Finding the Exact Solution of Special Nonlinear Partial Differential Equations by Homotopy Analysis Method

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Abstract

In this paper, the homotopy analysis method was used to solve nonlinear parabolic-hyperbolic partial differential equations. Examples are presented here to show the usability of the method for such equations. The results show that the HAM is very effective and convenient and that the obtained solutions of HAM have high accuracy.

Keywords: Homotopy analysis method, nonlinear parabolic-hyperbolic partial differential equations

Introduction

In 1992, Liao [1] employed the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method (HAM) [2,3]. This method has been successfully applied to solve many types of nonlinear problems [4,5]. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors, causing loss of accuracy, and requires large amounts of computer power and time. HAM is better since it does not involve discretization of the variables; hence, it is free from rounding off errors and does not require large amounts of computer memory or time. Here, the HAM is applied to solve nonlinear parabolic-hyperbolic partial differential equations. The Cauchy problem is considered in the nonlinear parabolic-hyperbolic partial differential equation of the following type.

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{\partial^2}{\partial t^2}-\Delta\right)u=F(u), \tag{1}$$

with initial conditions;

$$\frac{\partial^k u}{\partial t^k}(0,X)=\psi_k(X),\quad X=(x_1,x_2,\cdots,x_i),\qquad k=0,1,2 \tag{2}$$

where the nonlinear term is represented by $F(u)$, and Δ is the Laplace operator in R^n.

Basic idea of HAM

To illustrate the basic idea of the HAM, the following differential equation can be considered;

$$N[u(\tau)]=0, \tag{3}$$

where N is a nonlinear operator, τ denotes an independent variable, and $u(\tau)$ is an unknown function, respectively. For simplicity, all boundary or initial conditions are ignored. By means of generalizing the traditional homotopy method, Liao [2] constructs the so-called zero-order deformation equation.

$$(1-p)L[\phi(\tau;p)-u_0(\tau)] = p\,\hbar H(\tau)N[\phi(\tau;p)], \tag{4}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, and $u(\tau;p)$ is a unknown function, respectively. It is important that one has great freedom to choose auxiliary things in the HAM. Obviously, when $p = 0$ and $p = 1$, it holds that;

$$\phi(\tau;0) = u_0(\tau), \quad \phi(\tau;1) = u(\tau), \tag{5}$$

respectively. Thus, as p increases from 0 to 1, the solution $u(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $u(\tau;p)$ in Taylor series with respect to p;

$$\phi(\tau;p) = u_0(\tau) + \Sigma_{m=1}^{+\infty} u_m(\tau)p^m, \tag{6}$$

is obtained, where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}|_{p=0}. \tag{7}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (6) converges at $p = 1$, then

$$u(\tau) = u_0(\tau) + \Sigma_{m=1}^{+\infty} u_m(\tau), \tag{8}$$

is obtained, which must be one of the solutions of the original nonlinear equation, as proved by [3]. As $\hbar = -1$ and $H(\tau) = 1$, Eq. (4) becomes;

$$(1-p)L[\phi(\tau;p)-u_0(\tau)] + pN[\phi(\tau;p)] = 0, \tag{9}$$

which is used mostly in the HAM [6,7] as the solution obtained directly and without using Taylor series. According to the definition, the governing equation can be deduced from the zero-order deformation Eq. (4). Upon defining the vector;

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}, \tag{10}$$

differentiating Eq. (4) m-times with respect to the embedding parameter p, and then setting $p = 0$ and finally dividing them by $m!$, the so-called mth-order deformation equation is obtained;

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\, H(\tau)\Re_m(\vec{u}_{m-1}), \tag{11}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}}|_{p=0}, \tag{12}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{13}$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear Eq. (11) under the linear boundary conditions that come from the original problem, which can be easily solved by using symbolic computation software such as Matlab. For the convergence of the above method the reader is referred to Liao's work [2]. If Eq. (3) admits a unique solution, then this method will produce a unique solution. If Eq. (3) does not possess a unique solution, the HAM will give a solution among many other possible solutions.

Test examples

This section contains 4 examples of nonlinear parabolic-hyperbolic equations.

