Influence of Temperature Dependent Viscosity and Entropy Generation on the Flow of a Johnson-Segalman Fluid

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Abstract

In the present paper, we have discussed the pipe flow of a Johnson-Segalman fluid with temperature dependent viscosity. The series solutions of coupled nonlinear equations have been obtained for two cases of viscosity (Reynold and Vogels) by homotopy analysis method (HAM). The expressions for velocity, temperature, and entropy generation number have been presented graphically to discuss the physical significance of pertinent parameters of interest. The convergence of the series solution have been discussed by plotting $\hbar$ -curves.

Keywords: Temperature dependent viscosity, Johnson-Segalman fluid, HAM solutions, pipe flow

Introduction

In recent years, the study of non-Newtonion fluid is quite useful because of its variety of applications in various disciplines. Due to the complexity of non-Newtonion fluids there are many models of non-Newtonion fluids. An extensive study of non-Newtonion fluid models is available in the refs [1-10]. To be more specific heat transfer analysis plays an important role in non-Newtonion fluids especially in the handling and processing of coal-based slurries. In some situations, it is not necessary that the fluid viscosity is constant and it may vary with temperature or pressure. For example, in coal slurries the viscosity of the fluid may vary with temperature. Massoudi and Christie [11] have considered the effects of variable viscosity and viscous dissipation on the flow of a third grade fluid in a pipe. The influence of variable viscosity and viscous dissipation on the non-Newtonion flow have been analyzed analytically by Hayat *et al.* [12]. The approximate and analytical solutions of non-Newtonion fluid with variable viscosity have been reported by Yurusoy and Pakdemirli [13] and Pakdemirli and Yilbas [14]. Recently, Nadeem and Ali [15] have examined the analytical solutions for pipe flow of a fourth grade fluid with Reynold and Vogel's model viscosities.

With these studies in mind, the purpose of the present model is to discuss the flow of a Johnson-Segalman fluid in a pipe with variable viscosity. To the best of author's knowledge, the Johnson-Segalman fluid with variable viscosity is not yet reported. The governing equations of Johnson-Segalman fluid with variable temperature dependent viscosity have been modelled. It is worth mentioning that in the present situation the equations of motion and energy are coupled and each equation has an effect on the other whereas in the case of constant viscosity the momentum equation has an influence on the temperature but the temperature equation has no influence on the momentum equation. The highly nonlinear coupled equations are solved analytically with the help of homotopy analysis method (HAM). Some recent studies on HAM are cited in the refs [16-26]. The convergence of the HAM solution have been discussed by plotting $\hbar$ -curves. The expressions for velocity, viscous dissipation and entropy generation for two types of viscosities have been calculated and discussed graphically for different physical parameters.

Formulation of the problem

Let us consider a steady, incompressible flow of a Johnson-Segalman fluid having temperature dependent viscosity that is flowing in an infinitely long cylinder of radius R . The flow is induced by an applied pressure gradient in the z-direction which is taken as the axis of flow. The governing equations of motion in cylindrical coordinates are;

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{d}{dr}(rS_{rr}) - \frac{S_{\theta\theta}}{r}, \tag{1}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{2}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{d}{dr}(rS_{rz}) + \frac{1}{r}\frac{d}{dr}\left(\mu r\frac{dv}{dr}\right). \tag{3}$$

The energy equation in the presence of dissipation term for Johnson-Segalman fluid can be written as;

$$\mu\left(\frac{dv}{dr}\right)^2 - \eta m^2(1-a^2)\left(\frac{dv}{dr}\right)^4 + k\left(\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr}\right) + \eta\left(\frac{dv}{dr}\right)^2 = 0. \tag{4}$$

In the above equations, p is the pressure, μ is the variable viscosity, r represents the radial direction and $S_{rr}, S_{\theta\theta}, S_{rz}$ are stresses for the Johnson-Segalman fluid which are defined as [26];

$$S_{rr} = \frac{-m(1-a)\eta\left(\frac{dv}{dr}\right)^2}{1+m^2(1-a^2)\left(\frac{dv}{dr}\right)^2}, \tag{5}$$

$$S_{\theta\theta} = 0, \tag{6}$$

$$S_{rz} = \frac{\eta\frac{dv}{dr}}{1+m^2(1-a^2)\left(\frac{dv}{dr}\right)^2}, \tag{7}$$

where η is the dynamic viscosity, m is the relaxation time and a is the slip parameter. By using Eqs. (5) - (7), Eq. (3) takes the following form;

$$\begin{aligned}\frac{\partial p}{\partial z} &= \frac{d\mu}{dr}\frac{dv}{dr} + \frac{\mu}{r}\frac{dv}{dr} + \mu\frac{d^2v}{dr^2} - \eta m^2(1-a^2)\frac{1}{r}\left(\frac{dv}{dr}\right)^2 \\ &\quad -3\eta m^2(1-a^2)\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} + \eta\left(\frac{1}{r}\frac{dv}{dr} + \frac{d^2v}{dr^2}\right).\end{aligned} \tag{8}$$

The relevant boundary conditions are;

$$v(r)=0,\ \ \theta(r)=0 \text{ at } r=R, \tag{9}$$

$$\frac{dv}{dr}=0=\frac{d\theta}{dr},\ \ \text{at } r=0. \tag{10}$$

Introducing the non-dimensional quantities;

$$\bar{r}=\frac{r}{R},\ \ \bar{v}=\frac{v}{V_o},\ \ \bar{\mu}=\frac{\mu}{\mu_*},\ \ \bar{\theta}=\frac{\theta-\theta_w}{\theta_m-\theta_w}, \tag{11}$$

in which μ_*, θ_m and θ_w are constants and represent the constant viscosity, mean temperature and wall temperature, respectively.
After dropping the bars Eqs. (4) and (8) take the following form;

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\left[\frac{dv}{dr}+r\frac{d^2v}{dr^2}\right]-\frac{We^2}{r}\left(\frac{dv}{dr}\right)^2\left[\alpha(1-a^2)\frac{dv}{dr}+3\alpha(1-a^2)r\frac{d^2v}{dr^2}\right] +\alpha\left(\frac{1}{r}\frac{dv}{dr}+\frac{d^2v}{dr^2}\right)=C, \tag{12}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2\left[\mu+\alpha-\alpha We^2(1-a^2)\left(\frac{dv}{dr}\right)^2\right]=0. \tag{13}$$

The non-dimensional parameters involved in Eqs. (12) and (13) are;

$$\Gamma=\frac{\mu_* V_o^2}{k(\theta_m-\theta_w)},\quad We=\frac{mV_o}{R},\quad C_1=\frac{\partial p}{\partial z},\quad C=\frac{C_1R^2}{\mu_* V_o},\quad \alpha=\frac{\eta}{\mu_*}. \tag{14}$$

where Γ is the Brinkman number, We is the Weissenberg number, C_1 is the pressure drop in the axial direction, and α is the ratio of viscosities.
The boundary conditions take the form;

$$v(r)=0,\ \ \theta(r)=0 \text{ at } r=1, \tag{15}$$

$$\frac{dv}{dr}=0=\frac{d\theta}{dr},\ \ \text{at } r=0. \tag{16}$$

Solution of the problem

Following Nadeem and Ali [15], the solution is considered for 2 models of viscosity known as (i) Reynold's model of viscosity and (ii) Vogel's model of viscosity.

