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Article

Homotopy Analysis Method for Time-Fractional Schrödinger Equations

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Abstract

The Homotopy Analysis Method (HAM) is applied to tackle time-fractional Schrödinger equations. The proposed technique is fully compatible with the complexity of these problems and obtained results are highly encouraging. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the suggested algorithm.

Keywords: Homotopy Analysis Method, fractional Schrödinger partial differential equations, nonlinear problems

Introduction

Nonlinear partial differential equations [1-21] are of extreme importance in applied and engineering sciences. The thorough study of literature reveals that most of the physical phenomena are nonlinear in nature and hence there is a desire to find their appropriate solutions, see [1-21] and the references therein. Recently, scientists [16-21] have observed that there are a number of real time problems modeled by fractional nonlinear differential equations which are very hard to tackle. Inspired and motivated by ongoing research in this area, we apply a very reliable and efficient technique which is called the Homotopy Analysis Method (HAM) to find approximate solutions of time-fractional Schrödinger partial differential equations. It is observed that the proposed algorithm is fully synchronized with the complexity of these equations. The time- fractional Schrödinger partial differential equations are of the type:

$$iD_t^{\alpha}u(x,t) = -\frac{1}{2}u_{xx} + V_d(x)u(x,t) + \beta_d|u(x,t)|^2u(x,t), \qquad i^2=-1, \tag{1}$$

with initial conditions: $u(x,0)=f(x)$, where $V_d(x)$ is the trapping potential and β_d is a real constant, $0 \le \alpha \le 1$ and arise frequently in applied, physical and engineering sciences. Numerical results are very encouraging.

Definitions [16-21]

Definition 1 A real function *f(x)*, *x > 0*, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^∞ iff $f^m \in C_\mu$ $\mu \ge 1$ $m \in N$.

Definition 2 The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f \in C_\mu, \mu \ge -1$, is defined as,

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt,$$
$$\alpha > 0, x > 0\,,$$
$$J^0 f(x) = f(x).$$

Properties of the operator j^α can be found in (Caputo, 1967), we mention only the following.

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 3 The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^\alpha (x-t)^{m-\alpha-1} f^{(m)}(t)dt,$$

for $m-1 < \alpha \leq m, m \in Z, x > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1 if $m-1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then

$D_*^\alpha J^\alpha f(x) = f(x)$, and

$$J^\alpha D_*^\alpha f(x) = f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+) . x^k}{k!}, \quad x > 0.$$

Definition 4 For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial^m} d\tau, & for\ m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial^m}, & for\ \alpha = m \in N. \end{cases}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

Homotopy Analysis Method (HAM) [1-7, 19-21]

We apply the HAM [1-7,19-21] to the fractional Schrödinger Eq. (1). We consider the following differential equation

$$FD[u(x,t)] = 0, \tag{3}$$

where FD is a nonlinear operator for this problem, x and t denote independent variables, $u(x,t)$ is an unknown function.

In the frame of HAM [1-7,19-21], we can construct the following zeroth-order deformation:

$$(1-q)L\big(U(x,t;q) - u_0(x,t)\big) = q\hbar H(x,t) FD\big(U(x,t;q)\big), \tag{4}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function on the independent variables x, t and q.
Obviously, when, $q \neq 0$ and $q = 1$, it holds:

$$U(x,t;0) = u_0(x,t), U(x,t;1) = u(x,t), \tag{5}$$

respectively. Using the parameter q, we expand $U(x,t;q)$ in Taylor series as follows:

$$U(x,t;q) = u_0(x,t) + \sum_{m=1}^\infty u_m(x,t) q^m, \tag{6}$$

where: $u_m = \frac{1}{m!} \frac{\partial^m U(t;q)}{\partial^m q}|_{q=0}$. (7)

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(x,t)$ are selected such that the series (Eq. 6) is convergent at $q = 1$, then due to Eq. (5) we have:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^\infty u_m(x,t). \tag{8}$$

