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Modified Homotopy Analysis Method for Zakharov-Kuznetsov Equations

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Abstract

In this paper, we apply Modified Homotopy Analysis Method (MHAM) to find appropriate solutions to Zakharov-Kuznetsov equations, which are of utmost importance in applied and engineering sciences. The proposed modification is an elegant coupling of the Homotopy Analysis Method (HAM) and Taylor's series. Numerical results, coupled with graphical representation, explicitly reveal the complete reliability of the proposed algorithm.

Keywords: Homotopy analysis method, Taylor's series, exact solutions, MAPLE, Zakharov-Kuznetsov equations

Introduction

The rapid development of nonlinear sciences witnesses a wide range of analytical and numerical techniques by various scientists [1-19]. Most of the developed schemes have their limitations, like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems [1-11]. In a similar context, Liao [7-9] developed the Homotopy Analysis Method (HAM) which has been applied to a wide range of nonlinear problems of a physical nature; see [1-19] and the references therein. The basic motivation of the present study is the modification of the traditional HAM to tackle Zakharov-Kuznetsov equations. The proposed modification is made by combining the traditional HAM with Taylor's series. It is observed that the proposed modification is highly effective and absorbs some of the basic deficiencies of the original version of HAM. Moreover, this modified approach (MHAM) is more user-friendly and overcomes the complexities of selection of initial value. Several examples are given which reveal the efficiency and reliability of the proposed algorithm.

Analysis of homotopy analysis method (HAM)

The following differential equation

$$N[u(\tau)] = 0 \tag{1}$$

is considered where N is a nonlinear operator, τ denotes independent variables, and $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional Homotopy method, Liao constructs a so called zero - order deformation equation.

$$(1-p)L[\varphi(\tau;p)-u_0(\tau)] = phH(\tau)N[\varphi(\tau;p)], \quad (2)$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, and $u(\tau;p)$ is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary components in HAM. Obviously, when, $p = 0$ and $p = 1$, it holds;

$$\varphi(\tau;0) = u_0(\tau),\ \varphi(\tau;1) = u(\tau). \quad (3)$$

Thus, as p increases from 0 to 1, the solution $\varphi(\tau;p)$ varies from the initial guesses $u_0(\tau)$ to the solution $u(\tau)$. Expanding in the Taylor series with respect to p,

$$\varphi(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m, \quad (4)$$

where $u_m(\tau) = \frac{1}{m!}\frac{\partial^m \varphi(\tau;p)}{\partial p^m}$ at $p = 0$, (5)

is obtained. If the auxiliary linear operator, the initial guess, the auxiliary h, and the auxiliary function are so properly chosen, the above series converges at $p = 1$.

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \quad (6)$$

is then obtained. The vector is defined as;

$$\vec{u} = \{u_0(\tau), u_1(\tau), u_2(\tau), \ldots, u_0(\tau)\} \quad (7)$$

Differentiating Eq. (2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, the mth-order deformation equation is obtained;

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau)R_m(\vec{u}_{m-1}) \quad (8)$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\varphi(\tau;p)}{\partial p^{m-1}} \text{ at } p = 0$$

and

$$\chi_m = 0, m \leq 1,$$
$$= 1, m > 1,$$

Applying L^{-1} both sides of (8),

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + hL^{-1}[H(\tau)R_m(\vec{u}_{m-1})]$$

is obtained. This way, it is easy to obtain u_m for m ≥ 1, at mth- order;

$$u(\tau) = \sum_{m=0}^{M} u_m(\tau). \quad (9)$$

when $M \to \infty$, an accurate approximation of the original Eq. (1) is considered. For the convergence of the above method the reader is refered to is referred to Liao's work. If Eq. (1) admits a unique solution, then this method will produce a unique solution. If Eq. (1) does not possess a unique solution, the HAM will give a solution among many other (possible) solutions.

Numerical application

In this section, MHAM is applied to find appropriate solutions of Zakharov-Kuznetsov equations. The numerical results are very encouraging.

