WALAILAK JOURNAL
http://wjst.wu.ac.th Mathematics

A Fractional Finite Difference Method for Solving the Fractional Poisson Equation Based on the Shifted Grünwald Estimate

Abdollah BORHANIFAR* **and Sohrab VALIZADEH**

Department of Mathematics, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran

(*Corresponding author; e-mail: borhani@uma.ac.ir)

Received: 2 March 2012, Revised: 28 February 2013, Accepted: 3 July 2013

Abstract

In this study a fractional Poisson equation is scrutinized through finite difference using the shifted Grünwald estimate. A novel method is proposed numerically. The existence and uniqueness of the solution for the fractional Poisson equation is proved. Exact and numerical solutions are constructed and compared. Then numerical results show the efficiency of the proposed method.

Keywords: Fractional Poisson equation, Riemann-Liouville fractional derivative, Shifted Grünwald Estimate, Taylor's expansion of fractional order

Introduction and definitions

Mathematical modeling of many physical systems leads to linear and nonlinear fractional differential equations in various fields of physics and engineering. The numerical and analytical approximations of such systems have been intensively studied since the work of Momani and Odibat [1,2]. Recently, several mathematical methods including the Adomian decomposition method [3,4], variation iteration method [5], homotopy perturbation method [6,7] and fractional finite difference method [8,9] have been developed to obtain approximate analytical solutions. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations and some other methods give the solution in a series form which converges to the exact solution. Analysis of the wave equation and nonlinear equation in mathematical physics has been of considerable interest in the literature. These methods include Exp-Function [10-12], G'/G-function [13-15], and the DTM method [16,17]. The fractional Poisson equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics and other areas of mechanics and physics. In this paper, we examine finite difference methods to solve the fractional Poisson equation of the form;

$$\nabla^{\alpha} U(x,y) = \frac{\partial^{\alpha} U(x,y)}{\partial x^{\alpha}} + \frac{\partial^{\alpha} U(x,y)}{\partial y^{\alpha}} = f(x,y) \tag{1}$$

on a finite domain;

$$\Omega = \{(x,y) | (x,y) \in [0,1] \times [0,1]\},$$

with Dirichlet boundary conditions.

Here, we consider the case $1 \leq \alpha \leq 2$, where the parameter α is the fractional order of the spatial derivative. The function $f(x,y)$ in (1) is a real valued function $f: \mathbb{R}^2 \to \mathbb{R}$.

Riemann-Liouville fractional derivatives

Let function $f: \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow f(x)$, with continuous derivatives of the order n;

$$D_x^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)dt \tag{2}$$

be Riemann-Liouville fractional derivatives of order α, where n is an integer such that $n-1 \leq \alpha \leq n$. (see [18-20]);

Shifted Grünwald estimate

In the case of $1 \leq \alpha \leq 2$, we define the shifted Grünwald formula;

$$\frac{d^\alpha f}{dx^\alpha} = \lim_{M\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{M} g_k . f(x-(k-1)h)$$

which defines the following shifted Grünwald estimate to the fractional derivative;

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{h^\alpha} \sum_{k=0}^{M} g_k . f(x-(k-1)h) + O(h^\alpha) \tag{3}$$

where M is a positive integer, $h = \frac{R-L}{M}$, Γ is the gamma function, and the normalized Grünwald weights are defined by;

$$g_k = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)} \qquad \text{for } k = 0,1,2,\dots$$

or

$$g_0 = 1 \text{ and } g_k = (-1)^k \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \qquad \text{for } k = 1,2,3,\dots \tag{4}$$

Note that these normalized weights only depend on the order α and the index k (see [20-22]).