Let us define the vector:

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), \dots, u_n(x,t)\}. \tag{9}$$

Differentiating Eq. (7) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}), \tag{10}$$

where:

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}FD(U(t;q))}{\partial^{m-1}q}\Big|_{q=0}, \quad (11)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Finally, for the purpose of computation, we will approximate the HAM solution (8) by the following truncated series:

$$\emptyset_m(t) = \sum_{k=0}^{m-1} u_k(t). \quad (12)$$

Numerical examples

In this section, we apply the Homotopy Analysis Method (HAM) to solve time-fractional Schrödinger equations. Numerical results are very encouraging.

Example 1 Consider the following one-dimensional nonlinear time-fractional Schrödinger equation:

$$iD_t^\alpha u = -\frac{1}{2}u_{xx} - |u|^2 u, \text{ where } 0 < \alpha \leq 1, \quad (13)$$

with initial conditions:

$$u(x,0) = e^{\iota x}.$$

According to Eq. (4), the zeroth-order deformation can be given by:

$$(1-q)L\big(U(x,t;q) - u_0(x,t)\big) = q\hbar H(x,t)\left(D_t^\alpha U(x,t;q) - \frac{i}{2}\frac{\partial^2 U(x,t;q)}{\partial x^2} - iU^2\bar{U}\right).$$

We can start with an initial approximation $u_0(x,0) = e^{\iota x}$ and we choose the auxiliary linear operator:

$$L\big(U(x,t;q)\big) = D_t^\alpha U(x,t;q),$$

with the property $L(c) = 0$, where c is an integral constant. We also choose the auxiliary function to be:

$$H(x,t) = 1.$$

Hence, the mth-order deformation can be given by:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)R_m(\vec{u}_{m-1}),$$

where:

$$R_m(\vec{u}_{m-1}) = D_t^\alpha(u_{m-1}) - \frac{i}{2}\frac{\partial^2(u_{m-1})}{\partial x^2} - (u_{m-1}^2)(\bar{u}_{m-1}). \quad (14)$$

Now the solution of the mth-order deformation Eq. (14) for $m \geq 1$ becomes:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \quad (15)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = e^{\iota x},$$
$$u_1(x,t) = -\hbar\frac{1}{2}\mathrm{i}\, e^{\iota x}\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$u_2(x,t) = -\hbar\frac{i}{2}e^{ix}\frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2\frac{i}{2}e^{ix}\frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2\frac{1}{4}e^{ix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$\vdots$$

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,t) = e^{ix}\left[1 + \frac{\mathrm{i}}{2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right]. \quad (16)$$

For the special case $\alpha = 1$, we obtain from (16):

$$u(x,t) = e^{i(x+\frac{1}{2}t)}. \quad (17)$$

which is the exact solution of the Schrödinger equation. The results for the exact solution Eq. (17) and the approximate solution Eq. (16) are obtained using the Homotopy Analysis Method. For $\alpha = 0.25, 0.50, 0.75$ *and* 1, these are shown in **Figure 1**.

