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A Reliable Algorithm for Fractional Schrödinger Equations

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Abstract

In this paper, the Homotopy Perturbation Method (HPM) is applied to find exact solutions of time-fractional Schrödinger equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

Keywords: Homotopy perturbation method, fractional Schrödinger partial differential equations, nonlinear problems, exact solution

Introduction

Nonlinear partial differential equations [1-21] are of extreme importance in applied and engineering sciences. The through study of literature reveals that most physical phenomena are nonlinear in nature and hence there is a dire need to find their appropriate solutions; see [1-21] and the references therein. Recently, scientists have observed that a number of real time problems are modeled by fractional nonlinear differential equations [1,6-8,12,13,16,19,20] which are very hard to tackle. In the similar context, the Homotopy perturbation method (HPM) is applied to solve time-fractional Schrödinger partial differential equations [13,20].

$$D_t^\alpha u(x,t) + iu_{xx}(x,t) = 0$$
$$u(x,0) = f(x), \qquad i^2 = -1 \tag{1.1}$$

or

$$iD_t^\beta u(x,t) + u(x,t) - y|u(x,t)|^2 u(x,t) = 0, \tag{1.2}$$
$$u(x,0) = f(x) \qquad \iota^2 = -1$$

where $0 \le \alpha \le 1$. The fractional derivatives are considered in the Caputo sense. It is to be highlighted that such equations arise frequently in applied, physical and engineering sciences. The basic motivation of this paper is the extension of the Homotopy Perturbation Method (HPM) to find approximate solutions of time-fractional Schrödinger partial differential equations; see [13,20] and the references therein. It is observed that the proposed algorithm is fully synchronized with the complexity of fractional differential equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

Definitions [9,16-19]

Definition 2.1 A real-valued function *f(x)*, $x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^∞ if $f^m \in C_\mu\ \mu \ge 1\ m \in N$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order ≥ 0 , of a function $f \in C_\mu, \mu \ge -1$, is defined as;

$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt.\ \alpha > 0, x > 0,$
$J^0 f(x) = f(x).$ (2.1)

Properties of the operator j^α can be found in [12-14]; only the following is mentioned.

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t),$
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$

Definition 2.3 The fractional derivative of $f(x)$ in the Caputo sense is defined as;

$$D_*^\alpha f(x) = J^{m-\alpha} D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^\alpha (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (2.2)$$

for $m-1 < \alpha \leq m, m \in Z, x > 0, f \in C_{-1}^m$.

Also, two of its basic properties are needed here.

Lemma 2.1 if $m-1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then;

$$D_*^\alpha J^\alpha f(x) = f(x), \quad (2.3)$$

and

$$J^\alpha D_*^\alpha f(x) = f(x) \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+).x^k}{k!}, x > 0. \quad (2.4)$$

Homotopy perturbation method (HPM) [4-6,10-12,21]

The essential idea of this method is to introduce a Homotopy parameter, say p, which takes a value from 0 to 1. When p = 0, the system of equations is in a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of "deformation". Eventually at p = 1, the system takes the original form of the equation and the final stage of "deformation" gives the desired solution. To illustrate the basic concept of HPM, consider the following nonlinear system of differential equations.

The nonlinear differential Eq. (1.1) can be expressed in the operator form as;

$$D_t^\alpha u + R(u) + N(u) = 0 \quad (3.1)$$

subject to the initial conditions $u(x,0) = f(x)$, where D_t^α is the time-fractional differential operator, $N(u)$is the nonlinear operator and $R(u)$ is some linear operator. Rearranging Eq. (3.1) and applying the operators J^α, inverse of the operator D^α, to both sides of Eq. (3.1) yields;

$$u(x,t) = f(x) - J^\alpha[R(u) + N(u)]. \quad (3.2)$$

Assume the solution of Eq. (3.2) to be in the form.