Reynolds model

In this case, the viscosity is a function of temperature and is expressed in the form of an exponential function as [13];

$$\mu = e^{-M\theta}, \tag{17}$$

where M represents the Reynolds viscosity parameter.
Using the Maclaurin series expansion, the above expression can be written as;

$$\mu = 1 - M\theta + O(\theta^2). \tag{18}$$

Invoking Eq. (18) into Eqs. (12) and (13), we obtain;

$$-M\frac{d\theta}{dr}\frac{dv}{dr} + \frac{1}{r}\frac{dv}{dr} - \frac{M}{r}\theta\frac{dv}{dr} + \frac{d^2v}{dr^2} - M\theta\frac{d^2v}{dr^2} - \frac{\alpha We^2(1-a^2)}{r}\left(\frac{dv}{dr}\right)^3 - 3\alpha We^2(1-a^2)\frac{d^2v}{dr^2}\left(\frac{dv}{dr}\right)^2 + \frac{\alpha}{r}\frac{dv}{dr} + \alpha\frac{d^2v}{dr^2} = C, \tag{19}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2 - \Gamma M\theta\left(\frac{dv}{dr}\right)^2 - \alpha We^2(1-a^2)\Gamma\left(\frac{dv}{dr}\right)^4 + \alpha\Gamma\left(\frac{dv}{dr}\right)^2 = 0. \tag{20}$$

In order to solve Eqs. (19) and (20), using the homotopy analysis method, we choose;

$$v_{or}(r) = \frac{C}{4}(r^2 - 1), \tag{21}$$

$$\theta_{or}(r) = \frac{C^2\Gamma}{64}(-r^4 + 1), \tag{22}$$

as the initial guesses (for velocity and temperature in Reynold's model) and;

$$L_{vr}(v) = v'' + \frac{1}{r}v', \tag{23}$$

$$L_{\theta r}(\theta) = \theta'' + \frac{1}{r}\theta', \tag{24}$$

as the auxiliary linear operators which have the following properties;

$$L_{vr}(A_1 + B_1 \ln r) = 0, \tag{25}$$

$$L_{\theta r}(A_2 + B_2 \ln r) = 0, \tag{26}$$

where $A_{1,} A_{2,} B_1, B_2$ are constants.

Zeroth order deformation problem is defined as;

$$(1-q)L_{vr}[\overline{v}(r,q)-v_o(r)] = q\hbar_v N_{vr}[\overline{v}(r,q),\overline{\theta}(r,q)], \tag{27}$$

$$(1-q)L_{\theta r}[\overline{\theta}(r,q)-\theta_o(r)] = q\hbar_\theta N_{\theta r}[\overline{v}(r,q),\overline{\theta}(r,q)], \tag{28}$$

$$\overline{v}(r,q)=\overline{\theta}(r,q)=0, \quad r=1, \tag{29}$$

$$\frac{\partial\overline{v}(r,q)}{\partial r}=\frac{\partial\overline{\theta}(r,q)}{\partial r}=0, \quad r=0, \tag{30}$$

$$\begin{aligned} N_{vr}[\overline{v}(r,q),\overline{\theta}(r,q)] = & -M\frac{d\theta}{dr}\frac{dv}{dr}+\frac{1}{r}\frac{dv}{dr}-\frac{M}{r}\theta\frac{dv}{dr}+\frac{d^2v}{dr^2}-M\theta\frac{d^2v}{dr^2} \\ & -\frac{\alpha We^2(1-a^2)}{r}\left(\frac{dv}{dr}\right)^3-3\alpha We^2(1-a^2)\frac{d^2v}{dr^2}\left(\frac{dv}{dr}\right)^2 \\ & +\frac{\alpha}{r}\frac{dv}{dr}+\alpha\frac{d^2v}{dr^2}-C, \end{aligned} \tag{31}$$

$$\begin{aligned} N_{\theta r}[\overline{v}(r,q),\overline{\theta}(r,q)] = & \frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2-\Gamma M\theta\left(\frac{dv}{dr}\right)^2 \\ & -\alpha We^2(1-a^2)\Gamma\left(\frac{dv}{dr}\right)^4+\alpha\Gamma\left(\frac{dv}{dr}\right)^2, \end{aligned} \tag{32}$$

where $q\varepsilon[0,1]$ is the embedding parameter and $\hbar_v$ and $\hbar_\theta$ are auxiliary non-zero operators.

mth order deformation equations are defined as;

$$L_{vr}[v_m(r)-\chi_m v_{m-1}(r)] = \hbar_v R_{vr}(r), \tag{33}$$

$$L_{\theta r}[\theta_m(r)-\chi_m \theta_{m-1}(r)] = \hbar_\theta R_{\theta r}(r), \tag{34}$$

$$\begin{aligned} R_{vr} = & -M\sum_{k=0}^{m-1}v'_{m-1-k}\theta'_k+\frac{1}{r}v'_{m-1}-\frac{M}{r}\sum_{k=0}^{m-1}v'_{m-1-k}\theta_k+v''_{m-1} \\ & -M\sum_{k=0}^{m-1}v''_{m-1-k}\theta_k-\frac{\alpha We^2(1-a^2)}{r}\sum_{k=0}^{m-1}v'_{m-1-k}\sum_{l=0}^{k}v'_{k-l}v'_l \\ & -3\alpha We^2(1-a^2)\sum_{k=0}^{m-1}v'_{m-1-k}\sum_{l=0}^{k}v'_{k-l}v''_l+\frac{\alpha}{r}v'_{m-1}+\alpha v''_{m-1} \\ & -C(1-\chi_m), \end{aligned} \tag{35}$$

$$R_{\theta r} = \theta''_{m-1} + \frac{1}{r}\theta'_{m-1} + \Gamma\sum_{k=0}^{m-1} v'_{m-1-k} v'_k$$
$$-\Gamma M \sum_{k=0}^{m-1} v'_{m-1-k} \sum_{l=0}^{k} v'_{k-l}\theta_l + \alpha\Gamma\sum_{k=0}^{m-1} v'_{m-1-k} v'_k \tag{36}$$
$$-\alpha We^2(1-a^2)\Gamma\sum_{k=0}^{m-1} v'_{m-1-k} \sum_{l=0}^{k} v'_{k-l} \sum_{s=0}^{l} v'_{l-s} v'_s,$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{37}$$

We now use the symbolic software MATHEMATICA and solve the set of linear differential Eqs. (33) and (34), subject to relevant boundary conditions up to first few order of approximations. It is found that $v_m(r)$ and $\theta_m(r)$ can be written as;

$$v_m(r) = \sum_{n=0}^{2(2m+1)} a'_{m,n} r^n, \ \ m \ge 0, \quad \theta_m(r) = \sum_{n=0}^{4(m+1)} d'_{m,n} r^n, \ \ m \ge 0. \tag{38}$$

The solution thus can be defined as;

$$v(r) = \lim_{Q\to\infty}\left[\sum_{m=0}^{Q}\left(\sum_{n=0}^{2(2m+1)} a'_{m,n} r^n\right)\right], \tag{39}$$

$$\theta(r) = \lim_{Q\to\infty}\left[\sum_{m=0}^{Q}\left(\sum_{n=0}^{4(m+1)} d'_{m,n} r^n\right)\right]. \tag{40}$$