Example 1 Consider the ZK(2,2,2) equation;

$$u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0, \tag{10a}$$
$$u(x,y,0) = \frac{4}{3}\lambda \sinh^2(x+y), \tag{10b}$$

To solve the given Equation by HAM the linear operator;

$$L[\varphi(x,y,t;q)] = \frac{\partial}{\partial t}\big(\varphi(x,y,t;q)\big), \tag{11}$$

is chosen, with the property;

$$L[c_1 + tc_2] = 0, \tag{12}$$

where c_1 and c_2 are the integral constants. The inverse operator L^{-1} is given by;

$$L^{-1} = \int_0^t (.)dt, \tag{13}$$

and a nonlinear operator is defined as;

$$N[\varphi(x,y,t;q)] = \varphi(x,y,t;q)_t + (\varphi(x,y,t;q)^2)_x + \frac{1}{8}(\varphi(x,y,t;q)^2)_{xxx} + \frac{1}{8}(\varphi(x,y,t;q)^2)_{yyx}, \tag{14}$$

Using the above definition, the zeroth-order deformation equation is constructed.

$$(1-q)L[\varphi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\varphi(x,y,t;q)]. \tag{15}$$

For $q = 0$ and $q = 1$,

$$\varphi(x,y,t;0) = u_0(x,y,t), \qquad \varphi(x,y,t;1) = u(x,y,t). \tag{16}$$

can be written. Thus, the mth-order deformation equation is obtained.

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}), \tag{17}$$

with initial condition $u_m(x,y,0) = 0$.

where $$R_m(\vec{u}_{m-1}) = h\begin{bmatrix} \big(u_{m-1}(x,y,t)\big)_t + (\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t))_x \\ +\frac{1}{8}\big(\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t)\big)_{xxx} \\ +\frac{1}{8}\big(\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t)\big)_{yyx} \end{bmatrix}. \tag{18}$$

After applying Taylor's series, the result is;

$$R_m(\vec{u}_{m-1}) = h\Big[(u_{m-1}(x,y,t)+f_m)_t + (\textstyle\sum_{r=0}^{m-1}(u_r(x,y,t)+f_{r+1})\,(u_{m-1-r}(x,y,t)+f_{m-r}))_x + \frac{1}{8}(\sum_{r=0}^{m-1}(u_r(x,y,t)+f_{r+1})(u_{m-1-r}(x,y,t)+f_{m-r}))_{xxx} + \frac{1}{8}(\sum_{r=0}^{m-1}(u_r(x,y,t)+f_{r+1})(u_{m-1-r}(x,y,t)+f_{m-r}))_{yyx}\Big]. \quad (19)$$

Now the solutions of the mth-order deformation equation are;

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1, \quad (20)$$

an initial approximation is used at the start;

$$u_0(x,t) = \frac{4}{3}\sinh^2(y), \quad (21)$$

and by means of the iteration formula as discussed above, if $h=-1, H=1$, the others components can be directly as;

$$\begin{aligned} u_1 = &-\frac{32}{9}\lambda^2(-14xcosh^2(y) + 2x + 10xsinh(y)\,cosh(y)\,sinh(x+y)\,cosh(x+y) - 8\,sinh(x+y)\,cosh(x+y) - 5sinh(y)\,cosh(y) - 2x\cosh^2(x+y) + 10\,sinh(x+y)\,cosh^3(x+y) + 12xcosh^4(x+y) + 7\,sinh(y)\,cosh(y)\,cosh^2(x+sy) + 2\,sinh(x+y)\,cosh(x+y)\,cosh^2(y) + 4xcosh^2(y)cosh^2(x+y)t), \quad (22) \\ u_2 = &\frac{16}{27}\lambda^2 t(-84xcosh^2(y)cosh^2(x+y) - 18sinh(x+y)\,cosh(x+y)\,cosh^2(y) - 24x\sinh(y)\cosh(y)\,sinh(x+y)\,coh(x+y) + \cdots, \\ &\vdots \end{aligned}$$

The series form solution is given by;

$$u(x,t) = \frac{4}{3}sinh^2(y) + -\frac{32}{9}\lambda^2(-14x\cosh^2(y) + 2x + 10x\sinh(y)\cosh(y)\,sinh(x+y)\,cosh(x+y) - 8\,sinh(x+y)\,cosh(x+y) - 5sinh(y)\,cosh(y) - 2xcosh^2(x+y) + 10\,sinh(x+y)\,cosh^3(x+y) + \cdots \quad (23)$$

The rest of the components of the iteration formulae can be obtained using MAPLE.

