WALAILAK JOURNAL
http://wjst.wu.ac.th Mathematics

Homotopy Analysis Method for Fractional Reaction-Diffusion Equation with Ecological Parameters

Zaffar MEHMOOD[1], Raja MEHMOOD[1], Ahmet YILDIRIM[2] and Syed Tauseef MOHYUD-DIN[1]

[1]Department of Mathematics, HITEC University Taxila Cantt, Pakistan
[2]4146 Sk. No: 16 Zeytinalani Mah. Urla-Izmir-Turkey

(Corresponding author's e-mail: syedtauseefs@hotmail.com)

Abstract

This paper applies the Homotopy Analysis Method (HAM) to obtain analytical solutions of fractional reaction-diffusion equations with ecological parameters, which arise very frequently in mechanical engineering, control theory, solid mechanics and applied sciences. Numerical results reveal the complete compatibility of proposed algorithm for such problems. Some examples are presented to show the efficiency and simplicity of the method.

Keywords: Homotopy analysis method, fractional calculus, reaction-diffusion equation

Introduction

Differential equations arise in almost all areas of the applied and engineering sciences [1-18]. Several numerical and analytical techniques, including Homotopy Analysis (HAM), Perturbation, Modified Adomian's Decomposition (MADM), Finite Difference, Spline, and Variational Iteration (VIM) methods have been developed to solve such problems; see [1-18] and the references therein. Recently, many researchers have started working on a very special type of differential equations, which are called fractional differential equations [12-18] and are extremely important in physical problems related to applied and engineering sciences. It is to be highlighted that fractional systems of second-order obstacle problems arise very frequently in mechanical engineering, control theory, solid mechanics, mathematical modeling of contact, obstacle, unilateral, moving and free boundary value problems and applied sciences. Inspired and motivated by the ongoing research in this area, a very efficient and reliable technique, called the Homotopy Analysis Method (HAM) [1-10,15,16], is applied to obtain analytical solutions of fractional reaction-diffusion equations of initial and boundary value problems. The numerical results are very encouraging. The following nonlinear initial and boundary value parabolic problems are considered;

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u + au\left(1-\frac{u}{N}\right), & (x,t)\epsilon\ \Omega \times (0,T), \\ u(x,t)=0, & (x,t)\epsilon\ \partial\Omega \times (0,T), \\ u(x,y,z,0) = u_0(x,y,z) \geq 0, & x \in \Omega, \end{cases} \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $D > 0$ is the diffusion coefficient, $a > 0$ is the linear reproduction rate, $N > 0$ is the carrying capacity of the environment, and Ω is the region in 3D space [19].

Homotopy Analysis Method (HAM) [1-10,15,16]

The following equation is considered.

$$\widetilde{N}[u(\tau)] = 0, \tag{2}$$

where $\widetilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, all boundary and initial conditions are ignored. By means of HAM, Liao [6-10] constructed a zero-order deformation equation;

$$(1-p)\mathcal{L}[\emptyset(\tau;p) - u_0(\tau)] = p\hbar\widetilde{N}[\emptyset(\tau;p)], \tag{3}$$

where $\mathcal{L}$ is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar \neq 0$ is an auxiliary parameter and $p \in [0,1]$ is the embedding parameter. It is obvious that when p = 0 and 1, it holds that;

$$\mathcal{L}[\emptyset(\tau;0) - u_0(\tau)] = 0 \quad \Longrightarrow \quad \emptyset(\tau;0) = u_0(\tau), \tag{4}$$

$$\hbar\widetilde{N}[\emptyset(\tau;1)] = 0 \Longrightarrow \emptyset(\tau;1) = u(\tau), \tag{5}$$

respectively. The solution $\emptyset(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Liao [18] expanded $\emptyset(\tau;p)$ in a Taylor series about the embedding parameter;

$$\emptyset(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m, \tag{6}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \emptyset(\tau;p)}{\partial p^m}\,|p = 0 \tag{7}$$

The convergence of (3) depends on the auxiliary parameter $\hbar$. If this series is convergent at p = 1, then;

$$\emptyset(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau). \tag{8}$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \dots\dots\dots, u_n(\tau)\}. \tag{9}$$