Vogel's viscosity model

For Vogel's model, the viscosity is taken as [13];

$$\mu = \exp\left[\frac{A}{B+\theta} - \theta_w\right]. \tag{41}$$

Using the Maclaurin series expansion we can write;

$$\mu = \frac{C}{C^*}\left(1 - \frac{\theta A}{B^2}\right) + O(\theta^2), \tag{42}$$

where $C^* = \frac{C}{\mu_* \exp[\frac{A}{B} - \theta_w]}$ and A, B are Vogel's viscosity parameters.
Making use of Eq. (41), Eqs. (12) and (13) take the following form;

$$\begin{aligned}&-\frac{AC}{B^2C^*}\frac{d\theta}{dr}\frac{dv}{dr}+\frac{C}{rC^*}\frac{dv}{dr}+\frac{C}{C^*}\frac{d^2v}{dr^2}-\frac{AC}{B^2C^*}\frac{\theta}{r}\frac{dv}{dr}-\frac{AC}{B^2C^*}\theta\frac{d^2v}{dr^2}\\&-\frac{\alpha We^2(1-a^2)}{r}\left(\frac{dv}{dr}\right)^3-3\alpha We^2(1-a^2)\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}+\frac{\alpha}{r}\frac{dv}{dr}+\alpha\frac{d^2v}{dr^2}=C,\end{aligned} \tag{43}$$

$$\begin{aligned}&\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\frac{\Gamma C}{C^*}\left(\frac{dv}{dr}\right)^2-\frac{\Gamma AC}{B^2C^*}\theta\left(\frac{dv}{dr}\right)^2+\alpha\Gamma\left(\frac{dv}{dr}\right)^2\\&\qquad-\alpha We^2(1-a^2)\Gamma\left(\frac{dv}{dr}\right)^4=0.\end{aligned} \tag{44}$$

To calculate the HAM solution, we choose;

$$v_{ov}(r)=\frac{C^*}{4}(r^2-1), \tag{45}$$

$$\theta_{ov}(r)=\frac{CC^*\Gamma}{64}(-r^4+1), \tag{46}$$

as the initial guesses (for velocity and temperature in Vogel's model) and the auxiliary linear operators for velocity and temperature are;

$$L_{vv}(v)=v''+\frac{1}{r}v', \tag{47}$$

$$L_{\theta v}(\theta)=\theta''+\frac{1}{r}\theta', \tag{48}$$

which satisfies;

$$L_{vv}(A_3+B_3\ln r)=0, \tag{49}$$

$$L_{\theta v}(A_4+B_4\ln r)=0, \tag{50}$$

where A_3, A_4, B_3, B_4 are constants.
Zeroth order deformation problem is defined as;

$$(1-q)L_{vv}[\bar{v}(r,q)-v_o(r)]=q\hbar_v N_{vv}[\bar{v}(r,q),\bar{\theta}(r,q)], \tag{51}$$

$$(1-q)L_{\theta v}[\bar{\theta}(r,q)-\theta_o(r)] = q\hbar_\theta N_{\theta v}[\bar{v}(r,q),\bar{\theta}(r,q)], \tag{52}$$

$$\bar{v}(r,q)=\bar{\theta}(r,q)=0, \quad r=1, \tag{53}$$

$$\frac{\partial \bar{v}(r,q)}{\partial r}=\frac{\partial \bar{\theta}(r,q)}{\partial r}=0, \quad r=0, \tag{54}$$

where

$$\begin{aligned} N_{vv}[\bar{v}(r,q),\bar{\theta}(r,q)] = & -\frac{AC}{B^2C^*}\frac{d\theta}{dr}\frac{dv}{dr}+\frac{C}{rC^*}\frac{dv}{dr}+\frac{C}{C^*}\frac{d^2v}{dr^2}-\frac{AC}{B^2C^*}\frac{\theta}{r}\frac{dv}{dr} \\ & -\frac{AC}{B^2C^*}\theta\frac{d^2v}{dr^2}-\frac{\alpha We^2(1-a^2)}{r}\left(\frac{dv}{dr}\right)^3 \\ & -3\alpha We^2(1-a^2)\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}+\frac{\alpha}{r}\frac{dv}{dr}+\alpha\frac{d^2v}{dr^2}-C, \end{aligned} \tag{55}$$

$$\begin{aligned} N_{\theta v}[\bar{v}(r,q),\bar{\theta}(r,q)] = & \frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\frac{\Gamma C}{C^*}\left(\frac{dv}{dr}\right)^2-\frac{\Gamma AC}{B^2C^*}\theta\left(\frac{dv}{dr}\right)^2 \\ & +\alpha\Gamma\left(\frac{dv}{dr}\right)^2-\alpha We^2(1-a^2)\Gamma\left(\frac{dv}{dr}\right)^4. \end{aligned} \tag{56}$$

mth order deformation equations are defined as;

$$L_{vv}[v_m(r)-\chi_m v_{m-1}(r)] = \hbar_v R_{vv}(r), \tag{57}$$

$$L_{\theta v}[\theta_m(r)-\chi_m \theta_{m-1}(r)] = \hbar_\theta R_{v\theta}(r), \tag{58}$$

$$v_m(1)=\theta_m(1)=0, \tag{59}$$

$$\frac{dv_m(0)}{dr}=\frac{d\theta_m(0)}{dr}=0, \tag{60}$$

where

$$
\begin{aligned}
R_{vv} = & -\frac{AC}{B^2C^*}\sum_{k=0}^{m-1} v'_{m-1-k}\theta'_k + \frac{C}{rC^*}v'_{m-1} + \frac{C}{C^*}v''_{m-1} \\
& -\frac{AC}{rB^2C^*}\sum_{k=0}^{m-1} v'_{m-1-k}\theta_k - \frac{AC}{B^2C^*}\sum_{k=0}^{m-1} v''_{m-1-k}\theta_k \\
& -\frac{\alpha We^2(1-a^2)}{r}\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}v'_l \\
& -3\alpha We^2(1-a^2)\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}v''_l \\
& +\frac{\alpha}{r}v'_{m-1} + \alpha v''_{m-1} - C(1-\chi_m),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
R_{\theta v} = & \,\theta''_{m-1} + \frac{1}{r}\theta'_{m-1} + \frac{\Gamma C}{C^*}\sum_{k=0}^{m-1} v'_{m-1-k}v'_k \\
& -\frac{\Gamma AC}{B^2C^*}\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}\theta_l \\
& -\alpha We^2(1-a^2)\Gamma\sum_{k=0}^{m-1} v'_{m-1-k}\sum_{l=0}^{k} v'_{k-l}\sum_{s=0}^{l} v'_{l-s}v'_s \\
& +\alpha\Gamma\sum_{k=0}^{m-1} v'_{m-1-k}v'_k.
\end{aligned}
\tag{62}
$$

Adopting a similar procedure as discussed in the previous section, the solution is not shown here, however, the numerical data is shown through graphs.

Viscous dissipation and entropy generation

The entropy generation number for Johnson-Segalman fluid is defined as;

$$
N_S = \left(\frac{d\theta}{dr}\right)^2 + \Gamma\theta_o\left(\frac{dv}{dr}\right)^2\left[\mu + \alpha - \alpha We^2(1-a^2)\left(\frac{dv}{dr}\right)^2\right], \tag{63}
$$

where μ varies for Reynold's and Vogel's model.

Results and discussion

The analytical series solutions for both the Reynold's viscosity model and Vogel's viscosity model have been computed. The convergence regions of these series are strongly dependent upon the non-zero auxiliary parameters $\hbar$ which can be adjusted. To see the range of permissible values of $\hbar$, $\hbar$ -curves are plotted in **Figures 1 - 5** for various physical quantities. It is observed from these figures that the convergence region for the temperature profile is wider than the velocity profile in each case. The velocity and temperature profiles in Reynold's viscosity model for various values of viscosity parameter M, constant pressure C and Brinkman number Γ are displayed in **Figures 6 - 11**. It is observed that the velocity field increases with an increase in M. Also, the velocity is greatest at the center of the pipe, by the symmetry of the pipe (**Figure 6**). It is seen from **Figure 7** that the temperature profile also increases with an increase in M. The velocity and temperature for various values of C are shown in **Figures 8** and **9**. It is observed that both velocity and temperature increase with an increase in C with the maximum value occurring at the middle of the pipe. Almost similar observation appears for different values of Brinkman number Γ (**Figures 10** and **11**). The entropy generation number N_s in the Reynold's model for different values of M, Γ and C are presented in **Figures 12 - 14**. It is observed that with an increase in the physical parameters the entropy generation number increases and in the center of the pipe the values are almost the same.