Taylor's expansion of fractional order

Assume that the continuous function $f: \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow f(x)$, has a fractional derivative of order kα for any positive integer k and any α, $1 \leq \alpha \leq 2$. Then the following fractional Taylor's series holds

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)} f^{(k\alpha)}(x), \tag{5}$$

where $f^{(k\alpha)}(x)$ is the derivative of order kα for $f(x)$. Formally, one has;

$$f(x+h) = E_\alpha(h^\alpha D_x^\alpha) f(x), \tag{6}$$

where $E_\alpha(x)$ denotes the Mittag-Leffler function defined by the expression;

$$\mathrm{E}_{\alpha}(\mathrm{x}):=\sum_{\mathrm{k}=0}^{\infty}\frac{\mathrm{x}^{\mathrm{k}}}{\Gamma(1+\mathrm{k}\alpha)}. \tag{7}$$

We draw attention to the point that $\Gamma(1+k\alpha)\coloneqq(k\alpha)!$ (see [23,24]).

Approximating the fractional Poisson equation

To discuss discretizations, first consider the fractional Poisson problem (1) on the unit square $0\leq x\leq 1,\ 0\leq y\leq 1$ and suppose we have Dirichlet boundary conditions. We will use a uniform Cartesian grid consisting of grid points (x_i,y_j) where $x_i=ih$ for $i=1,2,\ldots m-1$ and $y_j=jk$ for $j=1,2,\ldots n-1$.

Let $u_{i,j}$ represent an approximation to $u(x_i,y_j)$. In order to discretize (1) we replace both the x- and y-derivatives with shifted Grünwald finite differences, which gives;

$$\frac{1}{h^{\alpha}}\sum_{s=0}^{i+1}g_s.u_{i-s+1,j}+\frac{1}{k^{\alpha}}\sum_{z=0}^{j+1}g_z.u_{i,j-z+1}=f_{i,j}. \tag{8}$$

For simplicity of notation we will consider the special case where $h=k$ hence $m=n$, though it is easy to handle the general case. We can then rewrite Eq. (8) as;

$$\frac{1}{h^{\alpha}}\left(\sum_{s=0}^{i+1}g_s.u_{i-s+1,j}+\sum_{z=0}^{j+1}g_z.u_{i,j-z+1}\right)=f_{i,j} \tag{9}$$

For $i,j=1,2,\ldots m-1$.

Consistency of the finite difference method

If u is replaced by U at the mesh points of the difference equation, where U is the exact solution of the fractional Poisson equation, the local truncation error T_{ij} at the (i,j) grid point is defined in the obvious way;

$$T_{ij}=\frac{1}{h^{\alpha}}\left(\sum_{s=0}^{i+1}g_s.U_{i-s+1,j}+\sum_{z=0}^{j+1}g_z.U_{i,j-z+1}\right)-f_{i,j}$$

and by splitting this into the α-order difference in the x- and y-derivatives using the fractional Taylor's expansion of fractional order (5) gives;

$$T_{ij}=\frac{1}{h^{\alpha}}\left(\sum_{s=0}^{i+1}g_s.\left(U_{i,j}+\frac{[(1-s)h]^{\alpha}}{\alpha!}\left(\frac{\partial^{\alpha}U}{\partial x^{\alpha}}\right)_{i,j}+\frac{[(1-s)h]^{2\alpha}}{(2\alpha)!}\left(\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}}\right)_{i,j}+\cdots\right)\right)$$
$$+\frac{1}{h^{\alpha}}\left(\sum_{z=0}^{j+1}g_z.\left(U_{i,j}+\frac{[(1-z)h]^{\alpha}}{\alpha!}\left(\frac{\partial^{\alpha}U}{\partial y^{\alpha}}\right)_{i,j}+\frac{[(1-z)h]^{2\alpha}}{(2\alpha)!}\left(\frac{\partial^{2\alpha}U}{\partial y^{2\alpha}}\right)_{i,j}+\cdots\right)\right)-f_{i,j} \tag{10}$$