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots. \quad (3.3)$$

Substituting (3.3) into (3.2);

$$u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots = f(x) - pJ^\alpha[R(u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots) + N(u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots). \quad (3.4)$$

Now equating the coefficients of like powers of p, the following forms are obtained.

$$\begin{aligned} p^0&: \quad u_0(x,t) = f(x) \\ p^1&: \quad u_1(x,t) = -J^\alpha[R(u_0) + N(u_0)], \\ p^2&: \quad u_2(x,t) = -J^\alpha[R(u_1) + N(u_1)], \\ p^3&: \quad u_3(x,t) = -J^\alpha[R(u_2) + N(u_2)] \\ &. \\ &. \\ &. \end{aligned} \quad (3.5)$$

Finally, the solution $u(x,t)$is approximated.

$u(x,t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots$
$u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots.$

Solution procedure

Example 4.1 Consider the following linear time-fractional Schrödinger equation;

$$D_t^{\alpha}u + iu_{xx} = 0 \tag{4.1}$$
where $0 < \alpha \leq 1$,

with initial conditions;

$u(x,0) = 1 + \cosh(2x).$
$u(x,t) = 1 + \cosh\,(2x) - iJ^{\alpha}(u_{xx})$
$$u_0 + u_1p^1 + u_2p^2 + u_3p^3 + \cdots = f(x) - ipJ^{\alpha}\left(\frac{\partial^2 u_0}{\partial x^2} + p^1\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + p^3\frac{\partial^2 u_3}{\partial x^2} + ..\right).$$

Consequently, the following approximants are obtained.

p^0: $u_0(x,t) = 1 + \cosh(2x),$
p^1: $u_1(x,t) = -4\text{i}\cosh(2x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$,
p^2: $u_2(x,t) = (4\text{i})^2\cosh(2x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$,
p^3: $u_3(x,t) = -(4\text{i})^3\cosh(2x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$,
$\vdots$.

The solution in the series form is given by;

$u = u_0 + u_1p^1 + u_2p^2 + u_3p^3 + \cdots$
$u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots$
$$u(x,t) = 1 + \cosh(2x)\,(1 - 4\text{i}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (4\text{i})^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (4\text{i})^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots). \tag{4.2}$$

For the special case $\alpha = 1$, the form (4.2) is obtained.

$$u(x,t) = 1 + \cosh(2x)\,e^{-4\iota t} \tag{4.3}$$

which is the exact solution of the Schrödinger equation.
Graphs for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in **Figure 1**.