The velocity and temperature in Vogel's model for different values of viscosity parameters A and B, Brinkman number Γ are displayed in **Figures 15 - 20**. It is observed that the velocity and temperature decreases with an increase in A but the behavior is opposite for an increase in the other physical parameter. The entropy generation number N_s for various values of A and B are shown in **Figures 21** and **22**. It is observed that N_s decreases with an increase in A and increases with an increase in B. The maximum value of the entropy generation number is at the pipe walls and gives a minimum value at the center of the pipe. The entropy generation number N_s against A and B for different values of pipe distance r are plotted in **Figures 23** and **24**. It is found from **Figure 23** that increasing parameter A lowers the total entropy generation number and **Figure 24** shows that entropy generation number increases upon increasing parameter B. The numerical data of various physical parameters for velocity and temperature are shown in **Tables 1 - 6**. These tables show that the maximum value of velocity and temperature are at the middle of the pipe.

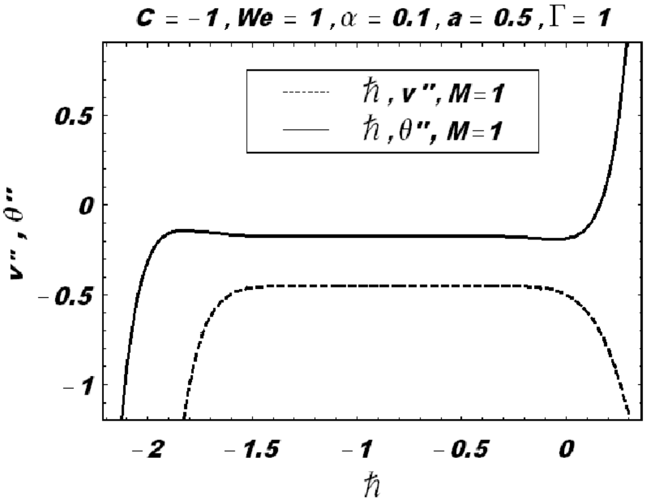

Figure 1 Convergence curve for velocity and temperature for *M* when other parameters are fixed.

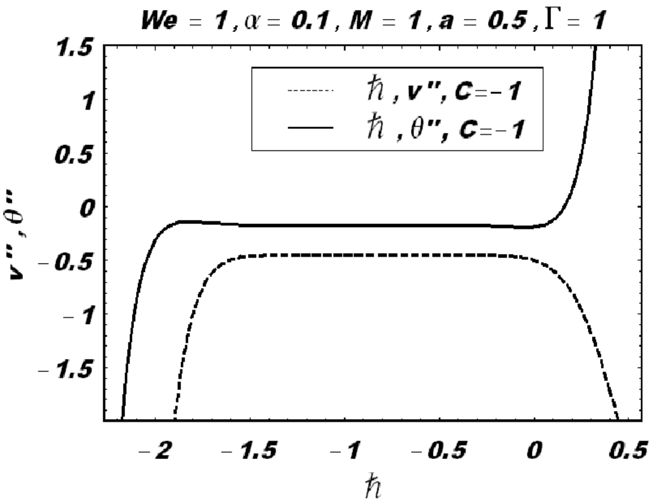

Figure 2 Convergence curve for velocity and temperature for *C* when other parameters are fixed.

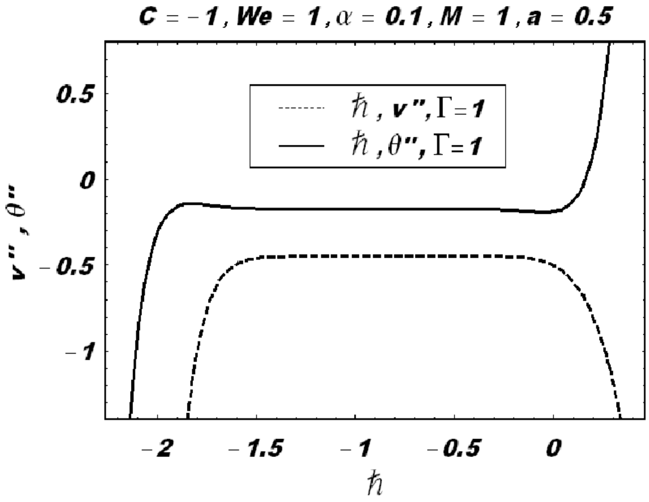

Figure 3 Convergence curve for velocity and temperature for Γ when other parameters are fixed.

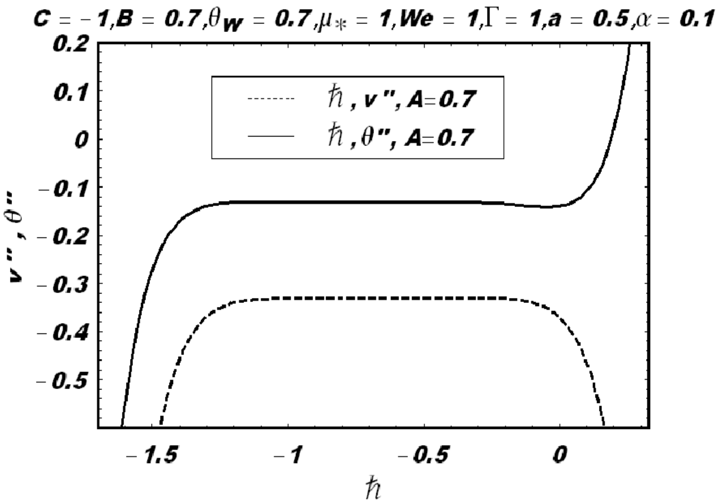

Figure 4 Convergence curve for velocity and temperature for *A* when other parameters are fixed.

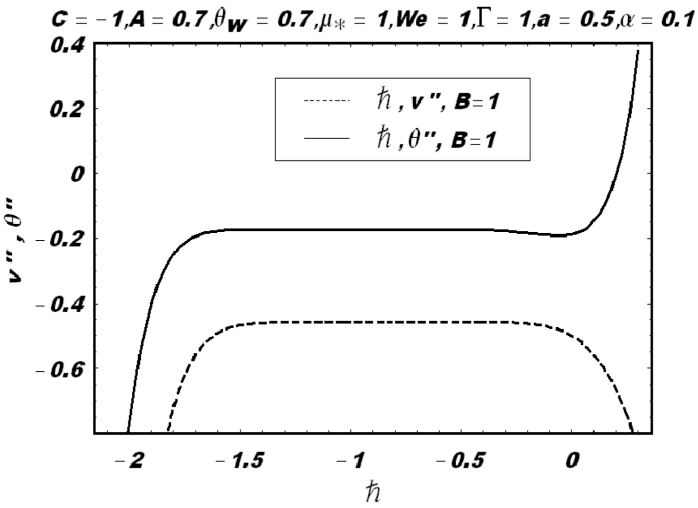

Figure 5 Convergence curve for velocity and temperature for *B* when other parameters are fixed.

