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Calculation of Oseen's Flow Past a Sphere using Indirect Boundary Element Method

Ghulam MUHAMMAD[1,*], Nawazish Ali SHAH[2] and Muhammad MUSHTAQ[2]

[1]*Department of Mathematics, GCS, Lahore, Pakistan*
[2]*Department of Mathematics, UET Lahore, Pakistan*

([*]Corresponding author's e-mail: g_muhammad123@hotmail.com)

Abstract

In this paper, a steady and incompressible Oseen's flow past a sphere is calculated using the indirect boundary element method (IDBEM). The surface of the sphere is discretized into quadrilateral elements, over which the velocity distribution is calculated. The computed results are compared with analytical results. It is found that both these results are in good agreement.

Keywords: Indirect boundary element method, Oseen's flow

Introduction

In recent past, well-known computational methods such as the finite difference method (FDM), the finite element method (FEM), and the boundary element method (BEM), have been applied for flow field calculations around objects. Out of these methods, BEM is a modern numerical technique in which only the surface of the body under consideration is discretized into different types of boundary elements [1]. BEM is well-suited to problems where the domain is exterior to the boundary, as in the case of flow past bodies. The most important features of BEM are the much smaller system of equations and the considerable reduction in data, which are essential to run a computer program efficiently. That is why BEM is more accurate, efficient and economical than other competitive computational methods. The study of flow past a sphere is of great practical importance in fluid dynamics. In Stokes' flow, the inertial effects become negligible, whereas the viscous effects become dominant, and in Oseen's flow, the inertial effects are also partially taken into consideration. So, the general Navier-Stokes' equations for steady Oseen's flow are greatly simplified. The first work on calculations of flow field around bodies was probably done by Hess and Smith [2]. The direct boundary element method (DBEM) for potential flow calculations around objects was applied in the past by Morino *et al*. [3]. In the recent past, boundary element methods have been applied by the author for calculation of Stoke's flow around the sphere [4].

Mathematical formulation of steady and incompressible Oseen's flow

The hydrodynamical equations governing the Oseen's flow are given as [5];

$$\left.\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\nabla^2 u \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\nabla^2 v \\ U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\nabla^2 w \end{aligned}\right] \quad (1)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (2)$$

$$\nabla^2 \phi = 0 \qquad (3)$$

$$p = \rho U \frac{\partial \phi}{\partial x} \qquad (4)$$

and

$$\nabla^2 p = 0. \qquad (5)$$

Steady Oseen's flow past a sphere

This problem was solved by Oseen and is often referred to as Oseen's flow. Oseen was the first who solved this problem analytically.

Let a solid sphere of radius 'a' be held fixed in a uniform stream U flowing steadily in the positive direction of the x-axis. Let the centre of the sphere be the origin of the coordinate system. Let the x-axis be in the direction of the uniform stream, as shown in **Figure 1**.

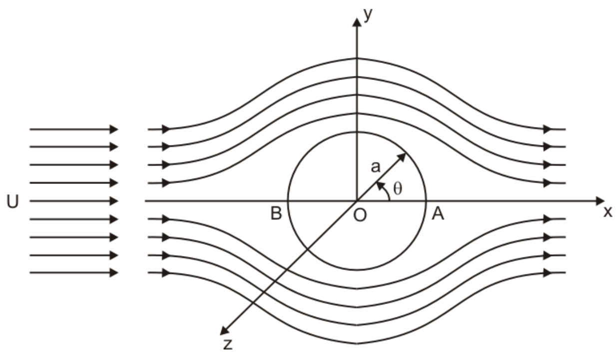


Figure 1 Oseen's flow past a sphere.

Velocity distribution

The velocity components are as under [6].