Example 3.1 *Consider the following nonlinear integro-differential equation;*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2u^2, \tag{14}$$

with the initial conditions;

$$u(0,x) = e^x, \quad \frac{\partial u}{\partial t}(0,x) = e^x, \quad \frac{\partial^2 u}{\partial t^2}(0,x) = e^x, \tag{15}$$

where the exact solution is $u(t,x) = e^{x+t}$. To solve Eq.(14) by means of the HAM, the linear operator is chosen as follows.

$$\mathcal{L}[\phi(t,x;p)] = \frac{\partial^3 \phi(t,x;p)}{\partial t^3}, \tag{16}$$

with the property $\mathcal{L}[c_1 + c_2 t + c_3 t^2] = 0$, where c_1, c_2, and c_3 are integral constants. The inverse operator $\mathcal{L}^{-1}$ is given by;

$$\mathcal{L}^{-1}(.) = \int_0^t \int_0^{t_1} \int_0^{t_2} (.) dt_3 dt_2 dt_1. \tag{17}$$

A nonlinear operator is now defined as;

$$\begin{aligned} \mathcal{N}[\phi(t,x;p)] = & \frac{\partial^3 \phi(t,x;p)}{\partial t^3} - \frac{\partial^3 \phi(t,x;p)}{\partial t \partial x^2} \\ & - \frac{\partial^4 \phi(t,x;p)}{\partial x^2 \partial t^2} + \frac{\partial^4 \phi(t,x;p)}{\partial x^4} \\ & - \left(\frac{\partial^2 \phi(t,x;p)}{\partial t^2}\right)^2 + \left(\frac{\partial^2 \phi(t,x;p)}{\partial x^2}\right)^2 + 2\phi^2(t,x;p). \end{aligned} \tag{18}$$

Using the above definition, the zeroth-order deformation equation is constructed;

$$(1-p)\mathcal{L}[\phi(t,x;p) - u_0(t,x)] = p\hbar\mathcal{H}(t,x)\mathcal{N}[\phi(t,x;p)]. \tag{19}$$

For $p = 0$ and $p = 1$,

$$\phi(t,x;0) = u_0(t,x), \qquad \phi(t,x;1) = u(t,x), \tag{20}$$

can be written; thus, the mth-order deformation equations are obtained;

$$\mathcal{L}[u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar\mathcal{H}(t,x)\Re_m(\vec{u}_{m-1}), \;\; (m \geq 1), \tag{21}$$

where

$$\begin{aligned}\Re_m(\vec{u}_{m-1}) = & \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial x^2 \partial t^2} \\ & + \frac{\partial^4 u_{m-1}}{\partial x^4} + \sum_{k=0}^{m-1} [\frac{\partial^2 u_k}{\partial x^2}\frac{\partial^2 u_{m-1-k}}{\partial x^2} \\ & - \frac{\partial^2 u_k}{\partial t^2}\frac{\partial^2 u_{m-1-k}}{\partial t^2} + 2u_k u_{m-1-k}].\end{aligned} \tag{22}$$

Now, for $m \geq 1$, the solution of the mth-order deformation Eq. (21);

$$u_m(t,x) = \chi_m u_{m-1}(t,x) + \hbar\mathcal{H}(t,x)\mathcal{L}^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{23}$$

Starting with an initial approximation $u_0(t,x) = \left(1 + t + \frac{t^2}{2}\right)e^x$, by means of the above iteration formula (13) if $\mathcal{H}(t,x) = 1, \hbar = -1$, the other components can be directly obtained as;

$$\begin{aligned} & u_1(t,x) = \frac{t^3}{3!}e^x, \qquad u_2(t,x) = \frac{t^4}{4!}e^x, \\ & u_3(t,x) = \frac{t^5}{5!}e^x, \qquad u_4(t,x) = \frac{t^6}{6!}e^x, \cdots \end{aligned} \tag{24}$$

Therefore, the solution of Example (3.1) can be readily obtained by;

$$u(t,x) = \sum_{m=0}^{+\infty} u_m(t,x) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)e^x. \tag{25}$$

Continuing the expansion to the last term gives the solution of Eq. (14) as follows;

$$u(t,x) = e^{x+t}, \tag{26}$$

which is the exact solution.

Example 3.2 *Consider the following nonlinear integro-differential equation;*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = u\left(\frac{\partial u}{\partial t}\right) + \left(\frac{\partial^2 u}{\partial t^2}\right)\left(\frac{\partial u}{\partial x}\right), \tag{27}$$

with the initial conditions;

$$\begin{aligned} & u(0,x) = \cos x, \qquad \frac{\partial u}{\partial t}(0,x) = -\sin x, \\ & \frac{\partial^2 u}{\partial t^2}(0,x) = -\cos x. \end{aligned} \tag{28}$$

To solve Eq. (27) by means of the HAM a nonlinear operator is defined as;

$$\mathcal{N}[\phi(t,x;p)] = \frac{\partial^3\phi(t,x;p)}{\partial t^3} - \frac{\partial^3\phi(t,x;p)}{\partial t\partial x^2} - \frac{\partial^4\phi(t,x;p)}{\partial x^2\partial t^2} + \frac{\partial^4\phi(t,x;p)}{\partial x^4} - \phi(t,x;p)\frac{\partial\phi(t,x;p)}{\partial t} - \frac{\partial^2\phi(t,x;p)}{\partial t^2}\frac{\partial\phi(t,x;p)}{\partial x}. \tag{29}$$