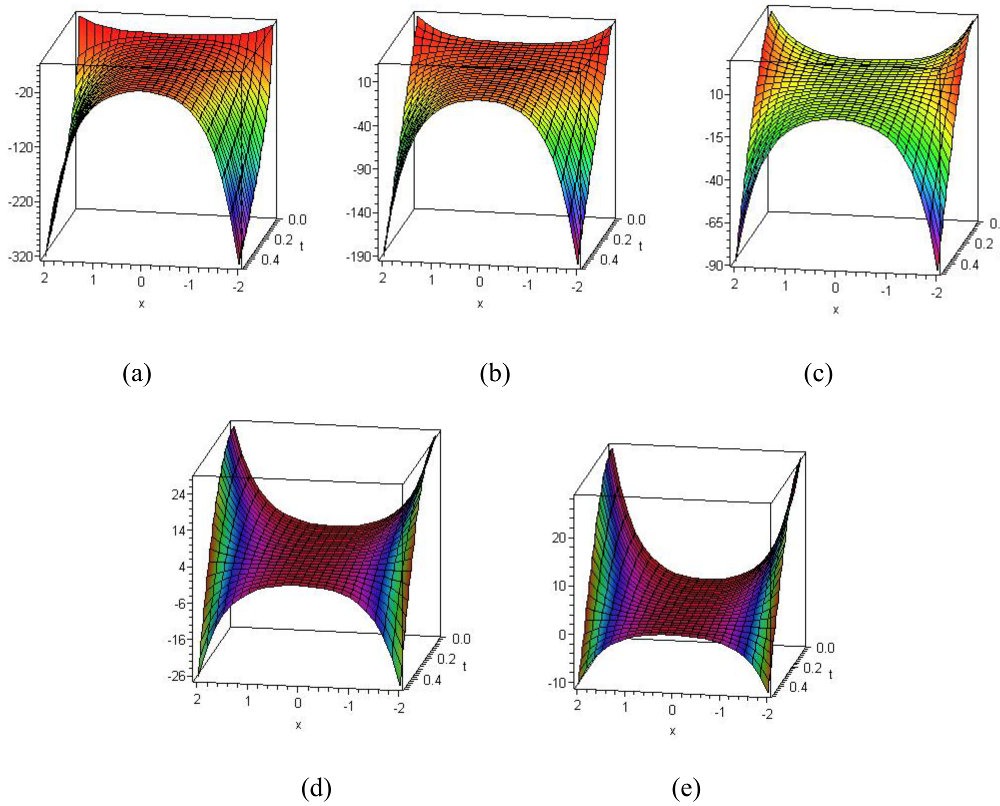


Figure 1 The surface shows solution $u(x,t)$for the Eq. (4.2) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution Eq. (4.3).

Example 5.2 Consider the following linear time-fractional Schrödinger equation;

$$D_t^\alpha u + iu_{xx} = 0 \qquad \text{where } 0 < \alpha \leq 1\,, \qquad (4.4)$$

with initial conditions;

$u(x,0) = e^{3\iota x}$.
$u(x,t) = e^{3ix} - iJ^\alpha(u_{xx})$
$u_0 + u_1 p^1 + u_2 p^2 + u_3 p^3 + \cdots = e^{3ix} - ipJ^\alpha(\frac{\partial^2 u_0}{\partial x^2} + p^1 \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + p^3 \frac{\partial^2 u_3}{\partial x^2} + ..)$.

Consequently, the following approximants are obtained.

p^0: $u_0(x,t) = e^{3\iota x}$,
p^1: $u_1(x,t) = 9\text{i}\, e^{3\iota x} \frac{t^\alpha}{\Gamma(\alpha+1)}$,
p^2: $u_2(x,t) = (9\text{i})^2\, e^{3\iota x} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$,
p^3: $u_3(x,t) = (9\text{i})^3\, e^{3\iota x} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$,
⋮.

The solution in the series form is given by;

$u = u_0 + u_1 p^1 + u_2 p^2 + u_3 p^3 + \cdots$
$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots$
$$u(x,t) = e^{3\iota x}\left(1 + 9i\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (9i)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (9i)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ \dots\right). \qquad (4.5)$$

For the special case $\alpha = 1$, the form (4.5) is obtained.

$$u(x,t) = e^{3\iota(x+3t)} \qquad (4.6)$$

which is the exact solution of the Schrödinger equation. Graphs for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in **Figure 2**.