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x}+\frac{1}{2k}\frac{\partial \chi}{\partial x}-\chi \\ v &= -\frac{\partial \phi}{\partial y}+\frac{1}{2k}\frac{\partial \chi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z}+\frac{1}{2k}\frac{\partial \chi}{\partial z} \end{aligned}\right] \qquad (6)$$

where $\chi = \dfrac{C e^{-k(r-x)}}{r}$

and

$$\phi = -U x + \frac{A_0}{r} + A_1 \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + A_2 \frac{\partial_2}{\partial x^2}\left(\frac{1}{r}\right) + \ldots\ldots. \tag{7}$$

For small values of k and r,

$$\chi = C\left(\frac{1}{r} - k + \frac{k x}{r} + \ldots\ldots.\right) \tag{8}$$

and

$$k = \frac{U}{2 \upsilon} \tag{9}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= -U + A_0 \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + A_1 \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + A_2 \frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right) + \ldots\ldots. \\
\frac{\partial \phi}{\partial y} &= A_0 \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + A_1 \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right) + A_2 \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right) + \ldots\ldots. \\
\frac{\partial \phi}{\partial z} &= A_0 \frac{\partial}{\partial z}\left(\frac{1}{r}\right) + A_1 \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right) + A_2 \frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right) - \ldots\ldots. \\
\frac{\partial \chi}{\partial x} &= C\left\{-\frac{x}{r^3} + k\left(\frac{r^2 - x^2}{r^3}\right) + \ldots\ldots.\right\} \\
\frac{\partial \chi}{\partial y} &= C\left\{-\frac{y}{r^3} + k\left(-\frac{x y}{r^3}\right) + \ldots\ldots.\right\} \\
\frac{\partial \chi}{\partial z} &= C\left\{-\frac{z}{r^3} + k\left(-\frac{x z}{r^3}\right) + \ldots\ldots.\right\} \\
\frac{\partial \chi}{\partial x} &= C\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + k \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \ldots\ldots.\right] \\
&= C\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + k \frac{\partial}{\partial r}\left(\frac{x}{r}\right)\frac{\partial r}{\partial x} + \ldots\ldots.\right] \\
&= C\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + k\left(\frac{r^2 - x^2}{r^3}\right) + \ldots\ldots.\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi &= \frac{C}{2k}\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + k\left(\frac{r^2 - x^2}{r^3}\right) + \ldots\ldots.\right] - C\left(\frac{1}{r} - k + \frac{k x}{r} + \ldots\ldots.\right) \\
&= -\frac{C}{2k}\left[-\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - k\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \ldots\ldots. + \left(\frac{2k}{r} - 2k^2 + \frac{2k^2 x}{r} + \ldots\ldots.\right)\right] \\
&= -\frac{C}{2k}\left[-\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - \frac{k}{r} + \frac{k x^2}{r^3} + \frac{2k}{r} + \ldots\ldots.\right] \\
&= -\frac{C}{2k}\left[-\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{k}{r} + \frac{k x^2}{r^3} + \ldots\ldots.\right] \\
&= -\frac{C}{2k}\left[-\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \left(\frac{4k}{3r} - \frac{1}{3}\frac{k}{r}\right) + \frac{k x^2}{r^3} + \ldots\ldots.\right] \\
&= -\frac{C}{2k}\left[\frac{4k}{3r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{1}{3}k\left(\frac{3x^2 - r^2}{r^3}\right) + \ldots\ldots.\right] \\
&= -\frac{C}{2k}\left[\frac{4k}{3r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{1}{3}k r^2\left(\frac{3x^2 - r^2}{r^5}\right) + \ldots\ldots.\right]
\end{aligned}$$

$$= -\frac{C}{2k}\left[\frac{4k}{3r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{x}{r}\right) + \ldots\ldots\right]$$

