, \qquad i = 0,1,\ldots, \lceil \nu \rceil - 1, \quad \nu > 0 \tag{12}$$

is obtained. Therefore, for $i = \lceil \nu \rceil, \lceil \nu \rceil + 1, \ldots, m$, and by using Eqs. (4) - (6) in Eq. (7);

$$\begin{aligned} D^{\nu} \, P_i^*(x) &= \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)! \; (k!)^2 \, L^k} \; D^{\nu} x^k \\ &= \sum_{k=\lceil \nu \rceil}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\,(k!)\,L^k \Gamma(k-\nu+1)} x^{k-\nu} \end{aligned} \tag{13}$$

is obtained. From Eqs. (11) - (13);

$$D^{\nu}(u_m(x)) = \sum_{i=\lceil \nu \rceil}^{m} \; \sum_{k=\lceil \nu \rceil}^{i} (-1)^{i+k} \frac{(i+k)!\, c_i}{(i-k)!\,(k!)\,L^k \Gamma(k-\nu+1)} x^{k-\nu} \tag{14}$$

is obtained. Now, $x^{k-\nu}$ can be expressed approximately in terms of shifted Legendre series, so;

$$x^{k-\nu} \cong \sum_{j=0}^{m} b_{kj} \, P_j^*(x) \tag{15}$$

is obtained, where, b_{kj} is obtained from (8) with $u(x) = x^{k-\nu}$ [11].
A combination of Eqs. (13)-(15) leads to the desired result.

The function $u_m(x)$ defined in (9) can be written in the following matrix form;

$$u_m(x) = \mathbf{P}(x)\mathbf{A}, \tag{16}$$

where $\mathbf{P}(x) = [P_0^*(x) \;\; P_1^*(x) \dots P_m^*(x)]$, $\quad \mathbf{A} = [c_0 \;\; c_1 \dots c_m]^T$.

Theorem 2

Let $\mathbf{P}(x)$ be a shifted Legendre vector defined in (16) and also $\nu > 0$; then, the matrix representation of $D^{\nu} u_m(x)$ has the following form;

$$D^{\nu} u_m(x) = \mathbf{P}(x) \;\; \mathbf{A}^{(\nu)}, \tag{17}$$

where $\mathbf{A}^{(\nu)}$ is the $(m+1)\times(m+1)$ computational matrix of fractional derivatives of order ν in the Caputo sense, and is defined as follows;

$$\mathbf{A}^{(\nu)} = \underset{\nu}{\mathbf{M}} \;\; \mathbf{A}, \tag{18}$$

where

$$\underset{\nu}{\mathbf{M}} = \begin{bmatrix} 0 & \cdots & 0 & \underset{\lceil\nu\rceil,k,0}{\Omega} & \underset{\lceil\nu\rceil+1,k,0}{\Omega} & \cdots & \underset{m,k,0}{\Omega} \\ 0 & \cdots & 0 & \underset{\lceil\nu\rceil,k,1}{\Omega} & \underset{\lceil\nu\rceil+1,k,1}{\Omega} & \cdots & \underset{m,k,1}{\Omega} \\ 0 & \cdots & 0 & \underset{\lceil\nu\rceil,k,2}{\Omega} & \underset{\lceil\nu\rceil+1,k,2}{\Omega} & \cdots & \underset{m,k,2}{\Omega} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \underset{\lceil\nu\rceil,k,m}{\Omega} & \underset{\lceil\nu\rceil+1,k,m}{\Omega} & \cdots & \underset{m,k,m}{\Omega} \end{bmatrix}_{(m+1)\times(m+1)}, \tag{19}$$

such that;

$$\underset{i,k,m}{\Omega} = \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i+k}\,(i+k)!\;(2m+1)\,(k-m-\nu+1)_m}{L^{\nu}\,(i-k)!\;k!\;\Gamma(k-\nu+1)(k-\nu+1)_{m+1}}, \qquad i = \lceil\nu\rceil, \lceil\nu\rceil+1, \dots, m.$$

Proof. Using Eq. (10), the following relation can be written;

$$D^{\nu} u_m(x) \cong$$

$$[P_0^*(x) \quad P_1^*(x) \quad \cdots \quad P_m^*(x)] \begin{bmatrix} \displaystyle\sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i+k}\,(i+k)!\;c_i}{L^{\nu}\,(i-k)!\;k!\;\Gamma(k-\nu+1)(k-\nu+1)} \\ \displaystyle\sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} \frac{3\,(-1)^{i+k}\,(i+k)!\;(k-\nu)\,c_i}{L^{\nu}\,(i-k)!\;k!\;\Gamma(k-\nu+1)(k-\nu+1)_2} \\ \vdots \\ \displaystyle\sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i+k}\,(i+k)!\;(2m+1)\,(k-m-\nu+1)_m c_i}{L^{\nu}\,(i-k)!\;k!\;\Gamma(k-\nu+1)(k-\nu+1)_{m+1}} \end{bmatrix}.$$

Making use of the previous relation gives $(m+1)\times(m+1)$ operational matrix of fractional derivative of order ν (18).