If the zeroth-order deformation equation Eq. (3) m-times is differentiated with respect to p and then divided by m! and finally set $p = 0$, the following mth-order deformation equation is obtained;

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\Re_m(\vec{u}_{m-1}), \tag{10}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\widetilde{N}[\emptyset(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0}, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \tag{12}$$

If each side of Eq. (12) is multiplied with $\mathcal{L}^{-1}$ each side of Eq. (12), the following m-th order deformation equation is obtained;

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \hbar\, \Re_m\, (\vec{u}_{m-1}). \tag{13}$$

Numerical application

In this section, three examples of nonlinear initial-boundary value parabolic problems for different cases of (1) in fractional order are solved to illustrate the implementation of the HAM.

Example 3.1 Consider the nonlinear parabolic problem from (1) as;

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\Delta u + au\left(1 - \frac{u}{N}\right), \quad (x,t) \epsilon\, \Omega \times (0,T) \qquad \text{where } 0 < \alpha \leq 1 \tag{14}$$

with an initial condition of;

$$u(x,y,z,0) = u_0(x,y,z) = xy + yz + xz \geq 0 \quad on \quad [0,1] \times [0,1] \times [0,1]. \tag{15}$$

The HAM is used to solve the system second-order boundary value problem. The zeroth-order deformation equation is;

$$(1-p)\mathcal{L}[\emptyset(\tau;p) - u_0(\tau)] = p\hbar\widetilde{N}[\emptyset(\tau;p)], \tag{16}$$

where $\mathcal{L}$ is the linear operator of the fractional order. Considering the initial conditions as;

$$u_0(x,y,z,t) = xy + yz + xz, \tag{17}$$

the m-th order deformation equation is;

$$\mathcal{L}(u_m - \chi_m u_{m-1}) = \hbar\, \Re_m\, (\vec{u}_{m-1}), \tag{18}$$

where $$\Re_m\, (\vec{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - D\Delta u_{m-1} - a u_{m-1}\left(1 - \frac{u_{m-1}}{N}\right), \tag{19}$$

The solution of m-th order deformation equation is;

$$u_m = \chi_m u_{m-1} + \mathcal{L}^{-1}[\hbar\, \Re_m\, (\vec{u}_{m-1})],$$

where $m \geq 1$.

Consequently, the first few terms of the HAM series solution are as follows;

$$u_0(x,y,z,t) = xy + yz + xz\ ,$$
$$u_1(x,y,z,t) = -\hbar\left\{a(xy + yz + zx)\left(1 - \frac{1}{N}(xy + yz + zx)\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\right\},$$
$$\begin{aligned} u_2(x,y,z,t) = -\hbar^2 \Bigg\{ & -\frac{2Da}{N}[(x+y)^2 + (y+z)^2 + (x+z)^2] \\ & + a^2(xy + yz + zx)\left(1 - \frac{1}{N}(xy + yz + zx)\right) \\ & - \frac{2a^2}{N}(xy + yz + zx)^2\left(1 - \frac{1}{N}(xy + yz + zx)\right)\Bigg\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned}$$

and so on. The HAM series solution is;

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$\begin{aligned} u(x,y,z,t) = xy + yz + xz - \hbar\left\{a(xy+yz+zx)\left(1-\frac{1}{N}(xy+yz+zx)\right)\right\}\frac{t^\alpha}{\Gamma(\alpha+1)} \\ -\hbar^2\left\{-\frac{2Da}{N}[(x+y)^2+(y+z)^2+(x+z)^2] \right. \\ +a^2(xy+yz+zx)\left(1-\frac{1}{N}(xy+yz+zx)\right) \\ \left. -\frac{2a^2}{N}(xy+yz+zx)^2\left(1-\frac{1}{N}(xy+yz+zx)\right)\right\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \end{aligned} \quad (20)$$

Now for $= 1$ and $\hbar = -1$;

$$\begin{aligned} u(x,y,z,t) = xy + yz + xz + \left\{a(xy+yz+zx)\left(1-\frac{1}{N}(xy+yz+zx)\right)\right\}t \\ -\left\{-\frac{2Da}{N}[(x+y)^2+(y+z)^2+(x+z)^2] \right. \\ +a^2(xy+yz+zx)\left(1-\frac{1}{N}(xy+yz+zx)\right) \\ \left. -\frac{2a^2}{N}(xy+yz+zx)^2\left(1-\frac{1}{N}(xy+yz+zx)\right)\right\}\frac{t^2}{2!} + \cdots. \end{aligned} \quad (21)$$

Table 1 Error table of Eq. (14) for different values of α and $D = N = 1$.