$$\begin{aligned}
u &= U - A_0\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - A_1\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) - A_2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\ldots\ldots \\
&\qquad - \frac{C}{2k}\left\{\frac{4}{3}\frac{k}{r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \ldots\ldots\right\} \\
&= U - A_0\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - A_1\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) - A_2\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right) - \ldots\ldots \\
&\qquad - \frac{2}{3}\frac{C}{r} + \frac{C}{2k}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - \frac{1}{6}Cr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) - \ldots\ldots \\
&= \left(U - \frac{2}{3}\frac{C}{r}\right) + \left(-A_0 + \frac{C}{2k}\right)\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \left(-A_1 - \frac{1}{6}Cr^2\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) - \ldots\ldots
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
v &= -A_0\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - A_1\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right) - A_2\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right) - \ldots\ldots \\
&\qquad - \frac{C}{2k}\left\{-\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \frac{1}{3}kr^2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) + \ldots\ldots\right\} \\
&= -A_0\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - A_1\frac{\partial^2}{\partial y\,\partial x}\left(\frac{1}{r}\right) - A_2\frac{\partial^3}{\partial y\,\partial x^2}\left(\frac{1}{r}\right) - \ldots\ldots \\
&\qquad + \frac{C}{2k}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - \frac{1}{6}Cr^2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) - \ldots\ldots \\
&= -A_0\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - A_1\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) - A_2\frac{\partial^3}{\partial y\,\partial x^2}\left(\frac{1}{r}\right) - \ldots\ldots\frac{C}{2k}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) \\
&\qquad - \frac{1}{6}Cr^2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) - \ldots\ldots \\
&= \left(-A_0 + \frac{C}{2k}\right)\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \left(-A_1 - \frac{1}{6}Cr^2\right)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) + \ldots\ldots
\end{aligned} \tag{11}$$

$$\begin{aligned}
w &= -A_0\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - A_1\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right) - A_2\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right) - \ldots\ldots \\
&\qquad - \frac{C}{2k}\left\{-\frac{\partial}{\partial z}\left(\frac{1}{r}\right) + \frac{1}{3}kr^2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right) + \ldots\ldots\right\} \\
&= -A_0\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - A_1\frac{\partial^2}{\partial z\,\partial x}\left(\frac{1}{r}\right) - A_2\frac{\partial^3}{\partial z\,\partial x^2}\left(\frac{1}{r}\right) - \ldots\ldots \\
&\qquad + \frac{C}{2k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - \frac{1}{6}Cr^2\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right) - \ldots\ldots \\
&= -A_0\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - A_1\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right) - A_2\frac{\partial^3}{\partial z\,\partial x^2}\left(\frac{1}{r}\right) - \ldots\ldots\frac{C}{2k}\frac{\partial}{\partial z} - \frac{1}{6}Cr^2\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right) - \ldots \\
&= \left(-A_0 + \frac{C}{2k}\right)\frac{\partial}{\partial z}\left(\frac{1}{r}\right) + \left(-A_1 - \frac{1}{6}Cr^2\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \ldots\ldots
\end{aligned} \tag{12}$$

Boundary conditions [7]

$u = 0, \quad v = 0, \quad w = 0 \quad \text{for} \quad r = a$

and

$u \to U \quad \text{for} \quad r \to \infty$
Using above boundary conditions;

$$U - \frac{2}{3}\,\frac{C}{a} = 0, \quad -A_0 + \frac{C}{2k} = 0 \ \text{ and } \ -A_1 - \frac{1}{6} C a^2 = 0, \ \ldots\ldots$$

$$\Rightarrow \ C = \frac{3}{2} U a, \quad A_0 = \frac{C}{2k} = \frac{1}{2k}\left(\frac{3}{2} U a\right) = \frac{3}{4k} U a$$

$$\begin{aligned} A_1 &= -\frac{1}{6} C a^2 \\ &= -\frac{1}{6}\left(\frac{3}{2} U a\right) a^2 = -\frac{1}{4} U a^3, \ \ldots\ldots \end{aligned}$$

Substituting the values of above constants in Eqs. (10) - (12),

$$\begin{aligned} u &= \left(U - \frac{U a}{r}\right) + \left(-\frac{3}{4k} U a + \frac{3}{4} U a\right)\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \left(\frac{1}{4} U a^3 - \frac{1}{4} U a r^2\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \ldots\ldots \\ v &= \left(-\frac{3}{4} U a + \frac{3}{4} U a\right)\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \left(\frac{1}{4} U a^3 - \frac{1}{4} U a r^2\right)\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{r}\right) + \ldots\ldots \\ w &= \left(-\frac{3}{4} U a + \frac{3}{4} U a\right)\frac{\partial}{\partial z}\left(\frac{1}{r}\right) + \left(\frac{1}{4} U a^3 - \frac{1}{4} U a r^2\right)\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{r}\right) + \ldots\ldots \end{aligned}$$