Now, (18) and Legendre collocation points x_i (i.e. the roots of $P^*_{m+1}(x)$) are substituted into (17) to get;

$$D^{\nu}u_m(x_i) = \mathbf{P}(x_i)\underset{\nu}{\mathbf{M}}\,\mathbf{A}, \quad i = 0,1,\ldots,m,$$

or in the compact form;

$$\mathbf{U}^{(\nu)} = \mathbf{P}\underset{\nu}{\mathbf{M}}\,\mathbf{A}, \tag{20}$$

where $\mathbf{U}^{(0)} = \mathbf{U} = \mathbf{P}\,\mathbf{A}$ and;

$$\mathbf{P} = [\mathbf{P}(x_0) \quad \mathbf{P}(x_1) \quad \cdots \quad \mathbf{P}(x_m)]^T, \quad \mathbf{U} = [u(x_0) \quad u(x_1) \quad \cdots \quad u(x_m)]^T.$$

To obtain the matrix representation of $u^r(x)$ using Legendre collocation points;

$$\begin{bmatrix} u^r(x_0) \\ u^r(\mathrm{x}_1) \\ \vdots \\ u^r(x_m) \end{bmatrix} = \begin{bmatrix} u(x_0) & 0 & \cdots & 0 \\ 0 & u(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u(x_m) \end{bmatrix}^{r-1} \begin{bmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_m) \end{bmatrix},$$

is obtained, which can be written in the following compact form (21);

$$\mathbf{U}^{r} = (\mathbf{U})^{r-1}\,\mathbf{U}, \tag{21}$$

where $\mathbf{U} = \text{Ł}\ \text{Ⱥ}$ such that;

$$\text{Ł} = \begin{bmatrix} P(x_0) & 0 & \cdots & 0 \\ 0 & P(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(x_m) \end{bmatrix}, \quad \text{Ⱥ} = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix}.$$

To obtain the matrix representation of $u^{\gamma_i}(x)\,u^{\beta_i}(x)$ using Legendre collocation points x_i;

$$\begin{bmatrix} u^{(\gamma_i)}(x_0)\,u^{(\beta_i)}(x_0) \\ u^{(\gamma_i)}(x_1)\,u^{(\beta_i)}(x_1) \\ \vdots \\ u^{(\gamma_i)}(x_m)\,u^{(\beta_i)}(x_m) \end{bmatrix} = \begin{bmatrix} u^{(\gamma_i)}(x_0) & 0 & \cdots & 0 \\ 0 & u^{(\gamma_i)}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{(\gamma_i)}(x_m) \end{bmatrix} \begin{bmatrix} u^{(\beta_i)}(x_0) \\ u^{(\beta_i)}(x_1) \\ \vdots \\ u^{(\beta_i)}(x_m) \end{bmatrix},$$

is obtained, which can be written in the following compact form;

$$\mathbf{U}^{(\gamma_i)}\,\mathbf{U}^{(\beta_i)} = \text{Ł}\,\underset{\gamma_i}{\mathbf{M}}\,\text{Ⱥ}\,\mathbf{P}\,\underset{\beta_i}{\mathbf{M}}\mathbf{A}, \tag{22}$$

where

$$\underset{\gamma_i}{\mathbf{M}} = \begin{bmatrix} \underset{\gamma_i}{M} & 0 & \cdots & 0 \\ 0 & \underset{\gamma_i}{M} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underset{\gamma_i}{M} \end{bmatrix}, \qquad \underset{\beta_i}{\mathbf{M}} = \begin{bmatrix} 0 & \cdots & 0 & \underset{\lceil\beta_i\rceil,k,0}{\Omega} & \underset{\lceil\beta_i\rceil+1,k,0}{\Omega} & \cdots & \underset{m,k,0}{\Omega} \\ 0 & \cdots & 0 & \underset{\lceil\beta_i\rceil,k,1}{\Omega} & \underset{\lceil\beta_i\rceil+1,k,1}{\Omega} & \cdots & \underset{m,k,1}{\Omega} \\ 0 & \cdots & 0 & \underset{\lceil\beta_i\rceil,k,2}{\Omega} & \underset{\lceil\beta_i\rceil+1,k,2}{\Omega} & \cdots & \underset{m,k,2}{\Omega} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \underset{\lceil\beta_i\rceil,k,m}{\Omega} & \underset{\lceil\beta_i\rceil+1,k,m}{\Omega} & \cdots & \underset{m,k,m}{\Omega} \end{bmatrix}.$$

The operational matrices of fractional derivatives for Chebyshev, Legendre and Jacobi polynomials are given in detail in [17,27,28] and are used with spectral methods for solving linear and nonlinear FDEs.