T	x	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$	
				u for frac ADM	u for ADM
0.2	0.25	3.67153	2.11381	1.60500	1.09500
	0.50	3.19074	2.08930	1.82000	1.38000
	0.75	1.65765	1.63300	1.85500	1.64500
	1.0	−1.22776	0.64385	1.68000	1.92000
0.4	0.25	6.06476	3.51596	2.22000	0.18000
	0.50	4.97276	2.99884	2.08000	0.32000
	0.75	1.92381	1.44686	1.42000	0.58000
	1.0	−3.68200	−1.42545	0.12000	1.08000
0.6	0.25	8.49447	5.40127	3.34500	−1.24500
	0.50	6.85192	4.36223	2.78000	−1.18000
	0.75	2.37235	1.55385	1.19500	−0.69500
	1.0	−5.84423	−3.54830	−1.68000	0.48000

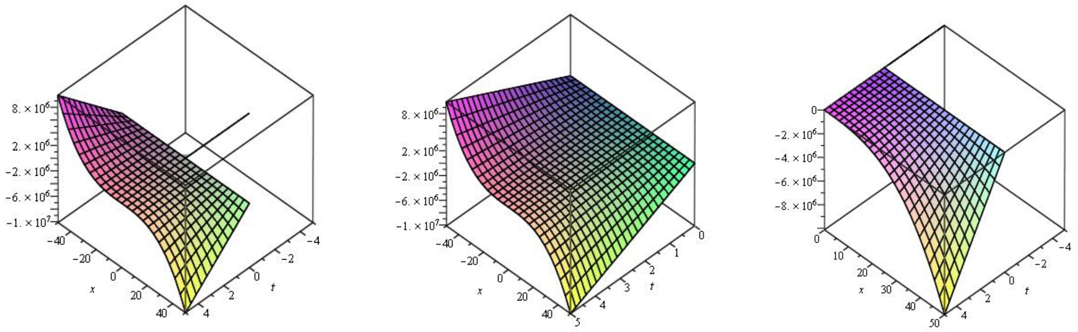

Figure 1 Graphical representation of solution of Eq. (14) for $D = a = N = 1$, $\alpha = 0.5$.

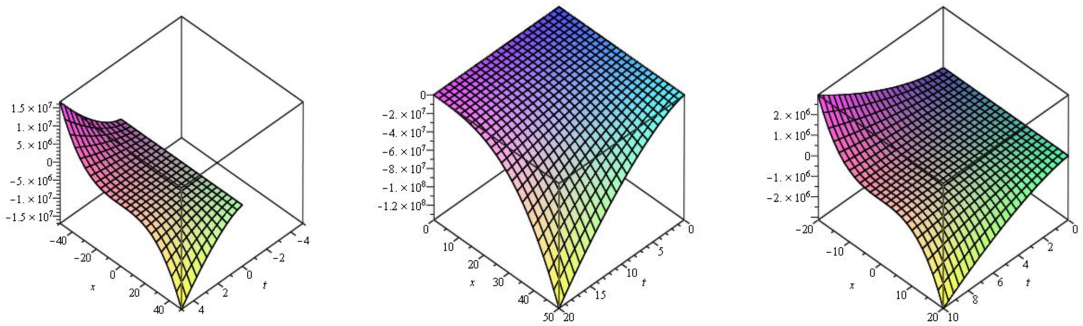

Figure 2 Graphical representation of solution of Eq. (14) for $D = a = N = 1$, $\alpha = 0.75$.