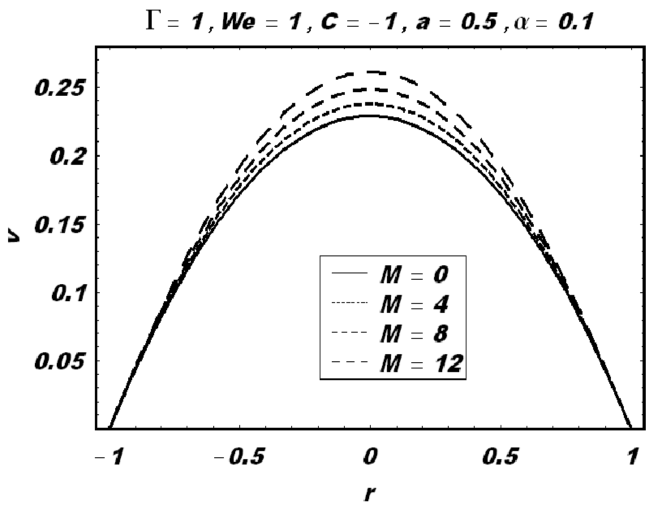

Figure 6 Velocity field for different values of *M* when other parameters are fixed.

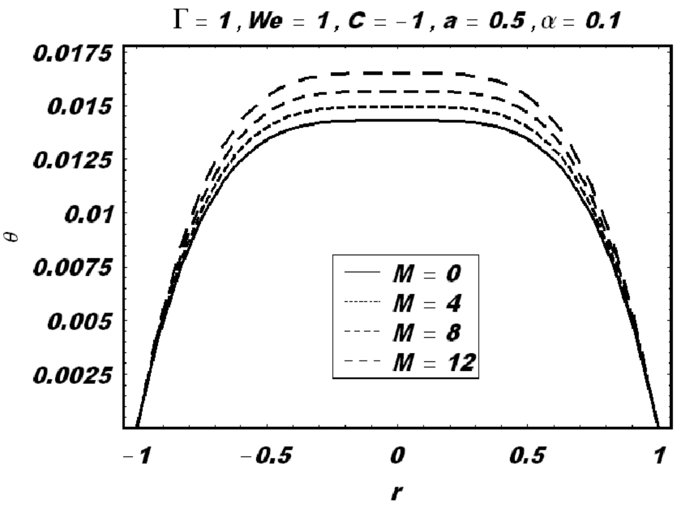

Figure 7 Temperature field for different values of *M* when other parameters are fixed.

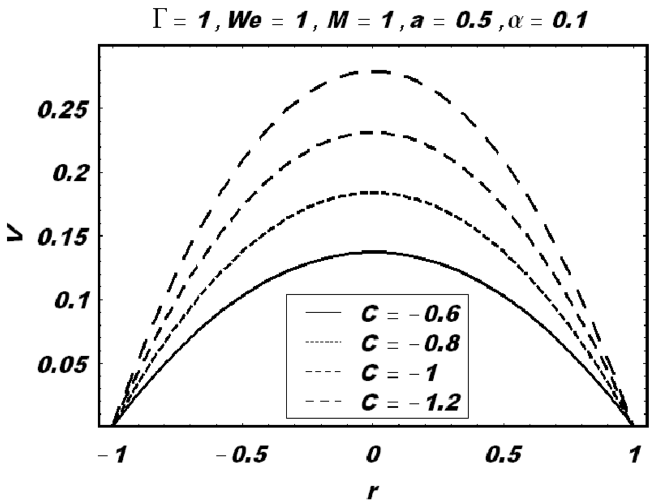

Figure 8 Velocity field for different values of *C* when other parameters are fixed.

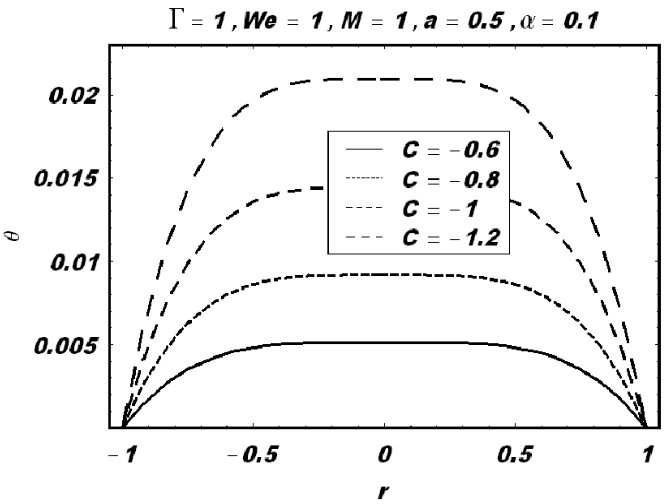

Figure 9 Temperature field for different values of *C* when other parameters are fixed.

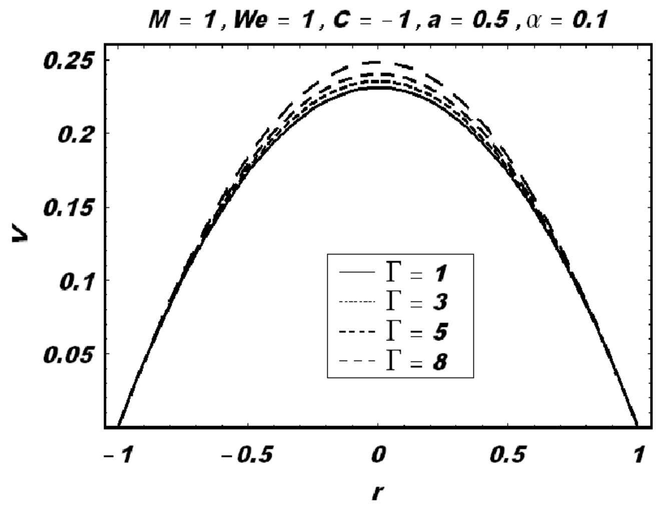

Figure 10 Velocity field for different values of Γ when other parameters are fixed.

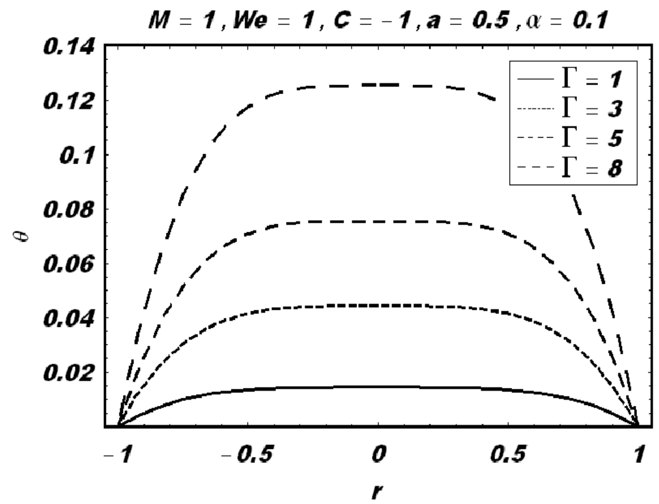

Figure 11 Temperature field for different values of Γ when other parameters are fixed.

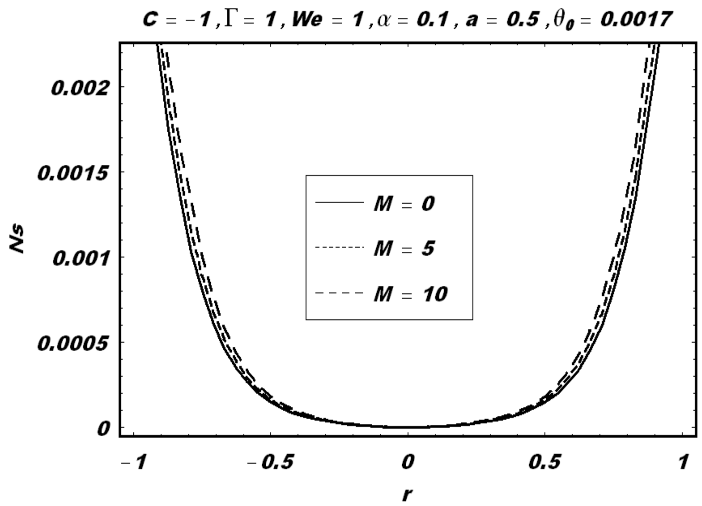

Figure 12 Entropy generation number for different values of *M* when other parameters are fixed.