Procedure of solution for the linear form of FDEs

To obtain the shifted Legendre solution of Eq. (1) under the mixed conditions (3), the following matrix method, based on computing Legendre coefficients, is used. Firstly, Legendre collocation points x_i are substituted into Eq. (1);

$$D^{\nu}u(x_i) + \sum_{i=1}^{k-1} R_i(x_i)D^{\beta_i}u(x_i) + R_k(x_i)u(x_i) = f(x_i), \quad i = 0,1,\dots,m; \tag{23}$$

this system (23) can be written in the following matrix form;

$$(\mathbf{P}\,\underset{\nu}{\mathbf{M}} + \sum_{i=1}^{k-1} \mathbf{R}_i\mathbf{P}\,\underset{\beta_i}{\mathbf{M}} + \mathbf{R}_k\mathbf{P})\mathbf{A} = \mathbf{F}, \tag{24}$$

where

$$\mathbf{R}_i = \begin{bmatrix} R_i(x_0) & 0 & \cdots & 0 \\ 0 & R_i(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_i(x_m) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{bmatrix}.$$

Eq.(24) is the main matrix equation for the solution of Eq. (1) and takes the following compact form:

$$\mathbf{WA} = \mathbf{F}, \quad \mathbf{W} = \mathbf{P}\,\underset{\nu}{\mathbf{M}} + \sum_{i=1}^{k-1} \mathbf{R}_i\mathbf{P}\,\underset{\beta_i}{\mathbf{M}} + \mathbf{R}_k\mathbf{P},$$

which corresponds to a system of $(m+1)$ linear algebraic equations with unknown Legendre coefficients c_j, $j = 0,1,\dots,m$. In addition, the matrix representation of the mixed conditions (3) has the following form;

$$\sum_{i=0}^{m-1} \varepsilon_{i\,j}\,\mathbf{P}(\tau_j)\underset{i}{\mathbf{M}}\,\mathbf{A} = \lambda_j, \quad j = 1,2,\dots,m, \quad \tau_j \in [0,L],$$

which can be written in the following compact form;

$$\mathbf{U}_j\mathbf{A} = \lambda_j, \quad \mathbf{U}_j = \sum_{i=0}^{m-1} \varepsilon_{i\,j}\,\mathbf{P}(\tau_j)\,\underset{i}{\mathbf{M}}.$$

Replacing m rows of the augmented matrix $[\mathbf{W};\mathbf{F}]$ by rows of the matrix $[\mathbf{U}_j, \lambda_j]$, and getting $[\mathbf{W};\mathbf{F}]$ or $\mathbf{WA} = \mathbf{F}$, which is a linear algebraic system, the unknown Legendre coefficients are obtained after solving it, and so the solution of Eq. (1) can be expressed as a truncated series from the shifted Legendre polynomials (9).

Procedure of solution for the non-linear form of FDEs

To obtain the shifted Legendre solution of Eq. (2) under the mixed conditions (3), the following matrix method, based on computing Legendre coefficients, is used. Firstly, Legendre collocation points are substituted into Eq. (2);

$$D^{\nu}u(x_i) + \sum_{i=0}^{k} \left[R_i(x_i)(u^i(x_i)D^{\beta_i}u(x_i)) + S_i(x_i)(D^{\gamma_i}u(x_i)D^{\beta_i}u(x_i))\right] = g(x_i), \tag{25}$$

this system (25) can be written in the following matrix form;

$$\mathbf{U}^{(\nu)} + \sum_{i=0}^{k} \left(\mathbf{R}_i\, \mathbf{U}^{i-1}\, \mathbf{U}\, \mathbf{U}^{(\beta_i)} + \mathbf{S}_i\, \mathbf{U}^{(\gamma_i)}\, \mathbf{U}^{(\beta_i)}\right) = \mathbf{G}, \tag{26}$$

and substituted by Eqs. (20), (22) into Eq. (26);

$$\left(\mathbf{P}\, \underset{\nu}{\mathbf{M}} + \sum_{i=0}^{k} \left(\mathbf{R}_i\, (\text{Ł}\, \mathbf{A})^{i-1}\, \mathbf{P}\, \mathbf{A}\, \mathbf{P}\, \underset{\beta_i}{\mathbf{M}} + \boldsymbol{S}_i\, \text{Ł}\, \underset{\gamma_i}{\mathbf{M}}\, \text{Ⱥ}\, \mathbf{P}\, \underset{\beta_i}{\mathbf{M}}\right)\right)\mathbf{A} = \mathbf{G}, \tag{27}$$

is obtained, where

$$\mathbf{S}_i = \begin{bmatrix} S_i(x_0) & 0 & \cdots & 0 \\ 0 & S_i(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_i(x_m) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_m) \end{bmatrix}.$$

Eq.(27) is the main matrix equation for the solution of Eq. (2), and can be written in the following compact form;

$$\mathbf{W}\mathbf{A} = \mathbf{F}, \quad \mathbf{W} = \mathbf{P}\, \underset{\nu}{\mathbf{M}} + \sum_{i=0}^{k} \left(\mathbf{R}_i\, (\text{Ł}\, \mathbf{A})^{i-1}\, \mathbf{P}\, \mathbf{A}\, \mathbf{P}\, \underset{\beta_i}{\mathbf{M}} + \mathbf{S}_i\, \text{Ł}\, \underset{\gamma_i}{\mathbf{M}}\, \text{Ⱥ}\, \mathbf{P}\, \underset{\beta_i}{\mathbf{M}}\right),$$

which corresponds to a system of $(m+1)$ non-linear algebraic equations with unknown Legendre coefficients $c_j,\ j = 0,1,\dots,m$. In addition, the matrix representation of the mixed conditions (3) has the following form;

$$\sum_{i=0}^{m-1} \varepsilon_{i\,j}\, \mathbf{P}(\tau_j) \underset{i}{\mathbf{M}}\, \mathbf{A} = \lambda_j, \quad j = 1,2,\dots,m, \quad \tau_j \in [0, L],$$

which can be written in the following compact form;

$$\mathbf{U}_j\mathbf{A} = \lambda_j, \quad \mathbf{U}_j = \sum_{i=0}^{m-1} \varepsilon_{i\,j}\mathbf{P}(\tau_j)\, \underset{i}{\mathbf{M}}.$$