Since $\sum_{k=0}^{\infty}g_k=0$ (11)
(see [8,9]), one can prove that;

$$\sum_{k=0}^{\infty} g_k (1-k)^\alpha = \alpha!. \tag{12}$$

Because $\sum_{k=0}^{\infty} g_k (1-k)^\alpha$ is α derivatives of $(1+x)^\alpha$ at $x=-1$. Thus from (10), (11) and (12) we can write;

$$T_{ij} = (\nabla^\alpha U - f)_{i,j} + h^\alpha \left(\sum_{s=0}^{\infty} g_s . \frac{[(1-s)]^{2\alpha}}{(2\alpha)!}\right)\left[\left(\frac{\partial^{2\alpha} U}{\partial x^{2\alpha}}\right)_{i,j} + \left(\frac{\partial^{2\alpha} U}{\partial y^{2\alpha}}\right)_{i,j}\right] + O(h^{2\alpha}) \tag{13}$$

But U is the exact solution of the differential Eq. (1) so $(\nabla^\alpha U - f)_{i,j} = 0$. This shows that the difference equation is consistent and the local truncation error vanishes as $h \to 0$. Therefore the proposed scheme is consistent.

Theorem 1. The solution of Eq. (9) exists and is unique.

Proof. We will apply a matrix equation system to the linear system of equations arising from the finite difference equations defined by (9), and will use the Greschgorin theorem [25] to determine that the solution of Eq. (9) is unique. The difference equations defined by (9), together with the Dirichlet boundary conditions, result in a linear system of equations of the form $AU = G$ where

$$A = \begin{bmatrix} B & g_0 I & 0 & & 0 & 0 & 0 \\ g_2 I & B & g_0 I & \cdots & 0 & 0 & 0 \\ g_3 I & g_2 I & B & & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \\ g_{m-3} I & g_{m-4} I & g_{m-5} I & & B & g_0 I & 0 \\ g_{m-2} I & g_{m-3} I & g_{m-4} I & \cdots & g_2 I & B & g_0 I \\ g_{m-1} I & g_{m-2} I & g_{m-3} I & & g_3 I & g_2 I & B \end{bmatrix}, B = \begin{bmatrix} 2g_1 & g_0 & 0 & & 0 & 0 & 0 \\ g_2 & 2g_1 & g_0 & \cdots & 0 & 0 & 0 \\ g_3 & g_2 & 2g_1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \\ g_{m-3} & g_{m-4} & g_{m-5} & & 2g_1 & g_0 & 0 \\ g_{m-2} & g_{m-3} & g_{m-4} & \cdots & g_2 & 2g_1 & g_0 \\ g_{m-1} & g_{m-2} & g_{m-3} & & g_3 & g_2 & 2g_1 \end{bmatrix}$$

in which A is an $(m-1)^2 \times (m-1)^2$ block matrix and each block B or I is itself an $(m-1) \times (m-1)$ matrix, and I is the $(m-1) \times (m-1)$ identity matrix. G and U are the $(m-1)^2 \times 1$ matrices as follows.

$$G = \begin{bmatrix} g_2 u_{1,0} + g_2 u_{0,1} \\ g_2 u_{2,0} + g_3 u_{0,1} \\ \vdots \\ g_2 u_{m-2,0} + g_{m-1} u_{0,1} \\ g_2 u_{m-1,0} + g_m u_{0,1} + g_0 u_{m,1} \\ \hline g_3 u_{1,0} + g_2 u_{0,2} \\ g_3 u_{2,0} + g_3 u_{0,2} \\ \vdots \\ g_3 u_{m-2,0} + g_{m-1} u_{0,2} \\ g_3 u_{m-1,0} + g_m u_{0,2} + g_0 u_{m,2} \\ \hline \vdots \\ g_{m-1} u_{1,0} + g_2 u_{0,m-2} \\ g_{m-1} u_{2,0} + g_3 u_{0,m-2} \\ \vdots \\ g_{m-1} u_{m-2,0} + g_{m-1} u_{0,m-2} \\ g_{m-1} u_{m-1,0} + g_m u_{0,m-2} + g_0 u_{m,m-2} \\ \hline g_m u_{1,0} + g_2 u_{0,m-1} + g_0 u_{1,m} \\ g_m u_{2,0} + g_3 u_{0,m-1} + g_0 u_{2,m} \\ \vdots \\ g_m u_{m-2,0} + g_{m-1} u_{0,m-1} + g_0 u_{m-2,m} \\ g_m u_{m-1,0} + g_m u_{0,m-1} + g_0 u_{m,m-1} + g_0 u_{m-1,m} \end{bmatrix}, \quad U = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{m-2,1} \\ u_{m-1,1} \\ \hline u_{1,2} \\ u_{2,2} \\ \vdots \\ u_{m-2,2} \\ u_{m-1,2} \\ \hline \vdots \\ u_{1,m-2} \\ u_{2,m-2} \\ \vdots \\ u_{m-2,m-2} \\ u_{m-1,m-2} \\ \hline u_{1,m-1} \\ u_{2,m-1} \\ \vdots \\ u_{m-2,m-1} \\ u_{m-1,m-1} \end{bmatrix}$$