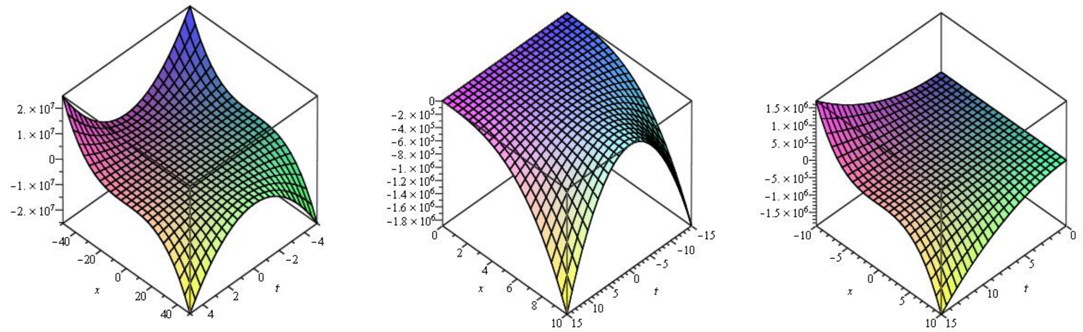

Figure 3 Graphical representation of solution of Eq. (14) for $D = a = N = 1$, $\alpha = 1$.

Example 3.2 Consider the nonlinear parabolic problem from (1) as;

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\Delta u + au\left(1 - \frac{u}{N}\right), \quad (x,t)\epsilon\ \Omega \times (0,T) \qquad \text{where } 0 < \alpha \leq 1 \tag{22}$$

with an initial condition of;

$$u(x,y,z,0) = u_0(x,y,z) = xy - z \geq 0 \quad on \quad [0,1] \times [0,1] \times [0,1]. \tag{23}$$

The HAM is used to solve the system second-order boundary value problem. The zeroth-order deformation equation is;

$$(1-p)\mathcal{L}[\emptyset(\tau;p)-u_0(\tau)]=p\hbar\widetilde{N}[\emptyset(\tau;p)], \tag{24}$$

where $\mathcal{L}$ is the linear operator of the fractional order. Considering the initial conditions as;

$$u_0(x,y,z,t)=xy-z\ , \tag{25}$$

The m-th order deformation equation is;

$$\mathcal{L}(u_m-\chi_m u_{m-1})=\hbar\,\Re_m\,(\vec{u}_{m-1})\ , \tag{26}$$

where

$$\Re_m\,(\vec{u}_{m-1})=\frac{\partial^\alpha u_{m-1}}{\partial t^\alpha}-D\Delta u_{m-1}-au_{m-1}\left(1-\frac{u_{m-1}}{N}\right). \tag{27}$$

The solution of the m-th order deformation equation is;

$$u_m=\chi_m u_{m-1}+\mathcal{L}^{-1}[\hbar\,\Re_m\,(\vec{u}_{m-1})], \tag{28}$$

where $m\geq 1$.

Consequently, the first few terms of the HAM series solution are as follows;

$$u_0(x,y,z,t)=xy-z\ , \tag{29}$$

$$u_1(x,y,z,t)=-\hbar\left\{a(xy-z)\left(1-\frac{1}{N}(xy-z)\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}, \tag{30}$$

$$u_2(x,y,z,t)=-\hbar^2\left\{-\frac{2Da}{N}[x^2+y^2+1]+a^2(xy-z)\left(1-\frac{1}{N}(xy-z)\right)-\frac{2a^2}{N}(xy-z)^2\left(1-\frac{1}{N}(xy-z)\right)\right\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{31}$$

and so on. The HAM series solution is;

$$u(x)=u_0(x)+u_1(x)+u_2(x)+\cdots \tag{32}$$

$$u(x,y,z,t)=xy-z-\hbar\left\{a(xy-z)\left(1-\frac{1}{N}(xy-z)\right)\right\}\frac{t^\alpha}{\Gamma(\alpha+1)}-\hbar^2\left\{-\frac{2Da}{N}[x^2+y^2+1]+a^2(xy-z)\left(1-\frac{1}{N}(xy-z)\right)-\frac{2a^2}{N}(xy-z)^2\left(1-\frac{1}{N}(xy-z)\right)\right\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots. \tag{33}$$

Now for $\alpha=1\ and\ \hbar=-1$ we get;

$$u(x,y,z,t) = xy - z + \left\{a(xy-z)\left(1-\frac{1}{N}(xy-z)\right)\right\}t - \left\{-\frac{2Da}{N}[x^2+y^2+1] + a^2(xy-z)\left(1-\frac{1}{N}(xy-z)\right) - \frac{2a^2}{N}(xy-z)^2\left(1-\frac{1}{N}(xy-z)\right)\right\}\frac{t^2}{2!} + \cdots \quad (34)$$