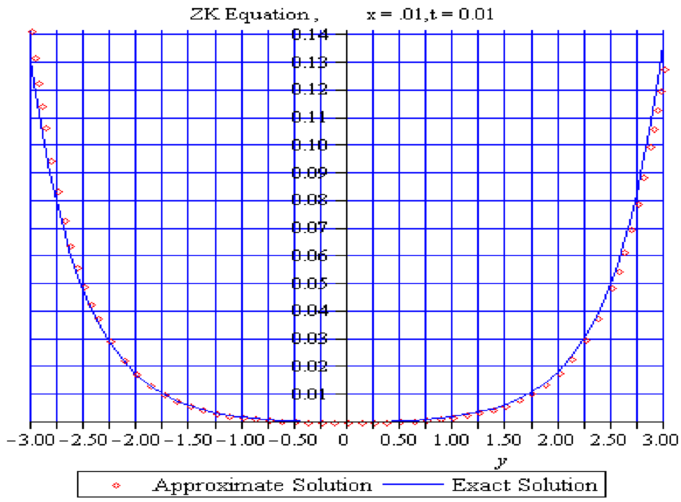

Figure 1 Graphical representation of exact and approximate solutions of Eq. (10) for different values of x and t.

Example 2 Consider the following ZK nonlinear PDE;

$$u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0, \tag{24a}$$
$$u(x,y,0) = -\frac{4}{3}\lambda \cosh^2(x+y), \tag{24b}$$

To solve the given Equation by HAM the linear operator;

$$L[\varphi(x,y,t;q)] = \frac{\partial}{\partial t}\big(\varphi(x,y,t;q)\big), \tag{25}$$

is chosen, with the property;

$$L[c_1 + tc_2] = 0, \tag{26}$$

where c_1 and c_2 are the integral constants. The inverse operator L^{-1} is given by;

$$L^{-1} = \int_0^t (.)dt, \tag{27}$$

A nonlinear operator can now be defined as;

$$N[\varphi(x,y,t;q)] = \varphi(x,y,t;q)_t + (\varphi(x,y,t;q)^2)_x + \frac{1}{8}(\varphi(x,y,t;q)^2)_{xxx} + \frac{1}{8}(\varphi(x,y,t;q)^2)_{yyx}, \tag{28}$$

Using the above definition, zeroth-order deformation equation can be constructed;

$$(1-q)L[\varphi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\varphi(x,y,t;q)]. \tag{29}$$

for $q = 0$ and $q = 1$,

$$\varphi(x,y,t;0) = u_0(x,y,t), \qquad \varphi(x,y,t;1) = u(x,y,t). \tag{30}$$

can be written. Thus, the mth-order deformation equation is obtained;

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}). \quad (31)$$

with initial condition $u_m(x,y,0) = 0$

$$\text{where } R_m(\vec{u}_{m-1}) = h\begin{bmatrix} \left(u_{m-1}(x,y,t)\right)_t + (\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t))_x \\ +\frac{1}{8}\left(\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t)\right)_{xxx} \\ +\frac{1}{8}\left(\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t)\right)_{yyx} \end{bmatrix}. \quad (32)$$

After applying Taylor's series, the result is;

$$R_m(\vec{u}_{m-1}) = h\Big[(u_{m-1}(x,y,t) + f_m)_t + (\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})\,(u_{m-1-r}(x,y,t) + f_{m-r}))_x + \frac{1}{8}(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})(u_{m-1-r}(x,y,t) + f_{m-r}))_{xxx} + \frac{1}{8}(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})(u_{m-1-r}(x,y,t) + f_{m-r}))_{yyx}\Big]. \quad (33)$$

Now the solutions of the mth-order deformation equation are;

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1, \quad (34)$$

an initial approximation is used at the start;

$$u_0(x,t) = -\frac{4}{3}\lambda cosh^2(y), \quad (35)$$

and by means of the iteration formula as discuss above, if $h = -1, H = 1$, the others components can be directly obtained as;

$$u_1 = -\frac{16}{9}\lambda^2(-3\cosh(y)\sinh(y) + 12cosh^3(y)\sinh(y) + 2x + 24cosh^4(y)x - 24cosh^2(y)x)t), \quad (36)$$

$$u_2 = -\frac{16}{27}\lambda^2 t(-9\cosh(y)\sinh(y) + 18cosh^3(y)\sinh(y) + 3x + 72cosh^4(y)x - 54cosh^2(y)x + 336\lambda\cosh(y)\sinh(y)\,tx + \cdots, \quad (37)$$

$$\vdots$$

The series form solution is given by;

$$u(x,t) = \begin{bmatrix} -\frac{4}{3}\cosh^2(y) - \frac{16}{9}\lambda^2(-3\cosh(y)\sinh(y) + 12cosh^3(y)\sinh(y) + 2x \\ +24\cosh^4(y)x - 24cosh^2(y)x)t) - \\ \frac{16}{27}\lambda^2 t(-9\cosh(y)\sinh(y) - \frac{16}{27}\lambda^2 t(-9\cosh(y)\sinh(y) + \cdots \end{bmatrix}. \quad (38)$$

The rest of the components of the iteration formulae can be obtained using MAPLE.