According to the Greschgorin theorem the eigenvalues of the matrix A lie in the union of the $K = (m-1)^2$ circles centered at $A_{i,i}$, with radius $r_i = \sum_{k=0,k\neq 1}^{K} |A_{i,k}|$. Here we have, $A_{i,i} = 2g_1 = -2\alpha$ and

$$r_i = \sum_{k=0,k\neq i}^{K} |A_{i,k}| = 2\sum_{k=0,k\neq i}^{i+1} |A_{i,k}| = 2\sum_{s=0,s\neq 1}^{i+1} g_s < -2g_1 = 2\alpha. \tag{14}$$

Because $\sum_{k=0}^{\infty} g_k = 0$ and $g_k > 0$ for $k \geq 2$ hence $\sum_{k=0}^{\infty} g_k < 0$ and so $\sum_{s=0,s\neq 1}^{i+1} g_s < -g_1$, therefore $A_{i,i} + r_i < 0$. We also have $A_{i,i} - r_i > -4\alpha$, with strict inequality holding true when α is not an integer. This implies that the eigenvalues of the matrix A are all less than 0 and larger than -4α. This is shown that matrix A is invertible. Inequality (14) shows that $-A$ is a strictly diagonal dominant and it is a positive defined matrix therefore A is non-singular and so $AU = G$ exist unique solution for any α belong to$(1,2)$ interval.■

For the classical Poisson equation, that is $\alpha = 2$ in (1), the resulting Grünwald estimate from (9) is the classical finite difference equation given by ($g_0 = 1, g_1 = -2, g_2 = 1$ and$g_3 = g_4 = \cdots = 0$)

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2} + \frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2} = f_{i,j} \ , i,j = 1,2,\ldots,m-1$$

The truncation error satisfies;

$$T_{ij} = (\nabla^2 U - f)_{i,j} + \frac{h^2}{12}\left[\left(\frac{\partial^4 U}{\partial x^4}\right)_{i,j} + \left(\frac{\partial^4 U}{\partial y^4}\right)_{i,j}\right] + O(h^4) \tag{15}$$

In this case, we will have matrixes A and B as follows for m grid;

$$A = \begin{bmatrix} B & I & 0 & & 0 & 0 & 0 \\ I & B & I & \cdots & 0 & 0 & 0 \\ 0 & I & B & & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & & B & I & 0 \\ 0 & 0 & 0 & \cdots & I & B & I \\ 0 & 0 & 0 & & 0 & I & B \end{bmatrix}, B = \begin{bmatrix} -4 & 1 & 0 & & 0 & 0 & 0 \\ 1 & -4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & & -4 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -4 & 1 \\ 0 & 0 & 0 & & 0 & 1 & -4 \end{bmatrix}$$

Note that $-A$ is a symmetric positive definite matrix and it is weak diagonally therefore A is non-singular and dominant and it is a positive defined matrix therefore A is non-singular and linear system of equations $AU = G$ (for $\alpha = 2$) have a unique solution.