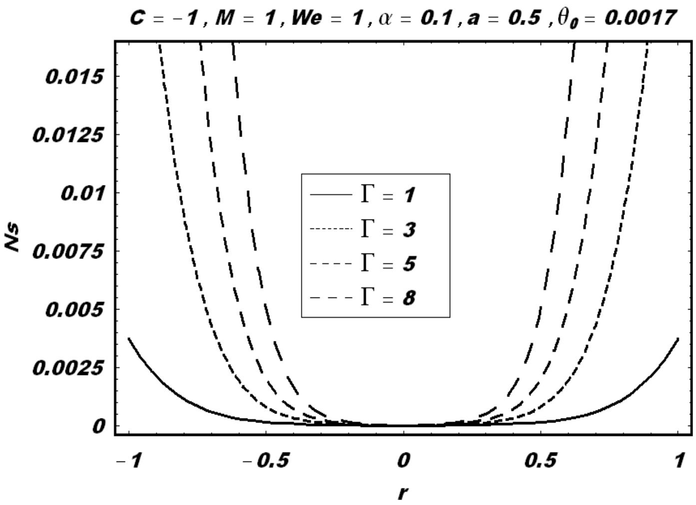

Figure 13 Entropy generation number for different values of Γ when other parameters are fixed.

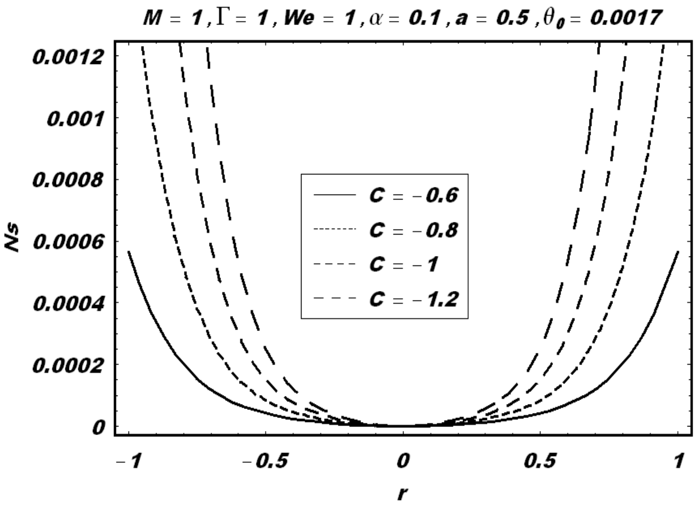

Figure 14 Entropy generation number for different values of *C* when other parameters are fixed.

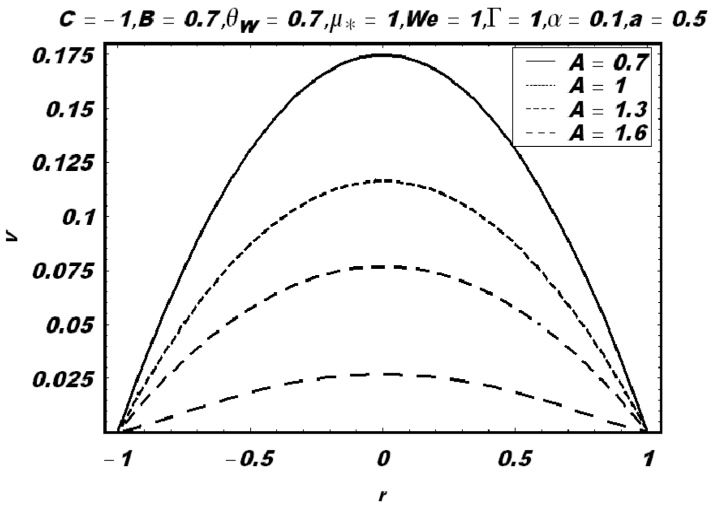

Figure 15 Velocity field for the case of Vogel's model for different values of *A* when other parameters are fixed.

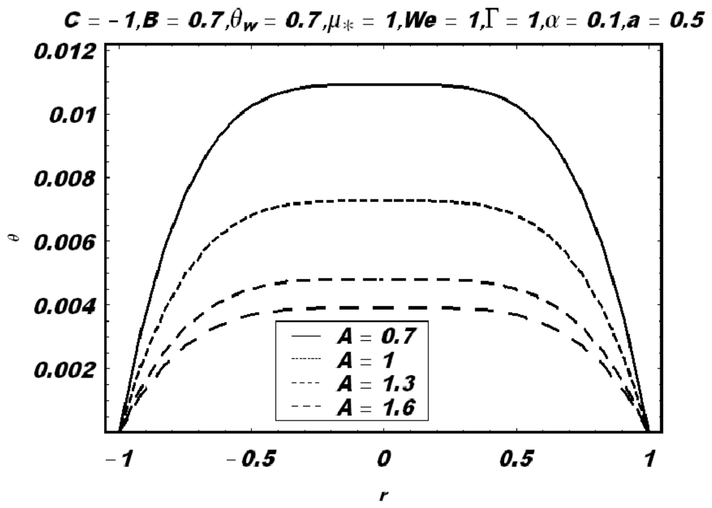

Figure 16 Temperature field for the case of Vogel's model for different values of *A* when other parameters are fixed.

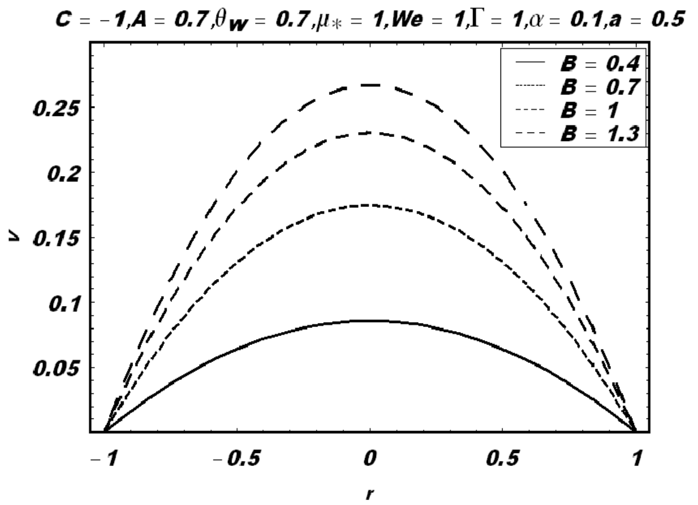

Figure 17 Velocity field for the case of Vogel's model for different values of B when other parameters are fixed.

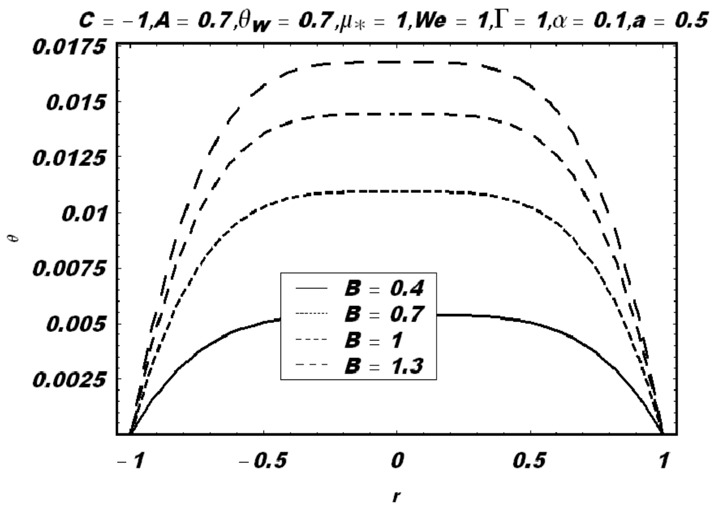

Figure 18 Temperature field for the case of Vogel's model for different values of B when other parameters are fixed.