or

$$\begin{aligned} u &= U\left(1 - \frac{a}{r}\right) + \frac{1}{4} U a\,(a^2 - r^2)\left(\frac{3x^2 - r^2}{r^5}\right) + \ldots\ldots \\ &= U\left\{\left(1 - \frac{a}{r}\right) + \frac{1}{4} a\,(a^2 - r^2)\left(\frac{3x^2 - r^2}{r^5}\right) + \ldots\ldots\right\} \end{aligned} \tag{13}$$

$$v = \frac{3}{4} U a \frac{(a^2 - r^2)\, x y}{r^5} + \ldots\ldots \tag{14}$$

$$w = \frac{3}{4} U a \frac{(a^2 - r^2)\, x z}{r^5} + \ldots\ldots \tag{15}$$

thus

$$V = \sqrt{u^2 + v^2 + w^2} \tag{16}$$

The velocity components in this case are truncated, so the analytical solution is taken approximately.

Equation of indirect boundary element method

For three-dimensional exterior flow problems, the equation of indirect boundary element method over the surface 'S' of the body is given by;

$$-\frac{1}{2}\Phi_i + \phi_\infty + \iint_{S-i} \Phi\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS = x_{\mathbf{i}} \tag{17}$$

where ϕ is velocity potential
ϕ_∞ is velocity potential at infinity
ϕ_i is velocity potential at the fixed point 'i'
Φ is total velocity potential
S–i is signifies that the point 'i' is excluded from the surface integral
r is distance of any point in the flow field from the centre of the sphere
x_i is the direction of flow

Discretization of sphere

The surface of the sphere is discretized into quadrilateral elements. The scheme of discretization is as shown in the **Figure 2**.

The indirect boundary element method is applied to calculate the slow flow solution around the sphere for which the analytical solution is available.

Consider the surface of the sphere in one octant to be divided into 3 quadrilateral elements by joining the centroid of the surface with the mid points of the curves in the coordinate planes as shown in **Figure 2** [8].

Then each element is divided further into 4 elements by joining the centroid of that element with the mid-point of each side of the element. Thus, one octant of the surface of the sphere is divided into 12 elements, and the whole surface of the body is divided into 96 boundary elements. The above mentioned method is adopted in order to produce a uniform distribution of element over the surface of the body.

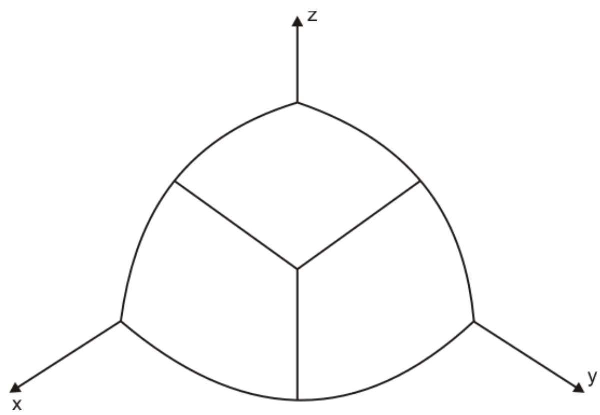


Figure 2 One octant of the surface of a sphere.

Figure 3 shows the method for finding the coordinate (x_p, y_p, z_p) of any point P on the surface of the sphere.

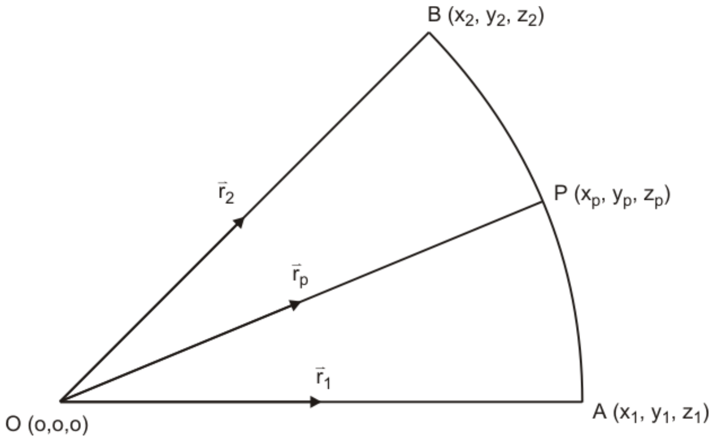


Figure 3 Method for Finding the Point P on the Surface of a Sphere.