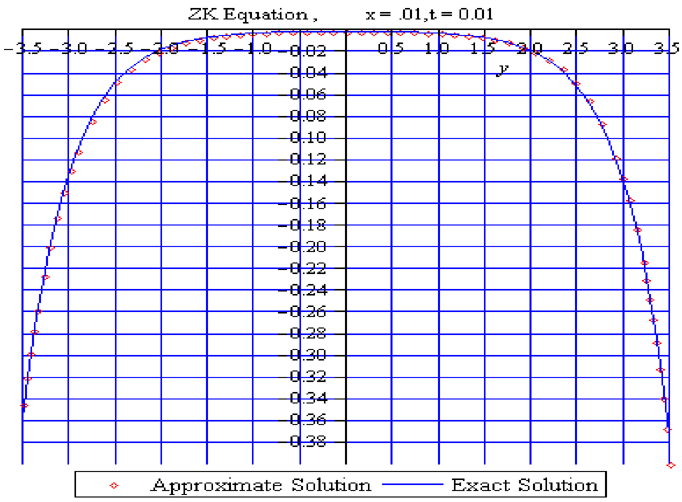

Figure 2 Graphical representation of exact and approximate solutions of Eq. (24) for different values of *x* and *t*.

Example 3 Consider the following nonlinear PDE;

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \tag{39a}$$
$$u(x, y, 0) = \frac{3}{2}\lambda \sinh\left(\frac{1}{6}(x + y)\right), \tag{39b}$$

To solve the given equation by HAM the linear operator;

$$L[\varphi(x, y, t; q)] = \frac{\partial}{\partial t}\big(\varphi(x, y, t; q)\big), \tag{40}$$

is chosen with the property;

$$L[c_1 + tc_2] = 0, \tag{41}$$

where c_1 and c_2 are the integral constants. The inverse operator L^{-1} is given by;

$$L^{-1} = \int_0^t (.)dt, \tag{42}$$

a nonlinear operator can be defined as;

$$N[\varphi(x, y, t; q)] = \varphi(x, y, t; q)_t + (\varphi(x, y, t; q)^2)_x + 2(\varphi(x, y, t; q)^2)_{xxx} + 2(\varphi(x, y, t; q)^2)_{yyx}, \tag{43}$$

using the above definition, the zeroth-order deformation equation is constructed;

$$(1 - q)L[\varphi(x, y, t; q) - u_0(x, y, t)] = qhH(x, y, t)N[\varphi(x, y, t; q)]. \tag{44}$$

for $q = 0$ and $q = 1$,

$$\varphi(x, y, t; 0) = u_0(x, y, t), \qquad \varphi(x, y, t; 1) = u(x, y, t). \tag{45}$$

can be written. Thus, we obtain the mth −order deformation equation;

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}). \tag{46}$$

with the initial condition $u_m(x,y,0) = 0$

where $$R_m(\vec{u}_{m-1}) = h\begin{bmatrix} \left(u_{m-1}(x,y,t)\right)_t + (\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t))_x \\ +2\left(\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t)\right)_{xxx} \\ +2\left(\sum_{r=0}^{m-1} u_r(x,y,t)\, u_{m-1-r}(x,y,t)\right)_{yyx} \end{bmatrix}. \tag{47}$$

After applying Taylor's series the result is;

$$R_m(\vec{u}_{m-1}) \ = h\begin{bmatrix} (u_{m-1}(x,y,t) + f_m)_t + (\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})\,(u_{m-1-r}(x,y,t) + f_{m-r}))_x \\ + 2(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})(u_{m-1-r}(x,y,t) + f_{m-r}))_{xxx} \\ +2(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})(u_{m-1-r}(x,y,t) + f_{m-r}))_{yyx} \end{bmatrix}. \tag{48}$$

Now the solutions of the mth −order deformation equation are;