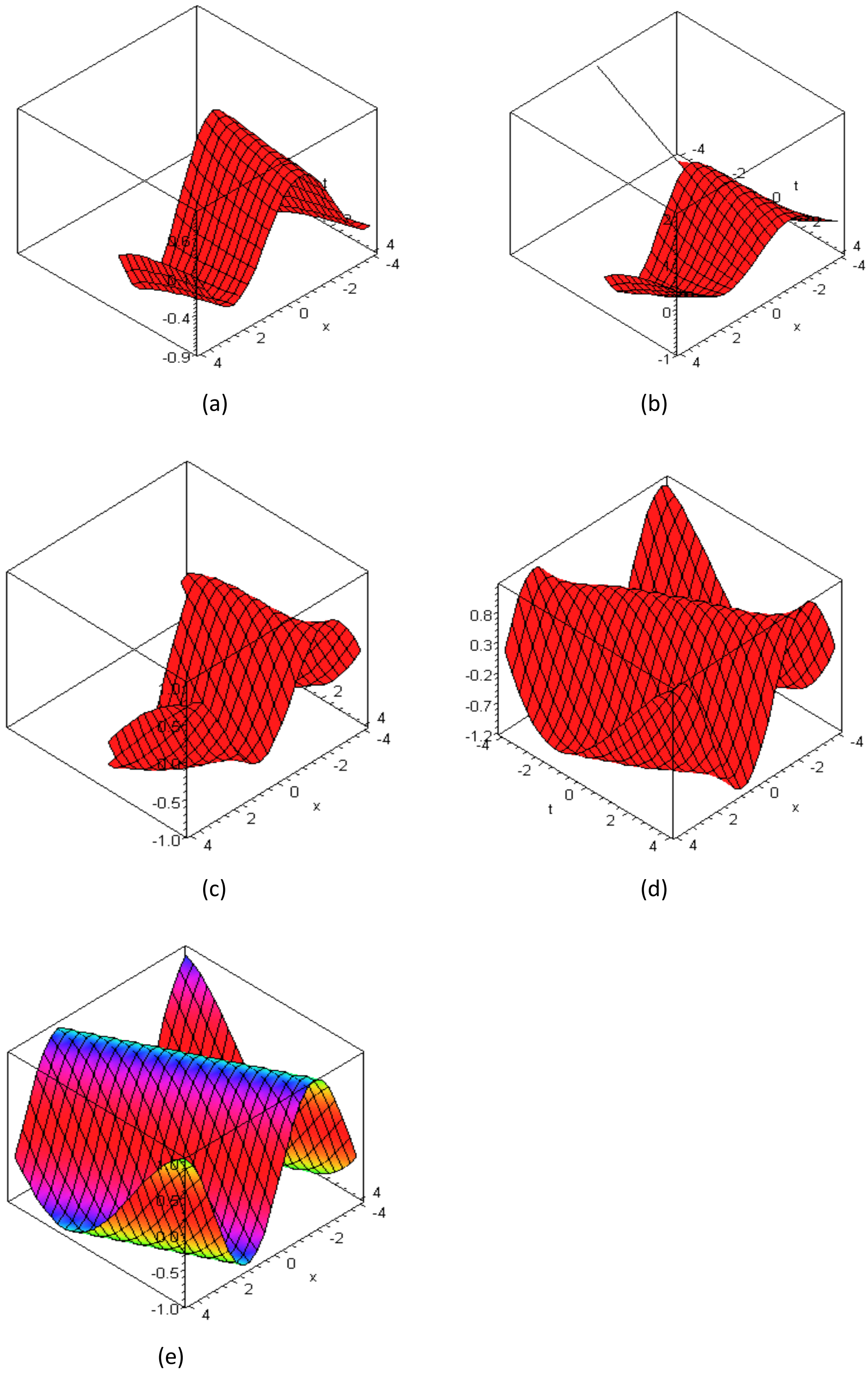


Figure 1 The surface shows solution $u(x,t)$ for the Eq. (16) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution Eq. (17).

Example 2: Consider the following one-dimensional nonlinear time-fractional Schrödinger equation:

$$iD_t^\alpha u = -\frac{1}{2}u_{xx} + ucos^2x - |u|^2u, \quad t \geq 0,$$
where $0 < \alpha \leq 1$, (18)

with initial conditions:

$$u(x,0) = sinx.$$

According to Eq. (4), the zeroth-order deformation can be given by:

$$(1-q)L\big(U(x,t;q) - u_0(x,t)\big) = q\hbar H(x,t)\begin{pmatrix} D_t^\alpha U(x,t;q) - \dfrac{i}{2}\dfrac{\partial^2 U(x,t;q)}{\partial x^2} \\ +iUcos^2x - iU^2\overline{U} \end{pmatrix}.$$