Thus, the mth-order deformation equations are obtained;

$$\mathcal{L}[u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar\mathcal{H}(t,x)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m \geq 1), \tag{30}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t\partial x^2} - \frac{\partial^4 u_{m-1}}{\partial x^2\partial t^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} - \sum_{k=0}^{m-1}\left(u_k\frac{\partial u_{m-1-k}}{\partial t} + \frac{\partial^2 u_k}{\partial t^2}\frac{\partial u_{m-1-k}}{\partial x}\right) \tag{31}$$

Now, for $m \geq 1$, the solution of the mth-order deformation Eq. (30);

$$u_m(t,x) = \chi_m u_{m-1}(t,x) + \hbar\mathcal{H}(t,x)\mathcal{L}^{-1}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{32}$$

Starting with an initial approximation $u_0(t,x) = \cos x - t\sin x - \frac{t^2}{2}\cos x$, by means of the above iteration formula (32), if $\mathcal{H}(t,x) = 1, \hbar = -1$ and after calculating the other terms, the results are obtained as follows.

$$u_1(t,x) = \frac{t^3}{3!}\sin x + \frac{t^4}{4!}\cos x,$$
$$u_2(t,x) = -\frac{t^5}{5!}\sin x - \frac{t^6}{6!}\cos x,$$
$$u_3(t,x) = \frac{t^7}{7!}\sin x + \frac{t^8}{8!}\cos x, \quad \cdots \tag{33}$$

Therefore, the solution of Example (3.2) can be readily obtained by;

$$u(t,x) = \sum_{m=0}^{+\infty} u_m(t,x) = cosx(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots) - \sin x(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots). \tag{34}$$

Continuing the expansion to the last term gives the solution of Eq.(27) as;

$$u(t,x) = cos(x+t), \tag{35}$$

which is the exact solution.

Example 3.3 *Consider the following equation;*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)u = \frac{\partial u}{\partial t} - 2u, \tag{36}$$

with the initial conditions;

$$u(0,x,y) = \sinh(x+y),$$
$$\frac{\partial u}{\partial t}(0,x,y) = 2\sinh(x+y),$$
$$\frac{\partial^2 u}{\partial t^2}(0,x,y) = 4\sinh(x+y) \tag{37}$$

subject to the exact solution $u(t,x,y) = sinh(x+y)e^{2t}$. To solve Eq.(36) by means of the HAM a nonlinear operator is defined as;

$$\begin{aligned}\mathcal{N}[\phi(t,x,y;p)] \\ = \frac{\partial^3\phi(t,x,y;p)}{\partial t^3} - \frac{\partial^3\phi(t,x,y;p)}{\partial t\partial x^2} - \frac{\partial^3\phi(t,x,y;p)}{\partial t\partial y^2} \\ - \frac{\partial^4\phi(t,x,y;p)}{\partial x^2\partial t^2} + \frac{\partial^4\phi(t,x,y;p)}{\partial x^4} + \frac{\partial^4\phi(t,x,y;p)}{\partial x^2\partial y^2} \\ + \frac{\partial^4\phi(t,x,y;p)}{\partial y^2\partial x^2} + \frac{\partial^4\phi(t,x,y;p)}{\partial y^4} - \frac{\partial\phi(t,x,y;p)}{\partial t} \\ +2\phi(t,x,y;p).\end{aligned} \tag{38}$$

Thus, the mth-order deformation equations are obtained;

$$\mathcal{L}[u_m(t,x,y) - \chi_m u_{m-1}(t,x,y)] = \hbar\mathcal{H}(t,x,y)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m \geq 1), \tag{39}$$

where

$$\begin{aligned}\mathcal{R}_m(\vec{u}_{m-1}) \\ = \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t\partial x^2} - \frac{\partial^3 u_{m-1}}{\partial t\partial y^2} - \frac{\partial^4 u_{m-1}}{\partial x^2\partial t^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} \\ + \frac{\partial^4 u_{m-1}}{\partial x^2\partial y^2} - \frac{\partial^4 u_{m-1}}{\partial y^2\partial t^2} + \frac{\partial^4 u_{m-1}}{\partial y^2\partial x^2} + \frac{\partial^4 u_{m-1}}{\partial y^4} - \frac{\partial u_{m-1}}{\partial t} \\ +2u_{m-1}.\end{aligned} \tag{40}$$

Now, for $m \geq 1$, the solution of the mth-order deformation Eq. (39) is;

$$u_m(t,x,y) = \chi_m u_{m-1}(t,x,y) + \hbar\mathcal{H}(t,x,y)\mathcal{L}^{-1}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{41}$$