Table 2 Error table of Eq. (22) for different values of α and $D = N = 1$.

t	X	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$	
				u *for frac ADM*	u *for ADM*
0.2	0.25	0.06893	−0.67878	−0.86437	−1.16062
	0.50	0.32153	−0.34035	−0.53000	−0.77000
	0.75	0.71105	0.02468	−0.20062	−0.42438
	1.0	1.20000	0.40370	0.12000	−0.12000
0.4	0.25	1.27583	−0.05883	−0.68250	−1.86750
	0.50	1.36476	0.23139	−0.32000	−1.28000
	0.75	1.76448	0.64351	0.07250	−0.82250
	1.0	2.40000	1.14184	0.48000	−0.48000
0.6	0.25	8.49447	0.86577	−0.20437	−2.87063
	0.50	6.85192	1.04137	0.13000	−2.03000
	0.75	2.37235	1.47386	0.56938	−1.44438
	1.0	−5.84423	2.09769	1.08000	−1.08000

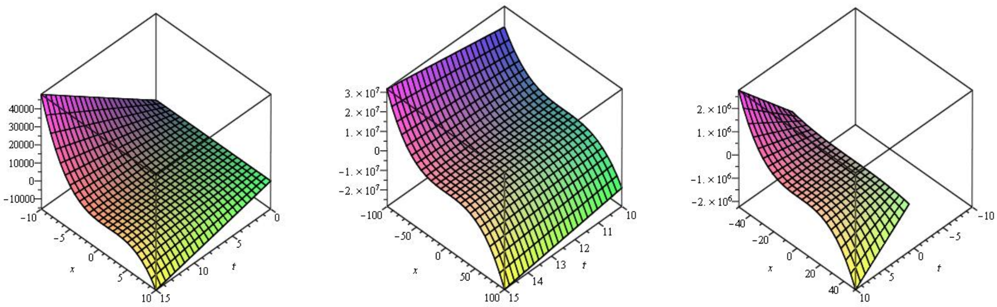

Figure 4 Graphical representation of solution of Eq. (22) for $D = a = N = 1$, $\alpha = 0.5$.

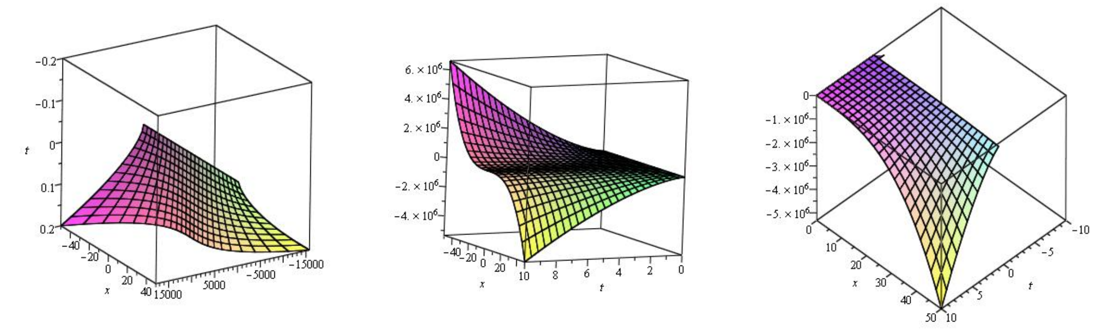

Figure 5 Graphical representation of Eq. (22) $D = a = N = 1\ , \alpha = 0.75$.