We can start with an initial approximation $u_0(x,0) = sinx$ and we choose the auxiliary linear operator:

$$L\big(U(x,t;q)\big) = D_t^\alpha U(x,t;q),$$

with the property $L(c) = 0$, where c is an integral constant. We also choose the auxiliary function to be:

$$H(x,t) = 1.$$

Hence, the mth-order deformation can be given by:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t)R_m(\vec{u}_{m-1}),$$

where:

$$R_m(\vec{u}_{m-1}) = D_t^\alpha(u_{m-1}) - \frac{i}{2}\frac{\partial^2(u_{m-1})}{\partial x^2} + i(u_{m-1})cos^2x - i(u_{m-1}^2)(\bar{u}_{m-1}). \quad (19)$$

Now the solution of the mth-order deformation Eq. (19) for $m \geq 1$ becomes:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \quad (20)$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,t) = sinx,$$
$$u_1(x,t) = \hbar\frac{3}{2}\mathrm{i}\,sinx\frac{t^\alpha}{\Gamma(\alpha+1)} ,$$
$$u_2(x,t) = \hbar\frac{3}{2}isinx\frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2\frac{3}{2}isinx\frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2\frac{9}{4}sinx\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$\vdots$$

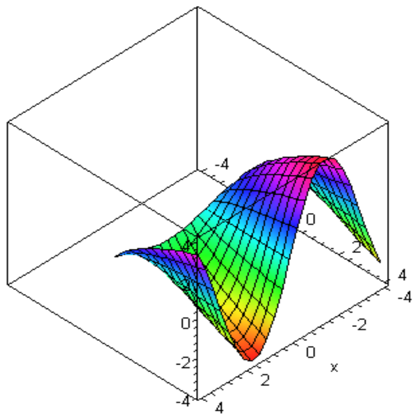
Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,t) = sinx[1 - \frac{3}{2}\mathrm{i}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{9}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots]. \quad (21)$$

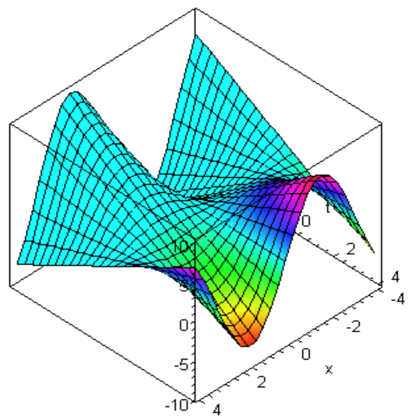
For the special case $\alpha = 1$, we obtain from Eq. (21):

$$u(x,t) = sinxe^{-\frac{3}{2}it}. \quad (22)$$

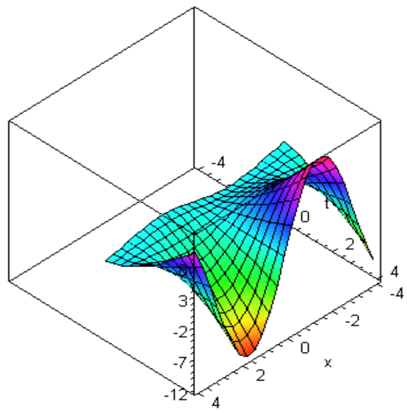
which is the exact solution of the Schrödinger equation. The results for the exact solution Eq. (22) and the approximate solution Eq. (21) are obtained using the Homotopy Analysis Method. For $= 0.25, 0.50, 0.75$ and 1 , these are shown in **Figure 2**.