Replacing m rows of the augmented matrix $[\mathbf{W}; \mathbf{F}]$ by rows of the matrix $[\mathbf{U}_j, \lambda_j]$, and getting $[\mathbf{W}; \mathbf{F}]$ or $\mathbf{WA} = \mathbf{F}$, which is a system of non-linear algebraic equations, the unknown Legendre coefficients are obtained after solving it, and so the solution of Eq. (2) can be expressed as a truncated series from the shifted Legendre polynomials (9).

Numerical simulation

In order to illustrate the effectiveness of the proposed method, it is implemented to solve the following examples of ordinary fractional differential equations.

Example 1:

Consider the non-homogenous fractional Bagley-Torvik equation of the linear form;

$$D^2u(x) + D^{\frac{3}{2}}u(x) + u(x) = 1 + x, \quad 0 \le x \le 1, \tag{28}$$

with the following initial conditions $u(0) = 1, \quad u'(0) = 1$.

The exact solution to the example (28) is $u(x) = 1 + x$.
The suggested method is applied with $m = 2$, and the solution $u(x)$ is approximated as follows;

$$u_2(x) = \sum_{i=0}^{2} c_i\, P_i^*(x). \tag{29}$$

For $m = 2$, a system of 3 linear algebraic equations is obtained, two of them from the initial conditions and the other from the main equation using the collocation point $x_0 = 0.5$ which is the root of $P_1^*(x) = 0$. Eq. (29) can be written in the following matrix form;

$$\mathbf{U}(x) = \mathbf{P}(x)\mathbf{A},$$

where $\mathbf{P}(x) = [P_0^*(x) \quad P_1^*(x) \quad P_2^*(x)]$, $\mathbf{A} = [c_0 \quad c_1 \quad c_2]^T$.

Using the procedure in section 4, the main matrix equation for this problem is;

$$(\mathbf{P}\underset{(2)}{\mathbf{M}} + \mathbf{P}\underset{(1.5)}{\mathbf{M}} + \mathbf{P})\mathbf{A} = \mathbf{F}, \tag{30}$$

where

$$\mathbf{P} = (1 \quad 0 \quad -0.5), \quad \underset{(2)}{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{(1.5)}{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 9.0270 \\ 0 & 0 & 5.4162 \\ 0 & 0 & -1.2896 \end{pmatrix}, \quad \mathbf{F} = (1.5).$$

The main matrices for the initial conditions are;

$$\mathbf{P}(0)\,\mathbf{A} = 1, \quad \mathbf{P}(0)\,\underset{(1)}{\mathbf{M}}\,\mathbf{A} = 1, \tag{31}$$

where $\mathbf{P}(0) = (1 \quad -1 \quad 1)$, $\underset{(1)}{\mathbf{M}} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$.

From Eqs. (30) - (31), the following system of linear algebraic equations;

$$c_0 + 21.1718\, c_2 = 1.5, \tag{32}$$

$$c_0 - c_1 + c_2 = 1, \tag{33}$$

$$2\,c_1 - 6\,c_2 = 1. \tag{34}$$

are obtained. Therefore, after solving this system (32)-(34), $c_0 = 1.5$, $c_1 = 0.5$, $c_2 = 0$ are obtained. So, the approximate solution;

$$u(x) \cong c_0\, P_0^*(x) + c_1\, P_1^*(x) + c_2\, P_2^*(x) = 1 + x,$$

is obtained, which coincides with the exact solution of this example.

Example 2:

Consider the following fractional Cauchy problem with variable coefficients of the linear form on [0,1];

$$D^{1.5}u(x) + 2\, Du(x) + 3\,\sqrt{x}D^{0.5}u(x) + (1-x)\, u(x) = \frac{2}{\Gamma(1.5)}\, x^{0.5} + 4\, x + \frac{4}{\Gamma(1.5)}\, x^2 + (1-x)x^2, \quad (35)$$

with the following initial conditions $u(0) = 0, \quad u'(0) = 0$.

The exact solution to the problem (35) is $u(x) = x^2$.

The suggested method is applied with $m = 3$, and the solution $u(x)$ is approximated as follows;

$$u_3(x) = \sum_{i=0}^{3} c_i\, P_i^*(x). \quad (36)$$

For $m = 3$, a system of 4 linear algebraic equations is obtained, two of them from the initial conditions and the other from the main equation using the 2 collocation points $x_0 = 0.211325$, $x_1 = 0.788675$ which are the roots of $P_2^*(x) = 0$. Eq. (36) can be written in the following matrix form;

$$\mathbf{U}(x) = \mathbf{P}(x)\mathbf{A},$$

where $\mathbf{P}(x) = [P_0^*(x) \quad P_1^*(x) \quad P_2^*(x) \quad P_3^*(x)]$, $\mathbf{A} = [c_0 \quad c_1 \quad c_2 \quad c_3]^T$.