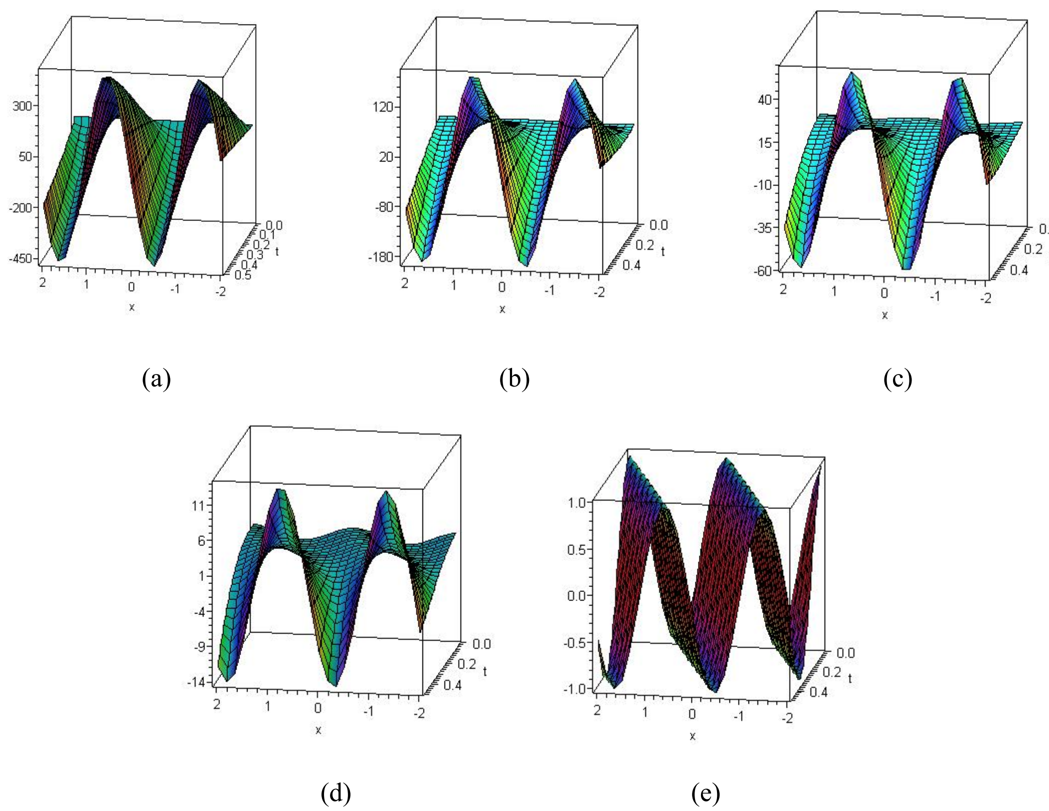


Figure 2 The surface shows solution $u(x,t)$for the Eq. (4.5) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution Eq. (4.6).

Example 5.3 Consider the following nonlinear time-fractional Schrödinger equation;

$$iD_t^{\alpha}u + u_{xx} + 2|u|^2 u = 0 \qquad \text{where } 0 < \alpha \leq 1\,, \qquad (4.7)$$

with initial conditions;

$$u(x,0) = e^{ix}$$
$$u(x,t) = e^{ix} + iJ^{\alpha}(u_{xx} + 2|u|^2 u)$$
$$u_0 + u_1 p^1 + u_2 p^2 + u_3 p^3 + \cdots = e^{ix} + (\frac{\partial^2 u_0}{\partial x^2} + p^1 \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_2}{\partial x^2} + p^3 \frac{\partial^2 u_3}{\partial x^2} + \cdots$$
$$+2\{(u_0 + p^1 u_1 + P^2 u_2 + \cdots)^2 (\bar{u}_0 + p^1 \bar{u}_1 + p^2 \bar{u}_2 + \ldots)\}).$$

Consequently, the following approximants are obtained.

p^0: $u_0(x,t) = e^{ix}$,
p^1: $u_1(x,t) = i\, e^{ix} \frac{t^{\alpha}}{\Gamma(\alpha+1)}$,
p^2: $u_2(x,t) = \mathrm{i}^2 e^{ix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$,
p^3: $u_3(x,t) = \mathrm{i}^3\, e^{ix} [5 - \frac{2\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$,
⋮

The solution in the series form is given by;