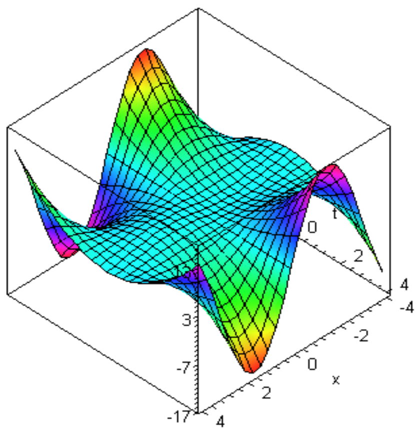
(a)



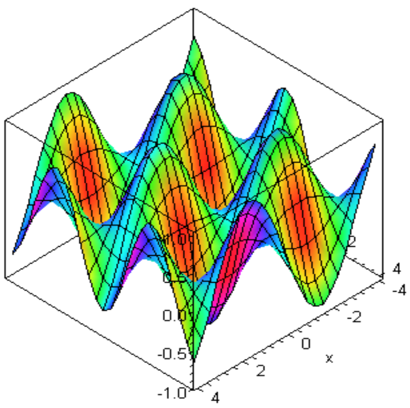
(b)



(c)



(d)



(e)

Figure 2 The surface shows solution $u(x,t)$ for the Eq. (21) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution Eq. (22).

Example 3: Consider the following two-dimensional nonlinear time-fractional Schrödinger equation:

$$iD_t^\alpha u = -\frac{1}{2}\left(u_{xx}+u_{yy}\right)+vu+|u|^2u, \quad (x,y)\in[0,2\pi]\times[0,2\pi], \quad \text{where } 0<\alpha\le 1, \tag{23}$$

with initial conditions:

$u(x,y,0) = sinxsiny.$

where $V(x,y) = 1 - sin^2xsin^2y$.

According to Eq. (4), the zeroth-order deformation can be given by:

$$(1-q)L\big(U(x,y,t;q)-u_0(x,y,t)\big) = q\hbar H(x,y,t)\begin{pmatrix} D_t^\alpha U(x,y,t;q)-\frac{i}{2}\left(\frac{\partial^2 U(x,y,t;q)}{\partial x^2}+\frac{\partial^2 U(x,y,t;q)}{\partial y^2}\right) \\ + \\ iVU(x,y,t;q)+iU^2(x,y,t;q)\bar{U}(x,y,t;q)\end{pmatrix}.$$

We can start with an initial approximation $u_0(x,y,0) = sinxsiny$ and we choose the auxiliary linear operator:

$L\big(U(x,y,t;q)\big) = D_t^\alpha U(x,y,t;q),$

with the property $L(c) = 0$, where c is an integral constant. We also choose the auxiliary function to be:

$H(x,y,t) = 1.$

Hence, the mth-order deformation can be given by:

$L[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = \hbar H(x,y,t)R_m(\vec{u}_{m-1}),$

where:

$$R_m(\vec{u}_{m-1}) = D_t^\alpha(u_{m-1})-\frac{i}{2}\left(\frac{\partial^2(u_{m-1})}{\partial x^2}+\frac{\partial^2(u_{m-1})}{\partial y^2}\right)+iV(u_{m-1})+i(u_{m-1}^2)(\bar{u}_{m-1}). \tag{24}$$

Now the solution of the mth-order deformation Eq. (24) for $m\ge 1$ becomes:

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t)+hL^{-1}[R_m(\vec{u}_{m-1})]. \tag{25}$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,y,t) = sinxsiny,$$
$$u_1(x,y,t) = \hbar 2\mathrm{i}\, sinxsiny\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$u_2(x,y,t) = \hbar 2\mathrm{i}\, sinxsiny\frac{t^\alpha}{\Gamma(\alpha+1)}+\hbar^2 2\mathrm{i}\, sinxsiny\frac{t^\alpha}{\Gamma(\alpha+1)}-\hbar^2 4\, sinxsiny\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$\vdots$.