Using the procedure in section 4, the main matrix equation for this problem is;

$$(\mathbf{P} \underset{(1.5)}{\mathbf{M}} + 2\,\mathbf{P} \underset{(1)}{\mathbf{M}} + \mathbf{G}\,\mathbf{P} \underset{(0.5)}{\mathbf{M}} + \mathbf{H}\,\mathbf{P})\mathbf{A} = \mathbf{F}, \quad (37)$$

where

$$\mathbf{P} = \begin{pmatrix} 1 & -0.57735 & 0 & 0.38490 \\ 1 & 0.57735 & 0 & -0.38490 \end{pmatrix}, \quad \underset{(1.5)}{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 9.02703 & -9.02703 \\ 0 & 0 & 5.41622 & 19.34364 \\ 0 & 0 & -1.2896 & 15.04505 \\ 0 & 0 & 0.60180 & -4.10319 \end{pmatrix},$$

$$\underset{(1)}{\mathbf{M}} = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \underset{(0.5)}{\mathbf{M}} = \begin{pmatrix} 0 & 1.50451 & -0.902703 & 1.28958 \\ 0 & 0.902703 & 1.93436 & -0.601802 \\ 0 & -0.214929 & 1.50451 & 2.22745 \\ 0 & 0.1003 & -0.41032 & 1.99058 \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} 0.78867 & 0 \\ 0 & 0.21132 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 1.37910 & 0 \\ 0 & 2.66422 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 2.11951 \\ 8.09776 \end{pmatrix}.$$

The main matrices for the initial conditions are;

$$\mathbf{P}(0)\ \mathbf{A} = 0, \quad \mathbf{P}(0) \underset{(1)}{\mathbf{M}}\ \mathbf{A} = 0, \quad (38)$$

where $\mathbf{P}(0) = (1 \quad -1 \quad 1 \quad -1)$.

From Eqs. (37) - (38), the following system of linear algebraic equations is obtained;

$$0.78868c_0 + 4.95401c_1 - 3.79951c_2 - 14.1566c_3 = 2.11951, \tag{39}$$

$$0.21133c_0 + 7.41602c_1 + 19.84180c_2 + 8.10778c_3 = 8.09776, \tag{40}$$

$$c_0 - c_1 + c_2 - c_3 = 0, \tag{41}$$

$$2c_1 - 6c_2 + 12c_3 = 0. \tag{42}$$

Therefore, after solving the system (39)-(42);

$c_0 = 0.33466, \quad c_1 = 0.50158, \quad c_2 = 0.16664, \quad c_3 = -0.0002,$
are obtained. So, the approximate solution for this problem;

$$u(x) \cong c_0\, P_0^*(x) + c_1\, P_1^*(x) + c_2\, P_2^*(x) + c_3\, P_3^*(x) = 1.00813\, x^2 - 0.00552882\, x^3,$$

is obtained. In **Figure 1**, the behavior of the obtained approximate solution with the exact solution is presented. Also, in **Table 1**, the absolute error between the exact solution and the obtained approximate solution with different values of m (m = 5, 7) is presented. From **Figure 1**, it can be seen that the approximate solution is in excellent agreement with the exact solution. From **Table 1**, it can also be confirmed that if m is increased, a more accurate approximate solution will be obtained.

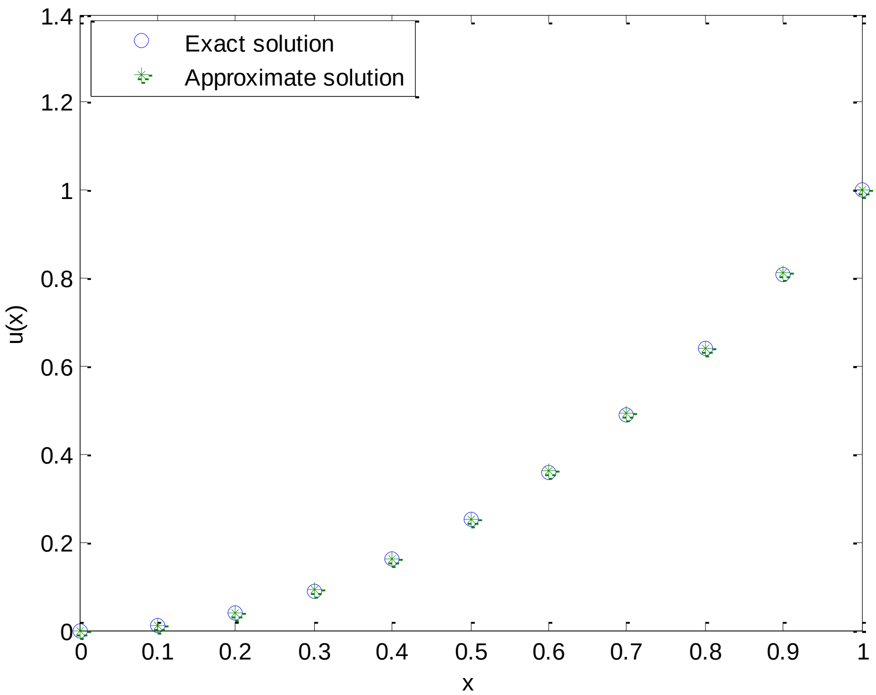

Figure 1 Comparison between the exact solution and the approximate solution with $m = 3$.