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1, \tag{49}$$

an initial approximation can be used at the start;

$$u_0(x,t) = \frac{3}{2}\lambda sinh\left(\frac{1}{6}y\right), \tag{50}$$

by means of the iteration formula as discuss above, if $h = -1, H = 1$, the other components can be directly obtained as;

$$u_1 = -\frac{1}{128}\lambda^3\begin{pmatrix} -8\, sinh\left(\frac{1}{6}y\right)x + 108\, sinh\left(\frac{1}{6}y\right)xcosh^2\left(\frac{1}{6}y\right)\times \\ -300\, cosh\left(\frac{1}{6}y\right) + 9x^2\, cos\, h^3\left(\frac{1}{6}y\right) + 348cosh^3\left(\frac{1}{6}y\right) - 2cosh\ \left(\frac{1}{6}y\right)x^2 \end{pmatrix}, \tag{51}$$

$$u_2 = \frac{1}{18432}\lambda^3 t\left(6048\, sinh\left(\frac{1}{6}y\right)x - 11232\, sinh\left(\frac{1}{6}y\right)xcosh^2\left(\frac{1}{6}y\right) + 10368\, cosh\left(\frac{1}{6}y\right) - 3888x^2 - 10368cosh^3\left(\frac{1}{6}y\right) + 3600cosh\left(\frac{1}{6}y\right)x^2 + \cdots\right), \tag{52}$$

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The series form solution is given by;

$$u(x,t) = \frac{3}{2}\lambda sinh\left(\frac{1}{6}y\right) - \frac{1}{128}\lambda^3\left(-8\sinh\left(\frac{1}{6}y\right)x + 108\sinh\left(\frac{1}{6}y\right)xcosh^2\left(\frac{1}{6}y\right) - 300\, cosh\left(\frac{1}{6}y\right) + 9x^2\, cos\, h^3\left(\frac{1}{6}y\right) + 348\cosh^3\left(\frac{1}{6}y\right) - 2cosh\left(\frac{1}{6}y\right)x^2\right) + \frac{1}{18432}\lambda^3 t(6048\, sinh\left(\frac{1}{6}y\right)x - \cdots. \tag{53}$$

The rest of the components of the iteration formulae can be obtained using MAPLE.

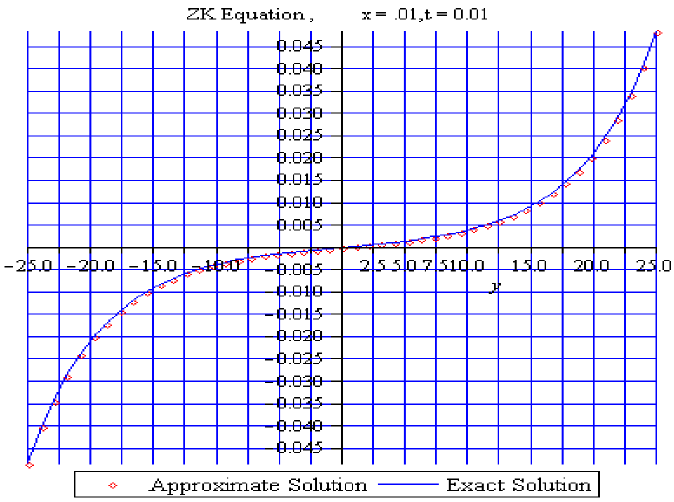

Figure 3 Graphical representation of exact and approximate solutions of Eq. (39) for different values of *x* and *t*.

Example 4 Consider the following nonlinear PDE;

$$u_t + (u^3)_x + \frac{1}{8}(u^3)_{xxx} + \frac{1}{8}(u^3)_{yyx} = 0, \tag{54a}$$

$$u(x,y,0) = \frac{3}{2}\lambda \cosh\left(\frac{1}{6}(x+y)\right), \tag{54b}$$

To solve the given Equation by HAM the linear operator;

$$L[\varphi(x,y,t;q)] = \frac{\partial}{\partial t}\left(\varphi(x,y,t;q)\right), \tag{55}$$

is chosen with the property;

$$L[c_1 + tc_2] = 0, \tag{56}$$

where c_1 and c_2 are the integral constants. The inverse operator L^{-1} is given by;

$$L^{-1} = \int_0^t (.)dt, \tag{57}$$

a nonlinear operator can be defined as;

$$N[\varphi(x,y,t;q)] = \varphi(x,y,t;q)_t + (\varphi(x,y,t;q)^2)_x + \frac{1}{8}(\varphi(x,y,t;q)^2)_{xxx} + \frac{1}{8}(\varphi(x,y,t;q)^2)_{yyx}, \tag{58}$$

using the above definition, the zeroth-order deformation equation can be constructed;

$$(1-q)L[\varphi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\varphi(x,y,t;q)]. \tag{59}$$

for $q = 0$ and $q = 1$,

$$\varphi(x,y,t;0) = u_0(x,y,t), \quad \varphi(x,y,t;1) = u(x,y,t). \tag{60}$$

can be written. Thus, we obtain the mth −order deformation equation;