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,y,t) = sinxsiny\left[1-2\mathrm{i}\frac{t^\alpha}{\Gamma(\alpha+1)}-4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right]. \tag{26}$$

For the special case $\alpha = 1$, we obtain from Eq. (26):

$$u(x,y,t) = sinxsinye^{-\frac{3}{2}\iota t}, \tag{27}$$

which is the exact solution of the Schrödinger equation. The results for the exact solution Eq. (25) and the approximate solution Eq. (27) are obtained using the Homotopy Analysis Method. For $\alpha = 0.25, 0.50, 0.75$ *and* 1, these are shown in **Figure 3**.

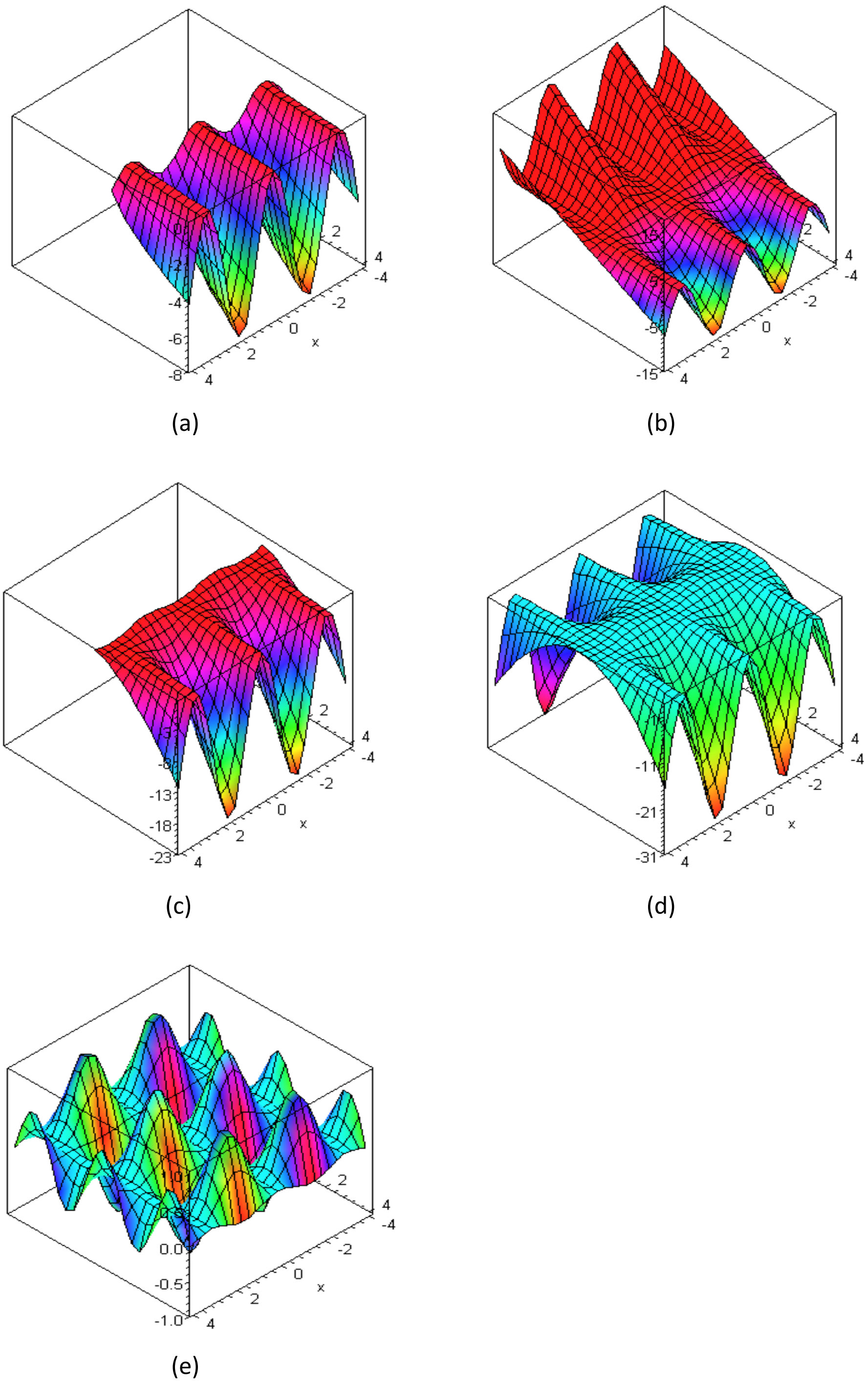


Figure 3 The surface shows solution $u(x,y,t)$ for the Eq. (25) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, and $x = y$ (e) exact solution Eq. (27).