Table 1 The absolute error between the exact solution and the obtained approximate solution with different values of m.

x	Absolute error at m = 5	Absolute error at m = 7
0.0	0.53741×10^{-5}	0.53741×10^{-7}
1.0	0.95347×10^{-5}	0.12458×10^{-7}
0.2	0.03245×10^{-5}	0.22400×10^{-7}
0.3	0.74185×10^{-5}	0.55879×10^{-7}
0.4	0.12365×10^{-5}	0.12254×10^{-7}
0.5	0.01548×10^{-5}	0.15670×10^{-7}
0.6	0.12355×10^{-5}	0.75346×10^{-7}
0.7	0.51597×10^{-5}	0.75398×10^{-7}
0.8	0.36985×10^{-5}	0.15975×10^{-7}
0.9	0.76841×10^{-5}	0.12578×10^{-7}
1.0	0.36587×10^{-5}	0.12358×10^{-7}

Example 3:

Consider the following fractional Bagley-Torvik equation of the non-linear form on $[0,1]$;

$$D^3u(x) + D^{2.5}u(x) + u^2(x) = x^4, \qquad (43)$$

with the following initial conditions $u(0) = 0$, $u'(0) = 0$ and $u''(0) = 2$.
The exact solution to the problem (43) is $u(x) = x^2$.
The suggested method is applied with $m = 3$, and the solution $u(x)$ is approximated as follows;

$$u_3(x) = \sum_{i=0}^{3} c_i\, P_i^*(x). \qquad (44)$$

For $m = 3$, a system of 4 non-linear algebraic equations is obtained, three of them from the initial conditions and the other from the main equation using the collocation point $x_0 = 0.5$. Eq. (44) can be written in the matrix form;

$$\mathbf{U}(\mathrm{x}) = \mathbf{P}(x)\mathbf{A},$$

where $\mathbf{P}(\mathrm{x}) = [P_0^*(x) \quad P_1^*(x) \quad P_2^*(x) \quad P_3^*(x)]$, $\mathbf{A} = [c_0 \quad c_1 \quad c_2 \quad c_3]^T$.

Using the procedure in section 5, the main matrix equation for this problem is;

$$(\mathbf{P}\underset{(3)}{\mathbf{M}} + \mathbf{P}\underset{(2.5)}{\mathbf{M}} + (\text{Ł}\ \text{Ⱥ})\,\mathbf{P})\mathbf{A} = \mathbf{F}, \qquad (45)$$

where $\mathbf{P} = (1 \quad 0 \quad -0.5 \quad 0)$, $\mathbf{A} = \text{Ⱥ}$, $\mathbf{F} = (0.0625)$, $\text{Ł} = \mathbf{P}$,

$$\underset{(3)}{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 & 120 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underset{(2.5)}{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 & 90.27033 \\ 0 & 0 & 0 & 54.16220 \\ 0 & 0 & 0 & -12.8958 \\ 0 & 0 & 0 & 6.018022 \end{pmatrix}.$$

The main matrices for the initial conditions are;

$$\mathbf{P}(0)\,\mathbf{A} = 0, \qquad \mathbf{P}(0)\,\underset{(1)}{\mathbf{M}}\,\mathbf{A} = 0, \qquad \mathbf{P}(0)\,\underset{(2)}{\mathbf{M}}\,\mathbf{A} = 2, \tag{46}$$

where

$$\mathbf{P}(0) = (1 \quad -1 \quad 1 \quad -1), \quad \underset{(1)}{\mathbf{M}} = \begin{pmatrix} 0 & 2 & 0 & 6 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underset{(2)}{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Eqs. (45) - (46), four non-linear algebraic equations;

$$c_0\,(c_0 - 0.5c_2) - 0.5(c_0 - 0.5c_2)\,c2 + 216.7182\,c_3 = 0.0625, \tag{47}$$

$$c_0 - c_1 + c_2 - c_3 = 0, \tag{48}$$

$$2c_1 - 6c_2 + 12c_3 = 0, \tag{49}$$

$$16c_2 - 60c_3 = 2. \tag{50}$$

are obtained. Therefore, after solving the system of Eqs. (47) - (50);
$c_0 = 0.33333, \quad c_1 = 0.5, \quad c_2 = 0.16666, \quad c_3 = 0,$

are obtained. So, the approximate solution;

$$u(x) \cong c_0\,P_0^*(x) + c_1\,P_1^*(x) + c_2\,P_2^*(x) + c_3\,P_3^*(x) = x^2,$$

is obtained, which is the exact solution for this example.