Numerical result

The following fractional Poisson equation;

$$\frac{\partial^\alpha U(x,y)}{\partial x^\alpha} + \frac{\partial^\alpha U(x,y)}{\partial y^\alpha} = f(x,y) \tag{16}$$

was considered on a finite domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the non-homogeneous function $f(x,y) = \Gamma(\alpha+1)(x^\alpha + y^\alpha)$ and boundary conditions;

$$\begin{cases} U(x,0) = U(0.y) = 0 \\ \quad U(x,1) = x^\alpha. \\ \quad U(1,y) = y^\alpha \end{cases} \tag{17}$$

This fractional Poisson equation has the exact solution $U(x,y) = (xy)^{\alpha}$, which can be verified by applying the fractional differential formula.

$$D_x^{\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \tag{18}$$

The fractional Poisson absolute-error is identified by $Error = \frac{1}{(m-1)^2}\sqrt{\sum_{i,j=1}^{m-1}\left(U_{i,j} - u_{i,j}\right)^2}$ in which $U_{i,j}$ and $u_{i,j}$ are exact and numerical solutions respectively. The values of absolute-error are shown in **Table 1** for the above example problem for different values of h and α. From **Table 1**, it can be seen that our fractional finite difference method guaranties the efficiency of the proposed method with $O(h^{\alpha})$. **Figures 1 - 4** having different values of h and α verify the efficiency of the proposed scheme.

Table 1 Absolute error for variation of h and α.

h	Fractional Poisson absolute error			Values of absolute error
	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
0.1000	0.0138	0.0086	0.0063	0.0053
0.0500	0.0076	0.0045	0.0029	0.0021
0.0250	0.0040	0.0023	0.0015	0.0010
0.0125	0.0020	0.0011	0.0008	0.0005

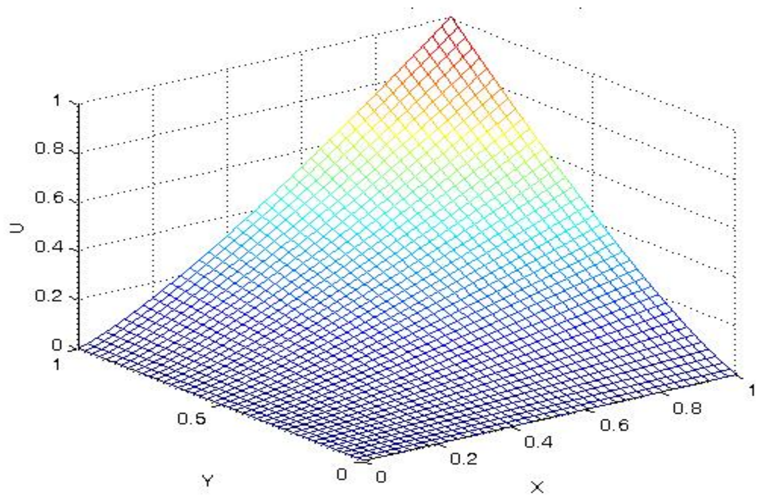

Figure 1 3-D meshed surface plot of approximate solution for $h = 0.025$, $\alpha = 1.25$ and error = 0.004.

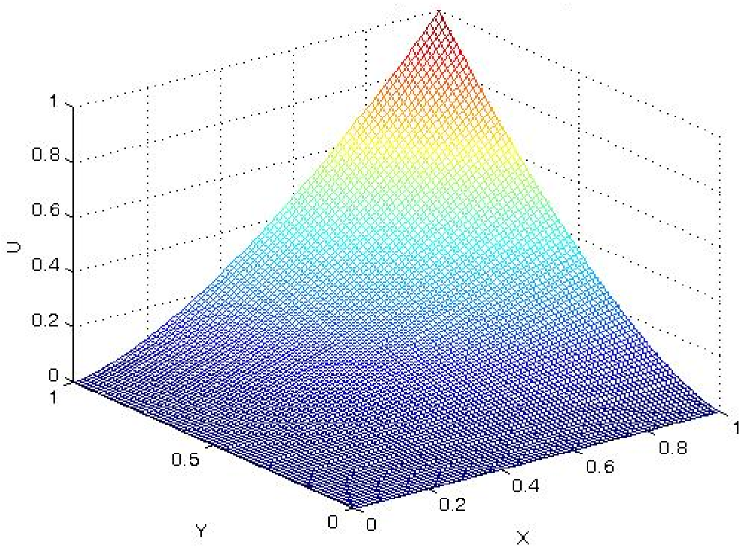

Figure 2 3-D meshed surface plot of approximate solution for $h = 0.0125$, $\alpha = 1.5$ and error = 0.0011.