From **Figure 3** the following equation is obtained.

$$\left|\vec{r_p}\right| = 1 \tag{18}$$

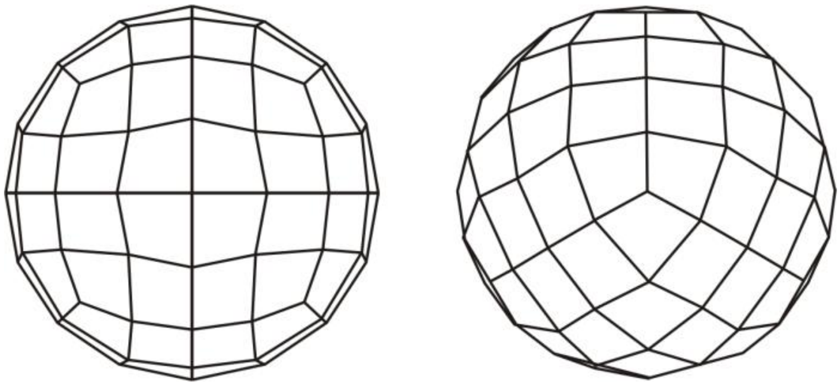
$$\vec{r_p} \cdot \vec{r_1} = \vec{r_p} \cdot \vec{r_2}$$
$$(\vec{r_1} - \vec{r_2}) \cdot \vec{r_p} = 0 \tag{19}$$

or in Cartesian form;

$$x_p^2 + y_p^2 + z_p^2 = 1$$
$$x_p\,(x_1 - x_2) + y_p\,(y_1 - y_2) + z_p\,(z_1 - z_2) = 0$$
$$x_p\,(y_1\,z_2 - z_1\,y_2) + y_p\,(x_2\,z_1 - x_1\,z_2) + z_p\,(x_1\,y_2 - x_2\,y_1) = 0 \tag{20}$$

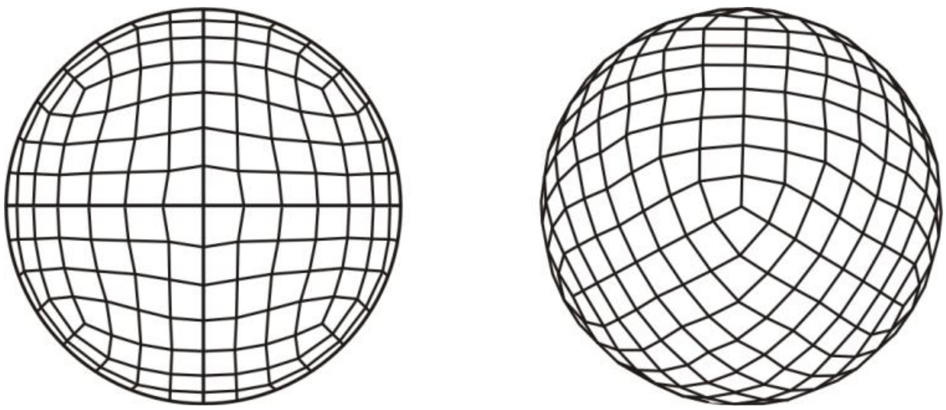
As the body possesses planes of symmetry, this fact may be used in the input to the program, and only the non-redundant portion needs to be specified by the input points. The other portions are automatically taken into account. The planes of symmetry are taken to be the coordinate planes of the reference coordinate system. The advantage of the use of symmetry is that it reduces the order of the resulting system of equations and consequently reduces the computing time in running a program. As a sphere is symmetric with respect to all 3 coordinate planes of the reference coordinate system, only one eighth of the body surface needs to be specified by the input points, while the other seven-eighths can be accounted for by symmetry.

The sphere is discretized into 96 and 384 boundary elements as shown below [9];



(a) (b)

Figure 4 Discretization of sphere into 96 boundary elements. The point of observation is (a) on the z-axis; (b) at 45° to all axes.