Example 4:

Consider the following fractional problem of the non-linear form on $[0,1]$;

$$D^4u(x) + D^{3.5}u(x) + u^3(x) = x^9, \tag{51}$$

with the following initial conditions $u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 0$ and $u'''(0) = 6$.

The exact solution to the problem (51) is $u(x) = x^3$.

The suggested method is applied with $m = 4$, and the solution $u(x)$ is approximated as follows;

$$u_4(x) = \sum_{i=0}^{4} c_i\,P_i^*(x). \tag{52}$$

For $m = 4$, a system of 5 non-linear algebraic equations is obtained, four of them from the initial conditions and the other from the main equation using the collocation point $x_0 = 0.5$ which is the root of $P_1^*(x) = 0$. Eq. (52) can be written in the matrix form;

$$\mathbf{U}(\mathrm{x}) = \mathbf{P}(x)\mathbf{A},$$

where $\mathbf{P}(\mathrm{x}) = [P_0^*(x_0) \quad P_1^*(x_0) \quad P_2^*(x_0) \quad P_3^*(x_0) \quad P_4^*(x_0)]$, $\mathbf{A} = [c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4]^T$.

Using the procedure in section 5, the main matrix equation for this problem is;

$$(\mathbf{P}\underset{(4)}{\mathbf{M}}+\mathbf{P}\underset{(3.5)}{\mathbf{M}}+(\text{Ł}\,\text{Ⱥ})^2\,\mathbf{P})\mathbf{A}=\mathbf{F}, \tag{53}$$

where $\underset{(4)}{\mathbf{M}}$ and $\underset{(3.5)}{\mathbf{M}}$ can be obtained by formula (19) as in the preceding examples;

$\mathbf{P}=(1 \quad 0 \quad -0.5 \quad 0 \quad 0.375)$, $\mathbf{A}=\text{Ⱥ}$, $\text{Ł}=\mathbf{P}$.

The main matrix relations for initial conditions are;

$$\mathbf{P}(0)\ \mathbf{A}=0, \quad \mathbf{P}(0)\underset{(1)}{\mathbf{M}}\ \mathbf{A}=0, \quad \mathbf{P}(0)\underset{(2)}{\mathbf{M}}\ \mathbf{A}=0, \quad \mathbf{P}(0)\underset{(3)}{\mathbf{M}}\ \mathbf{A}=6, \tag{54}$$

where $\mathbf{P}(0)=(1 \quad -1 \quad 1 \quad -1 \quad 1\,)$,

$$\underset{(1)}{\mathbf{M}}=\begin{pmatrix}0&2&0&2&0\\0&0&6&0&6\\0&0&0&10&0\\0&0&0&0&14\\0&0&0&0&0\end{pmatrix}, \quad \underset{(2)}{\mathbf{M}}=\begin{pmatrix}0&0&12&0&40\\0&0&0&60&0\\0&0&0&0&140\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\underset{(3)}{\mathbf{M}}=\begin{pmatrix}0&0&0&120&0\\0&0&0&0&840\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}.$$

From Eqs. (53) - (54), a system of 5 non-linear algebraic equations is obtained. Therefore, after solving this system;

$c_0=0.25$, $c_1=0.45$, $c_2=0.25$, $c_3=0.04999$, $c_4=0$,

are obtained. So, the approximate solution;

$$\mathrm{u}(x)\cong c_0\,P_0^*(x)+c_1\,P_1^*(x)+c_2\,P_2^*(x)+c_3\,P_3^*(x)+c_4\,P_4^*(x)=x^3,$$

is obtained, which is the exact solution for this example.

Example 5:

Consider the following fractional problem of the non-linear form on $[0,1]$;

$$D^\alpha u(x)+D^\gamma u(x)\,D^\beta u(x)+u^2(x)=f(x), \tag{55}$$

where $f(x)=\frac{6}{\Gamma(4-\alpha)}x^{3-\alpha}+\frac{36}{\Gamma(4-\beta)\Gamma(4-\gamma)}x^{6-\beta-\gamma}+x^6$, $\alpha\in[2,3]$, $\beta\in[1,2]$ and $\gamma\in[0,1]$, with the following initial conditions $u(0)=u'(0)=u''(0)=0$. The exact solution to the problem (55) is $u(x)=x^3$.
The suggested method is applied with $m=3$, and the solution $u(x)$ is approximated as follows;

$$u_3(x)=\sum_{i=0}^{3}c_i\,P_i^*(x). \tag{56}$$

For $m = 3$, a system of 4 non-linear algebraic equations is obtained, three of them from the initial conditions and the other from the main equation using the collocation point $x_0 = 0.5$ which is the root of $P_1^*(x) = 0$. Eq. (56) can be written in the matrix form;
$\mathbf{U}(\mathrm{x}) = \mathbf{P}(x)\mathbf{A}$,

where $\mathbf{P}(\mathrm{x}) = [P_0^*(x_0) \quad P_1^*(x_0) \quad P_2^*(x_0) \quad P_3^*(x_0)]$, $\mathbf{A} = [c_0 \quad c_1 \quad c_2 \quad c_3]^T$.
Using the procedure in section 5, the main matrix equation for this problem is;

$$(\mathbf{P} \underset{(2.98)}{\mathbf{M}} + \text{Ł} \underset{(0.98)}{\mathbf{M}} \,\text{Ⱥ}\, \mathbf{P} \underset{(1.98)}{\mathbf{M}} + \text{Ł}\,\text{Ⱥ}\,\mathbf{P})\mathbf{A} = \mathbf{F}, \tag{57}$$

where $\underset{(2.98)}{\mathbf{M}}$, $\underset{(1.98)}{\mathbf{M}}$ and $\underset{(0.98)}{\mathbf{M}}$ can be obtained by formula (19) as in the preceding examples;
$\mathbf{P} = (1 \quad 0 \quad -0.5 \quad 0)$, $\mathbf{A} = \text{Ⱥ}$, $\text{Ł} = \mathbf{P}$.