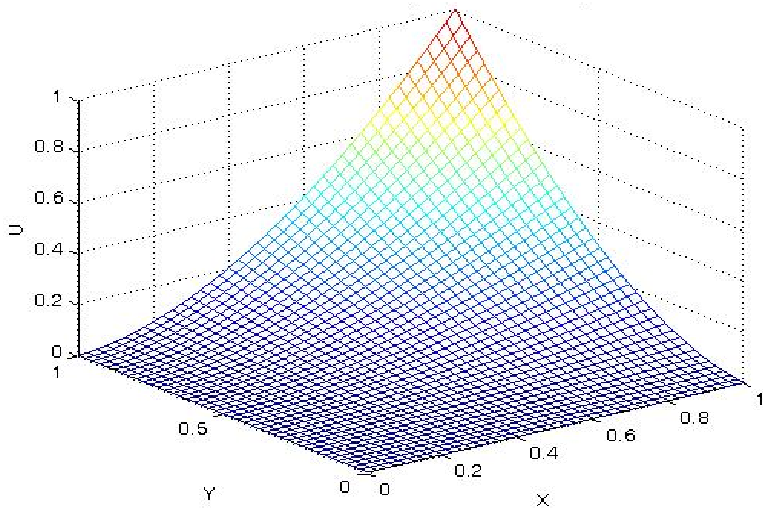

Figure 3 3-D meshed surface plot of approximate solution for $h = 0.025$, $\alpha = 1.75$ and error = 0.0015.

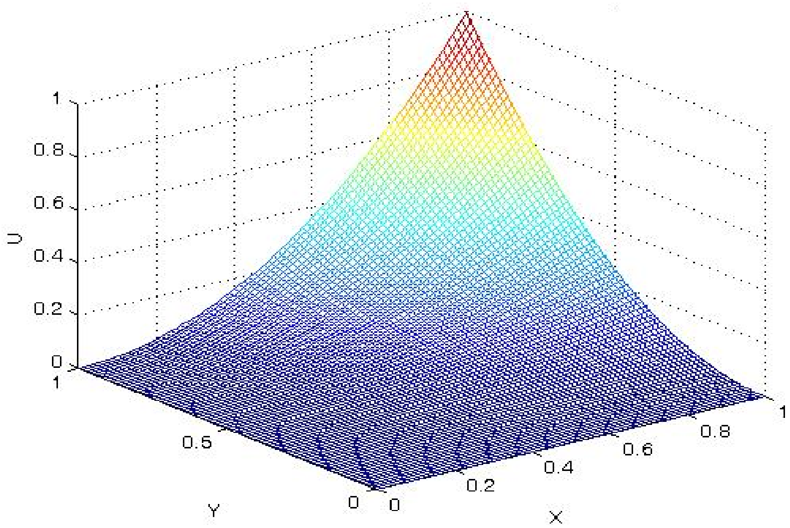

Figure 4 3-D meshed surface plot of approximate solution for $h = 0.0125$, $\alpha = 2$. and error = 0.0005.

Conclusions

The fractional finite difference method using a shifted Grünwald estimate is successful in finding the approximate solution for fractional Poisson equations. The anticipated method based on a shifted Grünwald approximation to the fractional derivative, is consistent of order α and the solution of the fractional finite difference method for discretization of fractional Poisson equations is unique.

Acknowledgements

This work is partially supported by Grant-in-Aid from the University of Mohaghegh Ardabili, Ardabil, Iran.