Starting with an initial approximation $u_0(t,x,y) = (1+2t+2t^2)\sinh(x+y)$, by means of the above iteration formula (41), if $\mathcal{H}(t,x,y) = 1, \hbar = -1$ and after calculating the other terms, the results are obtained as follows;

$$u_1(t,x,y) = \frac{4}{3}t^3\sinh(x+y),$$
$$u_2(t,x,y) = \frac{2}{3}t^4\sinh(x+y),$$
$$u_3(t,x,y) = \frac{4}{15}t^5\sinh(x+y),\cdots \tag{42}$$

Therefore, the solution of Example (3.3) can be readily obtained by;

$$u(t,x,y) = \sum_{m=0}^{+\infty} u_m(t,x,y)$$
$$= (1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \frac{4}{15}t^5$$
$$+ \cdots)\sinh(x+y) = (1 + 2\text{t} + \frac{(2\text{t})^2}{2!}$$
$$+ \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \cdots)\sinh(x+y). \quad (43)$$

Continuing the expansion to the last term gives the solution of Eq. (36) as;

$$u(t,x,y) = e^{2t}\sinh(x+y), \quad (44)$$

which is the exact solution.

Example 3.4 *Consider the following nonlinear partial differential equation;*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = -\left(\frac{1}{3}\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{1}{6}\frac{\partial^2 u}{\partial t^2}\right)^3$$
$$-16u, \quad (45)$$

with the initial conditions;

$$u(0,x) = -x^4, \quad \frac{\partial u}{\partial t}(0,x) = 0, \quad \frac{\partial^2 u}{\partial t^2}(0,x) = 0. \quad (46)$$

The exact solution for this example is $u(t,x) = -x^4 + 4t^3$. To solve Eq. (45) by means of the HAM, a nonlinear operator is defined as;

$$\mathcal{N}[\phi(t,x;p)]$$
$$= \frac{\partial^3 \phi(t,x;p)}{\partial t^3} - \frac{\partial^3 \phi(t,x;p)}{\partial t \partial x^2} - \frac{\partial^4 \phi(t,x;p)}{\partial x^2 \partial t^2} + \frac{\partial^4 \phi(t,x;p)}{\partial x^4}$$
$$+ \left(\frac{1}{3}\frac{\partial^2 \phi(t,x;p)}{\partial x^2}\right)^2 - \left(\frac{1}{6}\frac{\partial^2 \phi(t,x;p)}{\partial t^2}\right)^3 + 16\phi(t,x;p). \quad (47)$$

Thus, the mth-order deformation equations are obtained;

$$\mathcal{L}[u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar\mathcal{H}(t,x)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m \geq 1), \quad (48)$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial x^2 \partial t^2} + \frac{\partial^4 u_{m-1}}{\partial x^4}$$
$$+ \frac{1}{9}\sum_{k=0}^{m-1}\left(\frac{\partial^2 u_k}{\partial x^2}\frac{\partial^2 u_{m-1-k}}{\partial x^2}\right) - \frac{1}{216}\sum_{n=0}^{k}\left[\frac{\partial^2 u_{k-n}}{\partial t^2}\sum_{i=0}^{n}\frac{\partial^2 u_i}{\partial t^2}\frac{\partial^2 u_{n-i}}{\partial t^2}\right]$$
$$+16u_{m-1}. \quad (49)$$

Now, for $m \geq 1$, the solution of the mth-order deformation Eq. (48) is;

$$u_m(t,x) = \chi_m u_{m-1}(t,x) + \hbar\mathcal{H}(t,x)\mathcal{L}^{-1}[\mathcal{R}_m(\vec{u}_{m-1})]. \quad (50)$$

Starting with an initial approximation $u_0(t,x) = -x^4$, by means of the above iteration formula (50) if $\mathcal{H}(t,x) = 1, \hbar = -1$, and after calculating the other terms, the results are obtained as follows.

$$u_1(t,x) = 0, \qquad u_2(t,x) = 0,$$
$$u_3(t,x) = 4t^3, \qquad u_4(t,x) = 0, \quad \cdots \tag{51}$$

Therefore, the solution of Example (3.4) can be readily obtained by;

$$u(t,x) = \sum_{m=0}^{+\infty} u_m(t,x) = -x^4 + 4t^3, \tag{52}$$

which is the exact solution for this example.

Conclusions

In recent times, the HAM has been successfully applied to various linear and nonlinear problems in variant science. In this paper, the HAM was tested on some examples of special nonlinear partial differential equations to show the simplicity, effectiveness and straightforwardness of the method. Here, the HAM has been successfully applied in solving some nonlinear parabolic-hyperbolic partial differential equations. The examples used are further confirmation of the flexibility and potential of the HAM for complicated nonlinear problems in science and engineering.

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