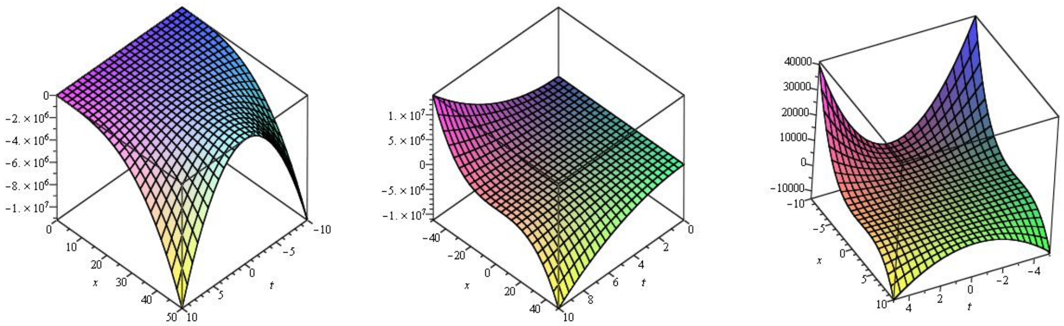

Figure 6 Graphical representation of solution of Eq. (22) for $D = a = N = 1\ , \alpha = 1$.

Example 3.3 Consider the nonlinear parabolic problem from (1) as;

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\Delta u + au\left(1 - \frac{u}{N}\right), \quad (x,t)\epsilon\ \Omega \times (0,T) \qquad \text{where } 0 < \alpha \leq 1 \tag{35}$$

with an initial condition of;

$$u(x,y,z,0) = u_0(x,y,z) = xyz \geq 0 \quad on \quad [0,1] \times [0,1] \times [0,1]. \tag{36}$$

The HAM is used for solving the system second-order boundary value problem. The zeroth-order deformation equation is;

$$(1-p)\mathcal{L}[\emptyset(\tau;p) - u_0(\tau)] = p\hbar\widetilde{N}[\emptyset(\tau;p)], \tag{37}$$

where $\mathcal{L}$ is the linear operator of the fractional order. Considering the initial conditions as;

$$u_0(x,y,z,t) = xyz, \tag{38}$$

the m-th order deformation equation is;

$$\mathcal{L}(u_m - \chi_m u_{m-1}) = \hbar\ \Re_m\ (\vec{u}_{m-1}), \tag{39}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - D\Delta u_{m-1} - a u_{m-1}\left(1 - \frac{u_{m-1}}{N}\right). \tag{40}$$

The solution of m-th order deformation equation is;

$$u_m = \chi_m u_{m-1} + \mathcal{L}^{-1}[\hbar\, \Re_m(\vec{u}_{m-1})], \tag{41}$$

where $m \geq 1$.

Consequently, the first few terms of the HAM series solution are as follows;

$$u_0(x,y,z,t) = xyz\,, \tag{42}$$

$$u_1(x,y,z,t) = -\hbar\left\{a(xyz)\left(1 - \frac{1}{N}(xyz)\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\right\}, \tag{43}$$

$$u_2(x,y,z,t) = -\hbar^2\left\{-\frac{2Da}{N}[(xy)^2 + (yz)^2 + (xz)^2] + a^2(xyz)\left(1 - \frac{1}{N}(xyz)\right) - \frac{2a^2}{N}(xyz)^2\left(1 - \frac{1}{N}(xyz)\right)\right\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{44}$$

and so on. The HAM series solution is;

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{45}$$

$$u(x,y,z,t) = xyz - \hbar\{a(xyz)\left(1 - \frac{1}{N}(xyz)\right)\frac{t^\alpha}{\Gamma(\alpha+1)}\} - \hbar^2\{-\frac{2Da}{N}[(xy)^2 + (yz)^2 + (xz)^2] + a^2(xyz)\left(1 - \frac{1}{N}(xyz)\right) - \frac{2a^2}{N}(xyz)^2\left(1 - \frac{1}{N}(xyz)\right)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \tag{46}$$

Now for $\alpha = 1$ *and* $\hbar = -1$;

$$u(x,y,z,t) = xyz + \left\{a(xyz)\left(1 - \frac{1}{N}(xyz)\right)\right\}t - \left\{-\frac{2Da}{N}[(xy)^2 + (yz)^2 + (xz)^2] + a^2(xyz)\left(1 - \frac{1}{N}(xyz)\right) - \frac{2a^2}{N}(xyz)^2\left(1 - \frac{1}{N}(xyz)\right)\right\}\frac{t^2}{2!} + \cdots \tag{47}$$

Table 1 Error table of Eq. (35) for different values of α and $D = N = 1$.