Example 4: Consider the following three-dimensional nonlinear time-fractional Schrödinger equation:

$$iD_t^\alpha u = -\frac{1}{2}\left(u_{xx} + u_{yy} + u_{yy}\right) + Vu + |u|^2 u, \quad (x, y, z) \in [0,2\pi] \times [0,2\pi] \times [0,2\pi], \tag{28}$$

where $0 < \alpha \leq 1$,

with initial conditions

$u(x, y, z, 0) = sinxsinysinz,$

where $V(x, y, z) = 1 - sin^2xsin^2ysin^2z$.

According to Eq. (4), the zeroth-order deformation can be given by:

$$(1-q)L\big(U(x,y,z,t;q) - u_0(x,y,z,t)\big) = q\hbar H(x,y,z,t)\left(D_t^\alpha U(x,y,z,t;q) - \frac{i}{2}\left(\frac{\partial^2 U(x,y,t;q)}{\partial x^2} + \frac{\partial^2 U(x,y,z,t;q)}{\partial y^2} + \frac{\partial^2 U(x,y,z,t;q)}{\partial z^2}\right) + iVU(x,y,z,t;q) + iU^2(x,y,z,t;q)\overline{U}(x,y,z,t;q)\right).$$

We can start with an initial approximation $u_0(x, y, z, 0) = sinxsinysinz$ and we choose the auxiliary linear operator:

$$L\big(U(x,y,z,t;q)\big) = D_t^\alpha U(x,y,z,t;q),$$

with the property $L(c) = 0$, where c is an integral constant. We also choose the auxiliary function to be: $H(x, t) = 1$.

Hence, the mth-order deformation can be given by:

$$L[u_m(x,y,z,t) - \chi_m u_{m-1}(x,y,z,t)] = \hbar H(x,y,z,t) R_m(\vec{u}_{m-1}),$$

where:

$$R_m(\vec{u}_{m-1}) = D_t^\alpha(u_{m-1}) - \frac{i}{2}\left(\frac{\partial^2(u_{m-1})}{\partial x^2} + \frac{\partial^2(u_{m-1})}{\partial y^2} + \frac{\partial^2(u_{m-1})}{\partial z^2}\right) + iV(u_{m-1}) + i(u_{m-1}^2)(\bar{u}_{m-1}). \tag{29}$$

Now the solution of the mth-order deformation Eq. (24) for $m \geq 1$ becomes:

$$u_m(x,y,z,t) = \chi_m u_{m-1}(x,y,z,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})].$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_0(x,y,z,t) = sinxsinysinz,$$
$$u_1(x,y,z,t) = \hbar\frac{5}{2}\mathrm{i}\, sinxsinysinz\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$u_2(x,y,z,t) = \hbar\frac{5}{2}\mathrm{i}\, sinxsinysinz\frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2\frac{5}{2}\mathrm{i}\, sinxsinysinz\frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2\frac{25}{4}\mathrm{i}\, sinxsinysinz\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

⋮.

Hence, the HAM series solution (for $\hbar = -1$) is:

$$u(x,y,z,t) = sinxsinysinz[1 - \frac{5}{2}\mathrm{i}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{25}{4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots]. \quad (30)$$

For the special case $\alpha = 1$, we obtain from Eq. (30):

$$u(x,y,z,t) = sinx\ siny\ sinze^{-\frac{5}{2}it}, \quad (31)$$

which is the exact solution of the Schrödinger equation. The results for the exact solution Eq. (30) and the approximate solution Eq. (31) are obtained using the Homotopy Analysis Method. For $\alpha = 0.25, 0.50, 0.75\ and\ 1$, these are shown in **Figure 4**.