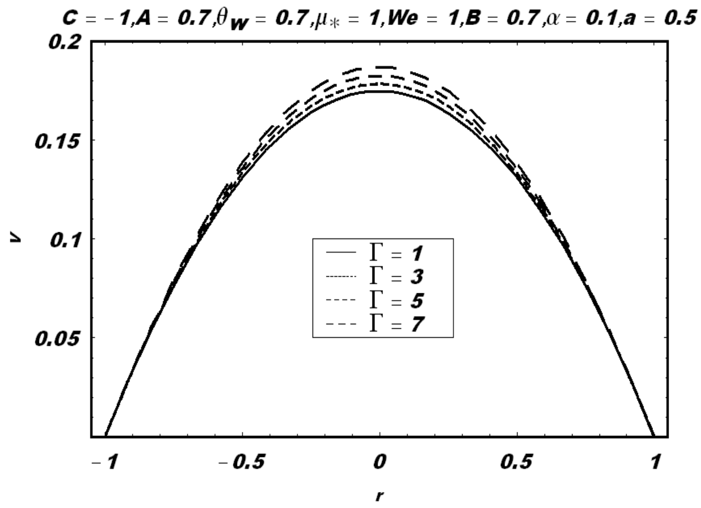

Figure 19 Velocity field for the case of Vogel's model for different values of Γ when other parameters are fixed.

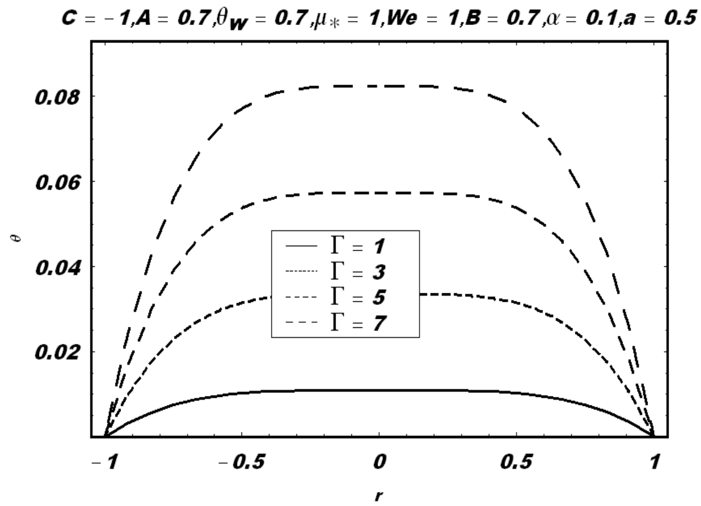

Figure 20 Temperature field for the case of Vogel's model for different values of Γ when other parameters are fixed.

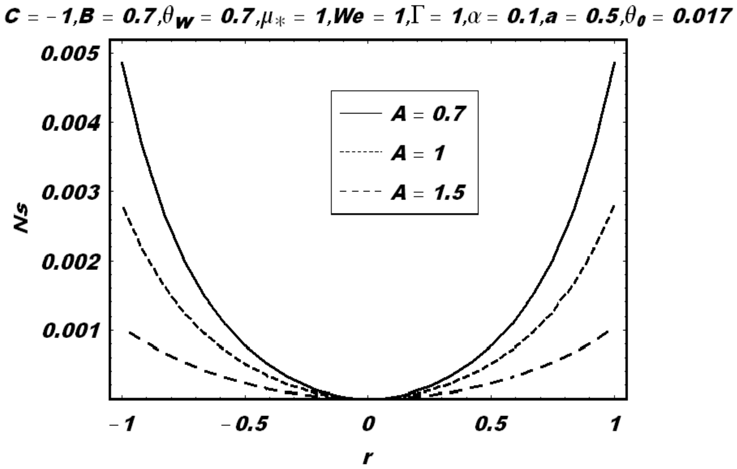

Figure 21 Entropy generation number for the case of Vogel's model for different values of *A* when other parameters are fixed.

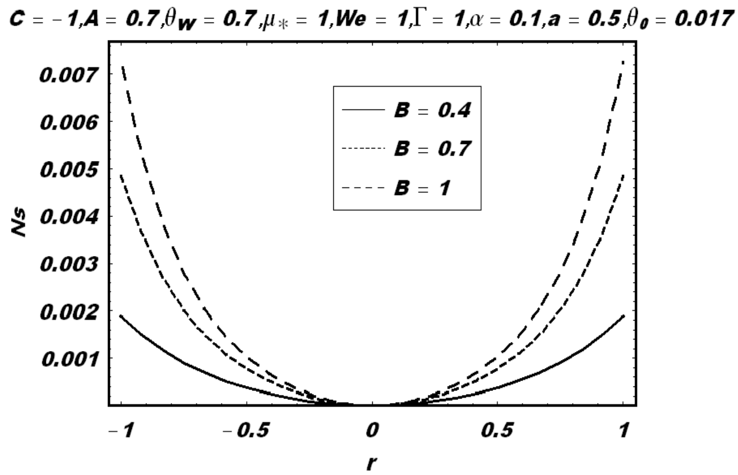

Figure 22 Entropy generation number for the case of Vogel's model for different values of *B* when other parameters are fixed.

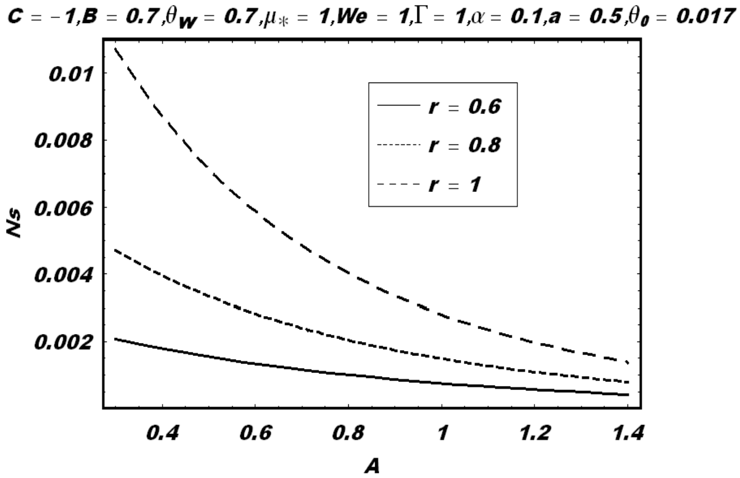

Figure 23 Entropy generation number for the case of Vogel's model for different values of r when other parameters are fixed.