t	**X**	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$	
				u *for frac ADM*	u *for ADM*
0.2	0.25	0.77587	0.45610	0.33063	0.24438
	0.50	1.22616	0.78320	0.61000	0.49000
	0.75	1.71337	1.10328	0.87438	0.70063
	1.0	2.20000	1.40370	1.12000	0.88000
0.4	0.25	1.24631	0.76296	0.49750	0.15250
	0.50	1.87841	1.20774	0.84000	0.36000
	0.75	2.62131	1.67926	1.17250	0.47750
	1.0	3.40000	2.14184	1.48000	0.52000
0.6	0.25	1.70763	1.14294	0.75063	-0.02563
	0.50	2.51851	1.73429	1.19000	-0.11000
	0.75	3.52013	2.40772	1.64438	0.08063
	1.0	4.60000	3.09769	2.08000	-0.08000

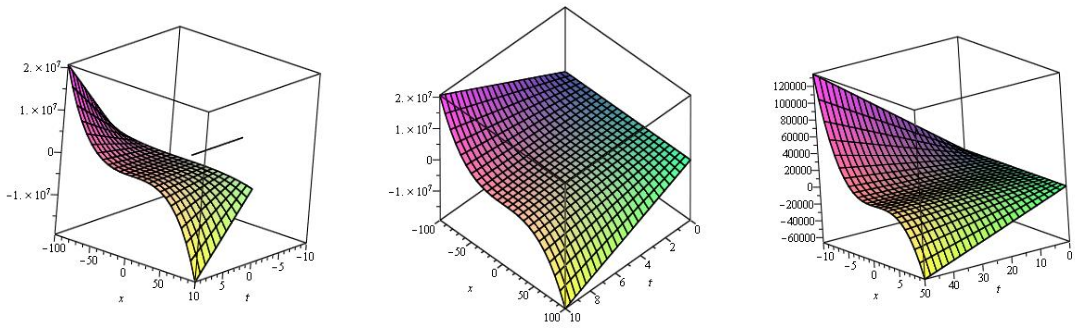

Figure 7 Graphical representation of solution of Eq. (35) for $D = a = N = 1$, $\alpha = 0.5$.

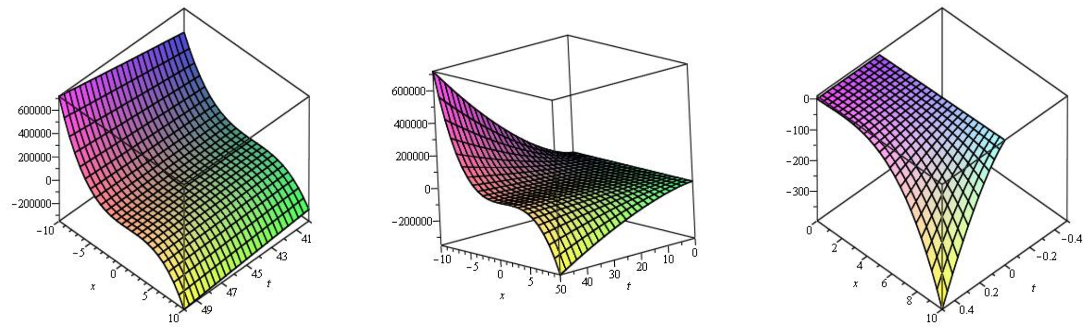

Figure 8 Graphical representation of solution of Eq. (35) for $D = a = N = 1$, $\alpha = 0.75$.

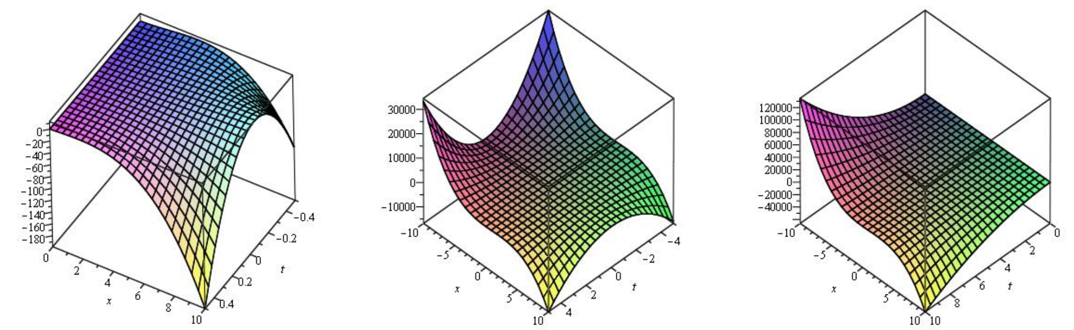

Figure 9 Graphical representation of solution of Eq. (35) for $D = a = N = 1$, $\alpha = 0.5$.