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\vec{u}_{m-1}). \tag{61}$$

with the initial condition $u_m(x,y,0) = 0$

where

$$R_m(\vec{u}_{m-1}) = h\begin{bmatrix} \left(u_{m-1}(x,y,t)\right)_t + \left(\sum_{r=0}^{m-1} u_r(x,y,t)\,u_{m-1-r}(x,y,t)\right)_x \\ +\frac{1}{8}\left(\sum_{r=0}^{m-1} u_r(x,y,t)\,u_{m-1-r}(x,y,t)\right)_{xxx} \\ +\frac{1}{8}\left(\sum_{r=0}^{m-1} u_r(x,y,t)\,u_{m-1-r}(x,y,t)\right)_{yyx} \end{bmatrix}. \tag{62}$$

After applying Taylor's series the result is;

$$R_m(\vec{u}_{m-1}) \;= h\begin{bmatrix} (u_{m-1}(x,y,t) + f_m)_t + \left(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})\,(u_{m-1-r}(x,y,t) + f_{m-r})\right)_x \\ +\frac{1}{8}\left(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})(u_{m-1-r}(x,y,t) + f_{m-r})\right)_{xxx} \\ +\frac{1}{8}\left(\sum_{r=0}^{m-1}(u_r(x,y,t) + f_{r+1})(u_{m-1-r}(x,y,t) + f_{m-r})\right)_{yyx} \end{bmatrix}. \tag{63}$$

Now the solutions of the mth −order deformation equation are;

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], m \geq 1, \tag{64}$$

an initial approximation can be used at the start;

$$u_0(x,t) = \frac{3}{2}\lambda cosh\left(\frac{1}{6}y\right), \tag{65}$$

by means of the iteration formula as discuss above, if $h = -1, H = 1$, the other components can be directly obtained can be directly obtained as;

$$u_1 = -\frac{1}{2048}\lambda^3\left(3588\, sinh\left(\frac{1}{6}y\right) cosh^2\left(\frac{1}{6}y\right) - 97\, sinh\left(\frac{1}{6}y\right)x^2 - 48\, sinh\left(\frac{1}{6}y\right) + 99x^2 sinh\left(\frac{1}{6}y\right) cos\,h^2\left(\frac{1}{6}y\right) - 1180 cos\,h\left(\frac{1}{6}y\right)x + 1188 cosh^3\left(\frac{1}{6}y\right)x\right), \tag{66}$$

$$u_2 = \frac{1}{4718592}\lambda^3 t\left(-165888\, sinh\left(\frac{1}{6}y\right) cosh^2\left(\frac{1}{6}y\right) + 4608\, sinh\left(\frac{1}{6}y\right)x^2 - \cdots,\right. \tag{67}$$

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.
..

The series form solution is given by;

$$u(x,t) = \frac{3}{2}\lambda cosh\left(\frac{1}{6}y\right) - \frac{1}{2048}\lambda^3\left(3588\, sinh\left(\frac{1}{6}y\right)cosh^2\left(\frac{1}{6}y\right) - 97\, sinh\left(\frac{1}{6}y\right)x^2 - 48\, sinh\left(\frac{1}{6}y\right) + 99x^2 sinh\left(\frac{1}{6}y\right)cos\,h^2\left(\frac{1}{6}y\right) - 1180 cos\,h\left(\frac{1}{6}y\right)x + 1188 cosh^3\left(\frac{1}{6}y\right)x\right) + \frac{1}{4718592}\lambda^3 t(-165888\, sinh\left(\frac{1}{6}y\right)cosh^2\left(\frac{1}{6}y\right) + \cdots, \quad (68)$$

The rest of the components of the iteration formulae can be obtained using MAPLE.