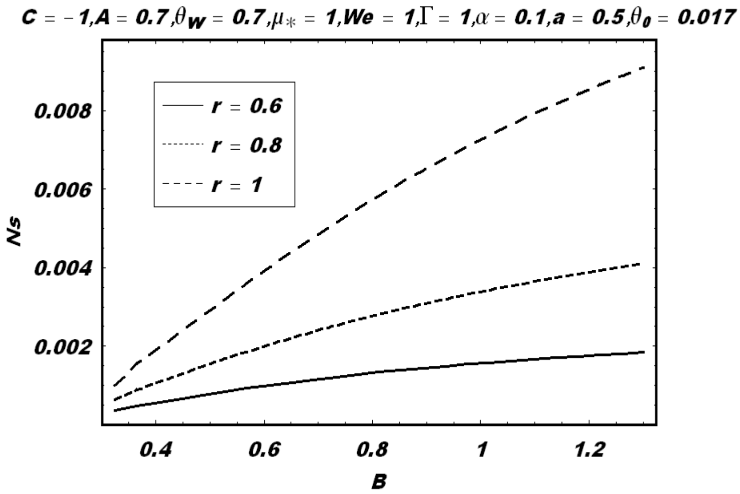

Figure 24 Entropy generation number for the case of Vogel's model for different values of r when other parameters are fixed.

Table 1 Variation in velocity and temperature for various values of α for fixed $C=-1,\ \Gamma=1,$ $We=1,\ M=1,\ a=0.5,$ in Reynolds viscosity model.

	α = 0		**α = 0.05**		**α = 0.1**	
r	**V**	**θ**	**V**	**θ**	**V**	**θ**
-1	0	0	0	0	0	0
-0.75	0.110024	0.0107939	0.105418	0.0103223	0.10108	0.00988116
-0.5	0.189182	0.0148178	0.18098	0.0141614	0.173318	0.0135493
-0.25	0.236788	0.0157475	0.226318	0.0150471	0.216585	0.0143946
0	0.252663	0.0158095	0.24142	0.0151061	0.230986	0.0144509
0.25	0.236788	0.0157475	0.226318	0.0150471	0.216585	0.0143946
0.5	0.189182	0.0148178	0.18098	0.0141614	0.173318	0.0135493
0.75	0.110024	0.0107939	0.105418	0.0103223	0.10108	0.00988116
1	0	0	0	0	0	0

Table 2 Variation in velocity and temperature for various values of ***We*** for fixed $C=-1,\ \Gamma=1,$ $\alpha=0.1,\ M=1,\ a=0.5,$ in Reynolds viscosity model.

	We = 1		**We = 2**		**We = 3**	
r	**V**	**θ**	**V**	**θ**	**V**	**θ**
-1	0	0	0	0	0	0
-0.75	0.10108	0.00988116	0.10516	0.0101457	0.115754	0.0107369
-0.5	0.173318	0.0135493	0.178826	0.0138639	0.192304	0.0145511
-0.25	0.216585	0.0143946	0.222414	0.0147139	0.23647	0.0154094
0	0.230986	0.0144509	0.236839	0.0147703	0.250938	0.0154659
0.25	0.216585	0.0143946	0.222414	0.0147139	0.23647	0.0154094
0.5	0.173318	0.0135493	0.178826	0.0138639	0.192304	0.0145511
0.75	0.10108	0.00988116	0.10516	0.0101457	0.115754	0.0107369
1	0	0	0	0	0	0

Table 3 Variation in velocity and temperature for various values of slip parameter ***a*** for fixed $C=-1$, $\Gamma=1$, $\alpha=0.1$, $M=1$, $We=1$, in Reynolds viscosity model.

	a = 0.4		**a = 0.5**		**a = 0.6**	
r	**V**	**θ**	**V**	**θ**	**V**	**θ**
-1	0	0	0	0	0	0
-0.75	0.101225	0.00989085	0.10108	0.00988116	0.100905	0.00986941
-0.5	0.173517	0.0135609	0.173318	0.0135493	0.173078	0.0135353
-0.25	0.216796	0.0144064	0.216585	0.0143946	0.216329	0.0143803
0	0.231198	0.0144627	0.230986	0.0144509	0.230729	0.0144366
0.25	0.216796	0.0144064	0.216585	0.0143946	0.216329	0.0143803
0.5	0.173517	0.0135609	0.173318	0.0135493	0.173078	0.0135353
0.75	0.101225	0.00989085	0.10108	0.00988116	0.100905	0.00986941
1	0	0	0	0	0	0

Table 4 Variation in velocity and temperature for various values of α for fixed $C=-1$, $A=0.7$, θ_w $=0.7$, $\mu_*=1$, $\Gamma=1$, $We=1$, $B=0.7$, $a=0.5$, in Vogel's viscosity model.

	α = 0		**α = 0.05**		**α = 0.1**	
r	**V**	**θ**	**V**	**θ**	**V**	**θ**
-1	0	0	0	0	0	0
-0.75	0.0815367	0.00800113	0.078806	0.00772818	0.076227	0.00747082
-0.5	0.140225	0.0109846	0.135426	0.0106067	0.130911	0.010251
-0.25	0.175525	0.011674	0.169447	0.0112714	0.163739	0.0108926
0	0.187297	0.0117199	0.180786	0.0113157	0.174677	0.0109353
0.25	0.175525	0.011674	0.169447	0.0112714	0.163739	0.0108926
0.5	0.140225	0.0109846	0.135426	0.0106067	0.130911	0.010251
0.75	0.0815367	0.00800113	0.078806	0.00772818	0.076227	0.00747082
1	0	0	0	0	0	0

Table 5 Variation in velocity and temperature for various values of β for fixed $C=-1,\ A=0.7,\ \theta_w=0.7,\ \mu_*=1,\ \Gamma=1,\ \alpha=0.1,\ B=0.7,\ a=0.5,$ in Vogel's viscosity model.

	We = 1		**We = 2**		**We = 3**	
r	**V**	**θ**	**V**	**θ**	**V**	**θ**
-1	0	0	0	0	0	0
- 0.75	0.076227	0.00747082	0.0774329	0.00755092	0.0797517	0.00769927
- 0.5	0.130911	0.010251	0.132565	0.0103469	0.135688	0.0105231
- 0.25	0.163739	0.0108926	0.165498	0.01099	0.168802	0.0111688
0	0.174677	0.0109353	0.176444	0.0110328	0.179762	0.0112117
0.25	0.163739	0.0108926	0.165498	0.01099	0.168802	0.0111688
0.5	0.130911	0.010251	0.132565	0.0103469	0.135688	0.0105231
0.75	0.076227	0.00747082	0.0774329	0.00755092	0.0797517	0.00769927
1	0	0	0	0	0	0

Table 6 Variation in velocity and temperature for various values of slip parameter ***a*** for fixed $C=-1,$ $A=0.7,\ \theta_w=0.7,\ \mu_*=1,\ \Gamma=1,\ \alpha=0.1,\ We=1,\ B=0.7,$ in Vogel's viscosity model.

	a = 0.4		**a = 0.5**		**a = 0.6**	
r	**V**	**θ**	**V**	**θ**	**V**	**θ**
-1	0	0	0	0	0	0
- 0.75	0.076273	0.00747391	0.076227	0.00747082	0.076171	0.00746705
- 0.5	0.130974	0.0102547	0.130911	0.010251	0.130833	0.0102464
- 0.25	0.163807	0.0108964	0.163739	0.0108926	0.163656	0.010888
0	0.174745	0.0109391	0.174677	0.0109353	0.174594	0.0109307
0.25	0.163807	0.0108964	0.163739	0.0108926	0.163656	0.010888
0.5	0.130974	0.0102547	0.130911	0.010251	0.130833	0.0102464
0.75	0.076273	0.00747391	0.076227	0.00747082	0.076171	0.00746705
1	0	0	0	0	0	0

Conclusions

The series solutions of coupled nonlinear equations have been obtained for 2 cases of viscosity (Reynold and Vogels) by HAM. The expressions for velocity, temperature, and entropy generation number have been presented graphically to discuss the physical significance of pertinent parameters of interest. The convergence of the series solution have been discussed by plotting $\hbar$ -curves.

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