References

[1] S Momani and Z Odibat. A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula. *J. Comput. Appl. Math.* 2008; **220**, 85-95.
[2] S Momani. A novel Non-perturbative analytical solutions of the space- and time-fractional Burgers equations. *Chaos Soliton. Fract.* 2006; **28**, 930-7.
[3] Q Yang, F Liu and I Turner. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. *Appl. Math. Model.* 2010; **34**, 200-18.
[4] M Grag and A Sharma. Solution of space-time Fractional Telegraph equation by Adomian decomposition method. *J. Inequalities Spec. Funct.* 2011; **1**, 1-7.
[5] Z Odibat and S Momani. Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlinear Sci. Numer. Simul.* 2006; **7**, 15-27.
[6] X Li, M Xu and X Jiang. Homotopy perturbation method to time-fractional diffusion equation with a moving boundary condition. *Appl. Math. Comput.* 2009; **208**, 434-9.
[7] L Song and H Zhang. Application of homotopy analysis method to fractional KDV-Burgers-Kuramoto equation. *Phys. Lett. A* 2007; **367**, 88-94.
[8] A Borhanifar and S Valizadeh. Numerical solution for fractional partial differential equations using Crank-Nicolson method with Shifted Grünwald estimate. *Walailak J. Sci. & Tech.* 2012; **9**, 433-44.
[9] MM Meerschaert and C Tadjeran. Finite difference approximations for two-sided space-fractional partial differential equations. *Appl. Numer. Math.* 2006; **56**, 80-90.

[10] A Borhanifar and MM Kabir. Soliton and periodic solutions for (3+1)-dimensional nonlinear evolution equations by Exp-function method. *Appl. Appl. Math.* 2009; **229**, 158-67.
[11] A Borhanifar, MM Kabir and L Maryam Vahdat. New periodic and soliton wave solutions for the generalized Zakharov system and (2+1)-dimensional Nizhnik-Novikov-Veselov system. *Chaos Soliton. Fract.* 2009; **42**, 1646-54.
[12] A Borhanifar, MM Kabir. New periodic and soliton solutions by application of Exp-function method for nonlinear evolution equations. *J. Comput. Appl. Math.* 2009; **229**, 158-67.
[13] A Borhanifar, A Zamiri. Application of G'/G-expansion method for the Zhiber-Shabat equation and other related equations. *Math. Comput. Model.* 2011; **54**, 2109-16.
[14] MM Kabir, A Borhanifar and R Abazari. Application of G'/G-expansion method to Regularized (RLW) equation. *Comput. Math. Appl.* 2011; **61**, 2044-7.
[15] A Borhanifar and R Abazari. Application General Solution of generalized (2+1)-Dimensional Kadomtsev Petviashvili (KP) equation by using the G'/G-expansion method. *J. Comput. Math.* 2011; **1**, 219-25.
[16] A Borhanifar and R Abazari. Numerical study of nonlinear Schodinger and coupled Schrodinger equations by differential transformation method. *Opt. Commun.* 2010; **283**, 2026-31.
[17] A Borhanifar and R Abazari. Numerical study of the solution of the Burgers and coupled Burgers equations by a differential transformation method. *Comput. Math. Appl.* 2010; **59**, 2711-22.
[18] P Agrawalar. Solution for a fractional diffusion-wave equation defined in a bounded domain, mechanical engineering and energy processes. *Nonlinear Dynam.* 2002; **29**, 145-55.
[19] J Munkhammar. *Riemann-Liouville Fractional Derivatives and the Taylor-Riemann Series*. Department of Mathematics Uppsala University. 2004.
[20] I Podlubny. *Fractional Differential Equations*. Academic Press, New York. 1999.
[21] K Miller and B Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley and Sons, New York, 1993.
[22] S Samko, A Kilbas and O Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, London, 1993.
[23] G Jumarie. Fractional Hamilton-Jacobi equation for the optimal control of non-random fractional dynamics with fractional cost function. *J. Appl. Math. Comput.* 2007; **23**, 215-28.
[24] G Jumarie. Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution. *J. Appl. Math. Comput.* 2007; **24**, 31-48.
[25] E Isaacson and HB Keller. *Analysis of Numerical Methods*. John Wiley & Sons Inc., New York, 1966.