The main matrix relations for initial conditions are;

$$\mathbf{P}(0)\ \mathbf{A} = 0, \qquad \mathbf{P}(0)\ \underset{(1)}{\mathbf{M}}\ \mathbf{A} = 0, \qquad \mathbf{P}(0)\ \underset{(2)}{\mathbf{M}}\ \mathbf{A} = 0, \tag{58}$$

where $\mathbf{P}(0) = (1 \quad -1 \quad 1 \quad -1)$.
From Eqs. (57) - (58), a system of 5 non-linear algebraic equations is obtained. Therefore, after solving this system;
$c_0 = 0.24971$, $c_1 = 0.44946$, $c_2 = 0.2497$, $c_3 = 0.04999$, $c_4 = 0.04994$,
are obtained. So, the approximate solution;
$\mathrm{u}(x) \cong c_0\, P_0^*(x) + c_1\, P_1^*(x) + c_2\, P_2^*(x) + c_3\, P_3^*(x) = 0.99883x^3$,
is obtained, which is the exact solution for this example.

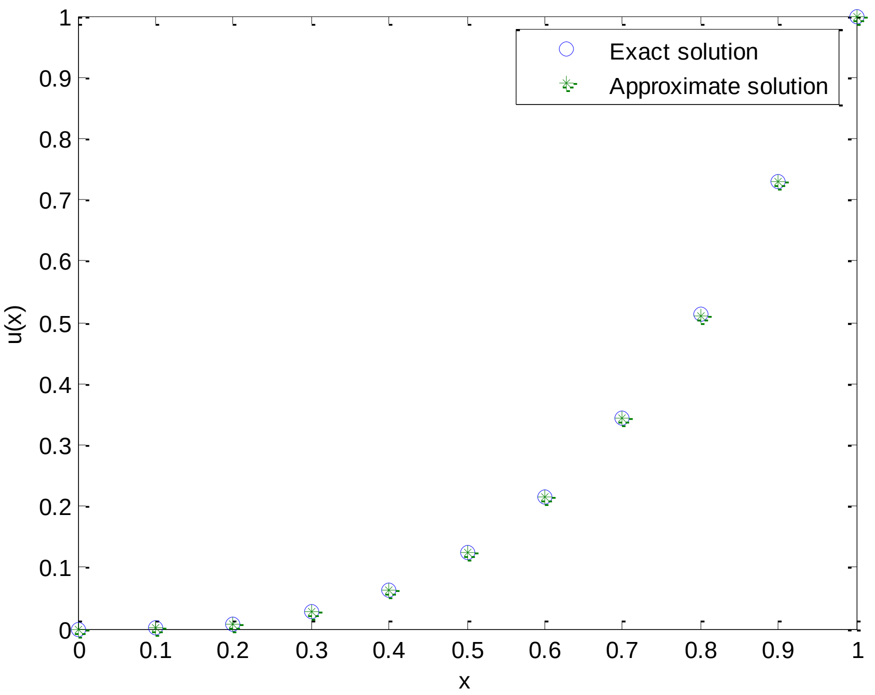

Figure 2 The behavior of the obtained approximate solution at ($\alpha = 2.98$, $\beta = 1.98$ and $\gamma = 0.98$) with the exact solution.

In **Figure 2**, the behavior of the obtained approximate solution at ($\alpha = 2.98$, $\beta = 1.98$ and $\gamma = 0.98$) is presented with the exact solution. From this figure, it can be seen that the approximate solution is in excellent agreement with the exact solution.

From the introduced examples in this section, it is obvious that when m increased, the approximate solution improved, as the errors are decreased, which is the main advantage of the proposed matrix method. This approach can also reformulated using the general Jacobi polynomials.

Conclusions and remarks

In this paper, a new Legendre approximation method for the solution of higher order fractional differential equations has been presented. These equations are transformed to a system of algebraic equations which provided a matrix representation. This new proposed method is non-differentiable, non-integral, straightforward, and well adapted to computer implementation. The solution is expressed as a truncated Legendre series, and so it can be easily evaluated for arbitrary values of x using any computer program without any computational effort. From illustrative examples, it can be seen that this matrix approach can obtain very accurate and satisfactory results. An important feature of this method is that an analytical solution can be obtained, such as has been demonstrated in examples 1, 3 and 4, when the exact solution is a polynomial and the errors are decreased when m is increased. All computational results are made using Mathematica program.

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