Conclusions

The HAM was applied to find appropriate solutions of fractional reaction-diffusion parabolic problems. It was observed that the proposed algorithm was fully compatible with the complexity of such problems. The numerical results explicitly revealed the complete reliability and efficiency of the proposed algorithm. Moreover, the obtained results are also fully compatible with the results obtained by the Variational Iteration Method (VIM) for $\alpha = 1$, but the HAM is easier to implement, since it is independent of the complexities arising in the calculation of Lagrange multipliers.

References

[1] S Abbasbandy. Approximate solution for the nonlinear model of diffusion and reaction in porous catalysts by means of the homotopy analysis method. *Chem. Eng. J*. 2008; **136**, 144-50.
[2] S Abbasbandy. Homotopy analysis method for heat radiation equations. *Int. Comm. Heat Mass Tran.* 2007; **34**, 380-7.
[3] S Abbasbandy. The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation. *Phys. Lett. A* 2007; **361**, 478-83.
[4] T Hayat, M Khan and S Asghar. Homotopy analysis of MHD flows of an Oldroyd 8- constant fluid. *Acta Mech.* 2004; **168**, 213-32.
[5] T Hayat, M Khan and S Asghar. Magneto hydrodynamic flow of an Oldroyd 6-constant fluid. *Appl. Math. Comput.* 2004; **155**, 417-25.
[6] SJ Liao. An approximate solution technique which does not depend upon small parameters (II): An application in fluid mechanics. *Int. J. Nonlinear Mech.* 1997, **32**, 815-22.
[7] SJ Liao. An explicit, totally analytic approximation of Blasius viscous flow problems. *Int .J. Nonlinear Mech.* 1999; **34**, 759-78.
[8] SJ Liao. On the analytic solution of magneto hydrodynamic flows of non-Newtonian fluids over a stretching sheet. *J. Fluid Mech.* 2003; **488**, 189-212.
[9] SJ Liao. An analytic approximate approach for free oscillations of self-excited systems. *Int. J. Nonlinear Mech.* 2004; **39**, 271-80.
[10] SJ Liao. On the homotopy analysis method for nonlinear problems. *Appl. Math. Comput*. 2004; **147**, 499-513.
[11] ST Mohyud-Din and MA Noor. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. *Int. J. Nonlinear Sci. Numer. Simulat.* 2008; **9**, 141-57.
[12] I Podlubny. *Fractional Differential Equations, An Introduction to Fractional Derivatives Fractional Differential Equations, some Methods of Their Solution and some of Their Applications*. 1st ed. Academic Press, San Diego, 1999, p. 1-340.
[13] M Caputo. Linear models of dissipation whose Q is almost frequency independent, Part II. *J. Roy. Astr. Soc.*1967; **13**, 529-35.
[14] YA Rossikhin and MV Shitikova. Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. *Appl. Mech. Rev*. 1997; **50**, 15-67.
[15] A Yildirim and ST Mohyud-Din. Analytical approach to space and time fractional Burger's equations. *Chin. Phys. Lett.* 2010; **27**, 090501.
[16] A Yildirim and ST Mohyud-Din. The numerical solution of the three-dimensional Helmholtz equations. *Chin. Phys. Lett.* 2010; **27**, 060201.
[17] S Momani. An algorithm for solving the fractional convection-diffusion equation with nonlinear source term. *Commun. Nonlinear. Sci. Numer. Simulat.* 2007; **12**, 1283-90.
[18] Z Odibat and S Momani. Numerical solution of Fokker-Planck equation with space and time-fractional derivatives. *Phys. Lett. A* 2007; **369**, 349-58.
[19] JD Murray. *Mathematical Biology*. 2nd ed. Springer-Verlag, Berlin, 1993, p. 1-554.