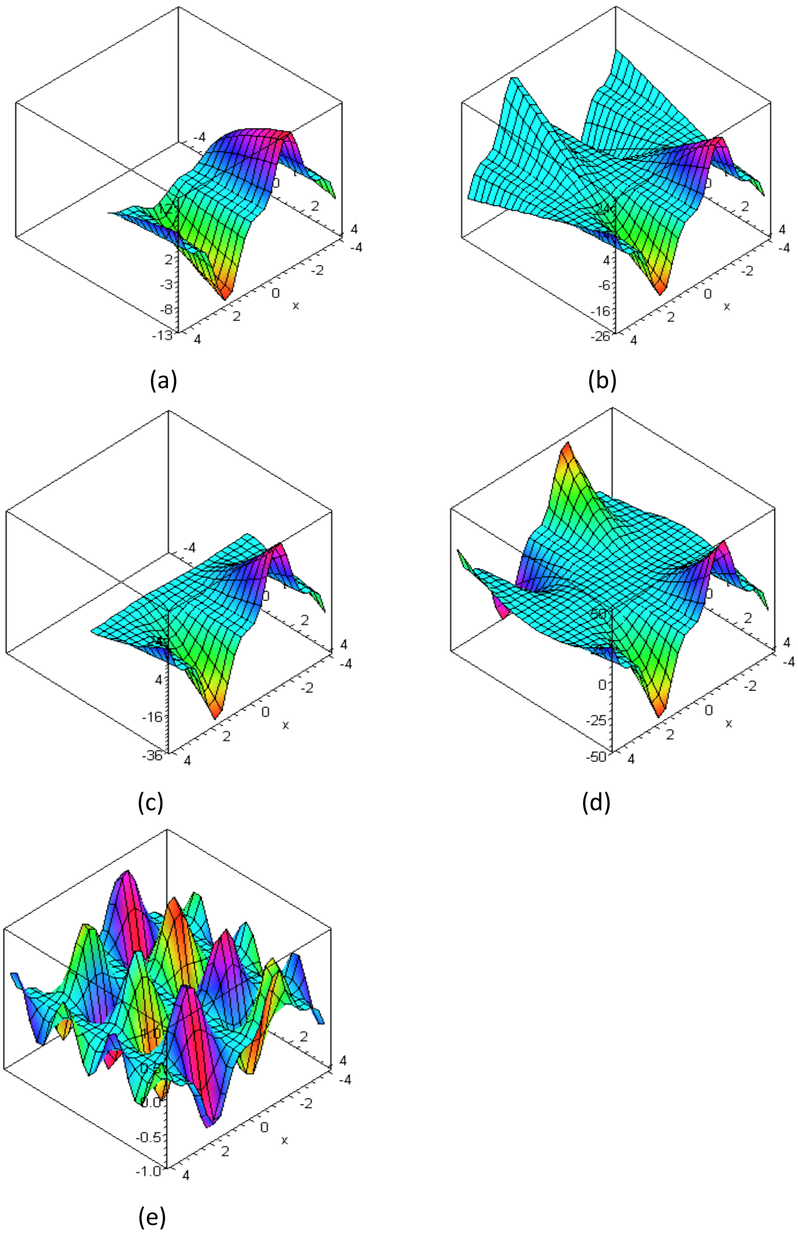


Figure 4 The surface shows solution $u(x,y,z,t)$for the Eq. (30) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, and $x = y = z$. (e) exact solution Eq. (31).

Conclusions

The Homotopy Analysis Method (HAM) has been implemented to find appropriate solutions of time-fractional nonlinear Schrödinger equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

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