(a) (b)

Figure 5 Discretization of sphere into 384 boundary elements. The point of observation is (a) on the z-axis; (b) at 45° to all axes.

The sphere is discretized into 96 and 384 boundary elements and the computed results are compared with analytical solutions for the sphere using Fortran programming.

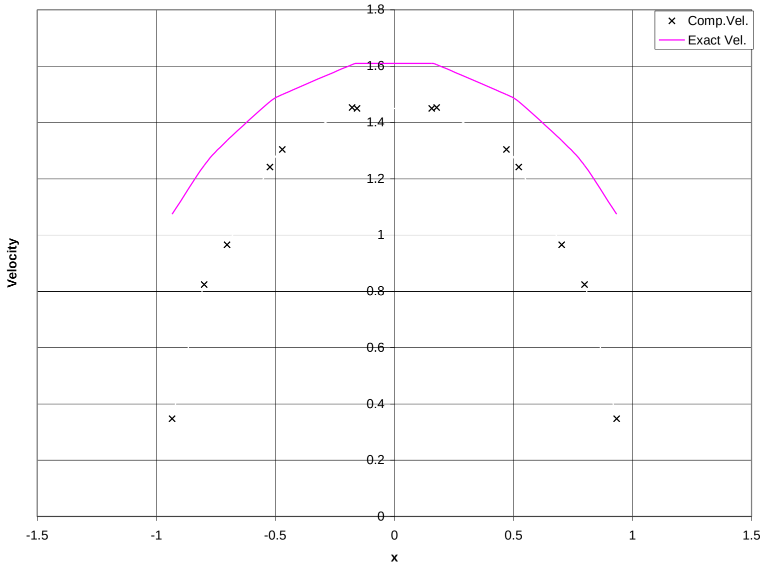


Figure 6 Comparison of computed and analytical velocity distributions over the surface of the sphere using 96 boundary elements.

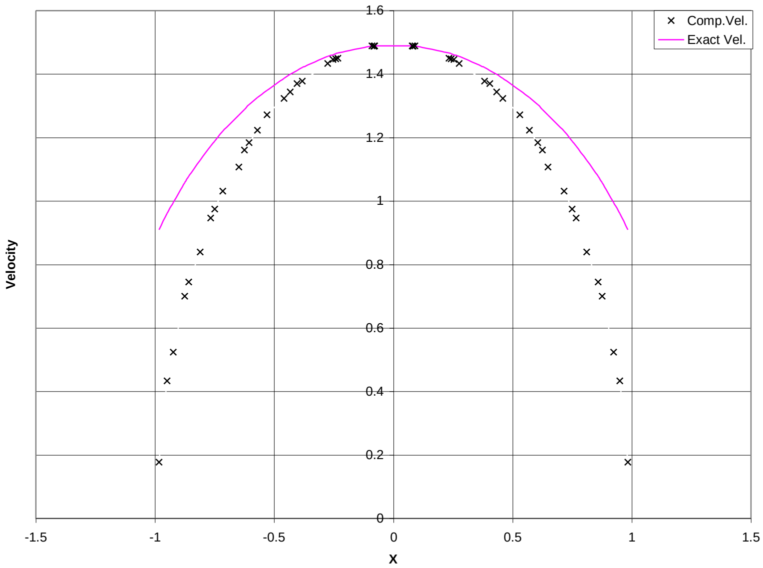


Figure 7 Comparison of computed and analytical velocity distributions over the surface of the sphere using 384 boundary elements.

Since the streamlines are symmetrical around the sphere, the **Figures 6** and **7** shown above are symmetrical on both sides. At the top of **Figure 7**, the computed results are convergent with the approximate analytical results and as can be seen, the computed results are slightly different with the analytical ones due to increase of viscous effects and truncation velocity components.

Conclusions

The indirect boundary element method has been used to calculate Oseen's flow past a sphere using different numbers of boundary elements. The computed velocities obtained in this way were compared with approximate analytical velocities for this flow over the boundary of the sphere. From the above figures, it is concluded that the computed values are in good agreement with the approximate analytical values for the sphere.

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