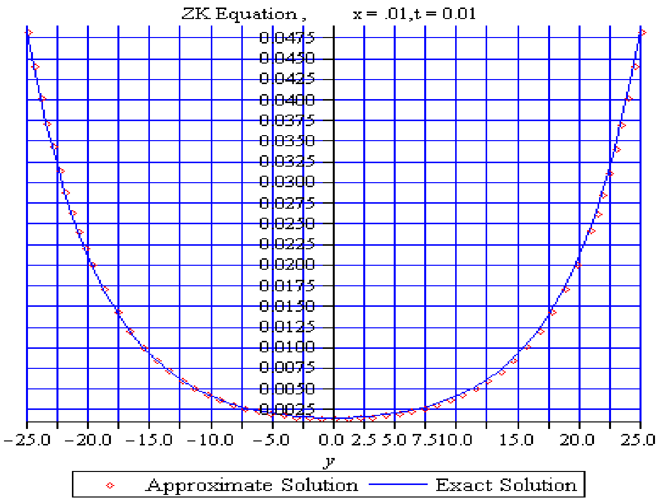

Figure 4 Graphical representation of exact and approximate solutions of Eq. (4) for different values of *x* and *t*.

Conclusions

MHAM is applied to find appropriate solutions of nonlinear partial differential equations (PDEs). The proposed modified version is fully capable to cope with the nonlinearity of the physical problems. The suggested technique can be a nice addition in the existing techniques of solving nonlinear problems of a versatile physical nature.

References

[1] S Abbasbandy. Homotopy analysis method for generalized Benjamin-Bona-Mahony equation. *Z. Angew. Math. Phys.* 2008; **59**, 51-62.
[2] S Abbasbandy and FS Zakaria. Soliton solutions for the fifth-order K-dV equation with the homotopy analysis method. *Nonlinear Dyn.* 2008; **51**, 83-7.
[3] S Abbasbandy. Homotopy analysis method for the Kawahara equation. *Nonlinear Anal. B* 2010; **11**, 307-12.
[4] S Abbasbandy, E Shivanian and K Vajravelu. Mathematical properties of curve in the framework of the homotopy analysis method. *Anal. Meth.* 2011; **16**, 4268-75.
[5] S Abbasbandy. The application of homotopy analysis method to nonlinear equations arising in heat transfer. *Phys. Lett. A* 2006; **360**, 109-13.
[6] S Abbasbandy. The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation. *Phys. Lett. A* 2007; **361**, 478-83.

[7] SJ Liao. *Beyond Perturbation: Introduction to the Homotopy Analysis Method. In*: CRC Press, Boca Raton, Chapman and Hall, 2003.
[8] SJ Liao. On the homotopy analysis method for nonlinear problems. *Appl. Math. Comput.* 2004; **47**, 499-513.
[9] SJ Liao. Comparison between the homotopy analysis method and homotopy perturbation method. *Appl. Math. Comput*. 2005; **169**, 1186-94.
[10] ME Gurtin and RC Maccamy. On the diffusion of biological population. *Math. Biosci.* 1977; **33**, 35-49.
[11] WSC Gurney and RM Nisbet. The regulation of in homogenous populations. *J. Theor. Biol.* 1975; **52**, 441-57.
[12] YG Lu. Holder estimates of solution of biological population equations. *Appl. Math. Lett.* 2000; **13**, 123-6.
[13] Y Tan and S Abbasbandy. Homotopy analysis method for quadratic Riccati differential equation. *Comm. Nonlinear Sci. Numer. Simulat*. 2008; **13**, 539-46.
[14] T Hayat, M Khan and S Asghar. Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid. *Acta Mech.* 2004; **168**, 213-32.
[15] T Hayat and M Khan. Homotopy solutions for a generalized second-grade fluid past a porous plate. *Nonlinear Dyn.* 2005; **42**, 395-405.
[16] ST Mohyud-Din and A Yildirim. The numerical solution of three dimensional Helmholtz equations. *Chin. Phys. Lett.* 2010; **27**, 060201.
[17] A Yildirim and ST Mohyud-Din. Analytical approach to space and time fractional Burger's equations. *Chin. Phys. Lett.* 2010; **27**, 090501.
[18] T Hayat, M Khan and S Asghar. Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid. *Acta Mech.* 2004; **168**, 213-32.
[19] T Hayat and M Khan. Homotopy solutions for a generalized second-grade fluid past a porous plate. *Nonlinear Dyn.* 2005; **42**, 395-405.