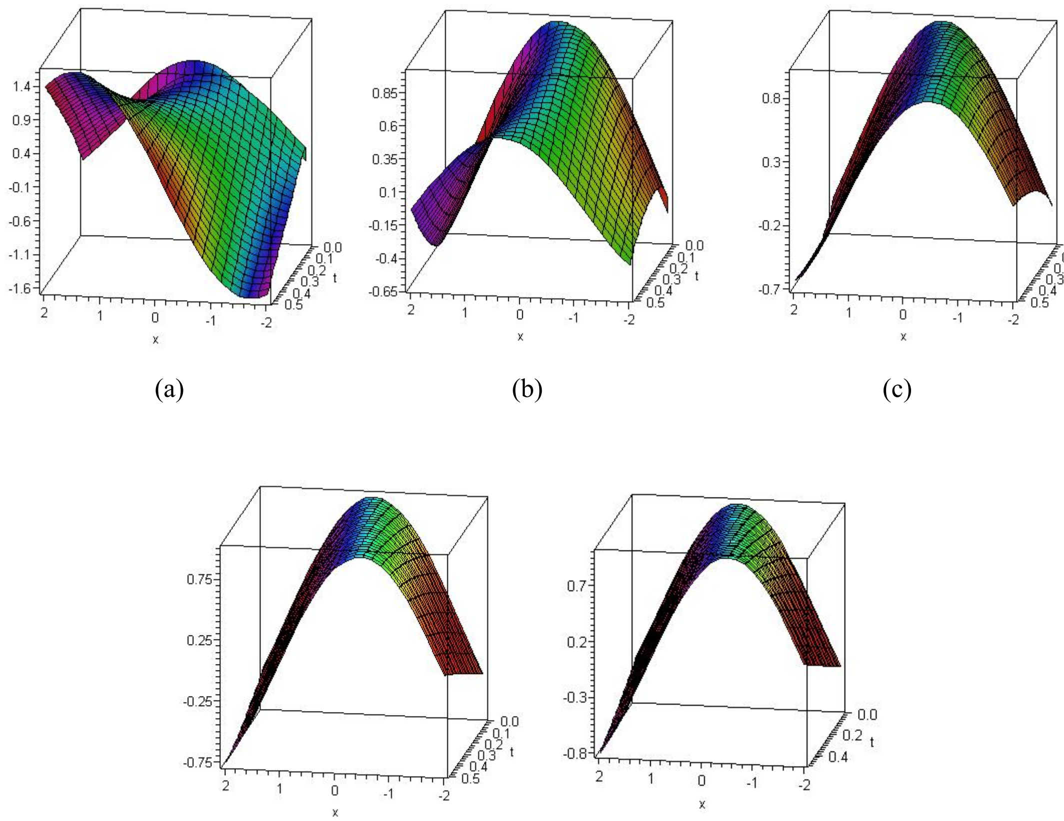
$$u = u_0 + u_1 p^1 + u_2 p^2 + u_3 p^3 + \cdots$$
$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$u(x,t) = e^{ix}(1 + \mathrm{i}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \mathrm{i}^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \mathrm{i}^3 [5 - \frac{2\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}] \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots). \quad (4.8)$$

For the special case $\alpha = 1$, the form Eq. (4.8) is obtained.

$$u(x,t) = e^{\iota(x+t)}, \quad (4.9)$$

which is the exact solution of the Schrödinger equation. Graphs for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in **Figure 3**.



(a) (b) (c)

(d) (e)

Figure 3 The surface shows solution $u(x,t)$ when (a) $\alpha = 0.25$, (b) α 0.50, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution Eq. (4.9).

Example 5.4 Consider the following nonlinear time-fractional Schrödinger equation;

$$iD_t^{\alpha}u + u_{xx} - 2|u|^2u = 0 \qquad \text{where } 0 < \alpha \leq 1\,, \tag{4.10}$$

with initial conditions;

$$u(x,0) = e^{ix}.$$
$$u(x,t) = e^{ix} + iJ^{\alpha}(u_{xx} - 2|u|^2u)$$
$$u_0 + u_1p^1 + u_2p^2 + u_3p^3 + \cdots = e^{ix} + ipJ^{\alpha}(\frac{\partial^2 u_0}{\partial x^2} + p^1\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + p^3\frac{\partial^2 u_3}{\partial x^2} + \cdots$$
$$-2\{(u_0 + p^1u_1 + P^2u_2 + \cdots)^2(\bar{u}_0 + p^1\bar{u}_1 + p^2\bar{u}_2 + \ldots)\})$$

Consequently, the following approximants are obtained.

p^1: $\quad u_1(x,t) = -3\mathrm{i}\, e^{ix}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$,

p^2: $\quad u_2(x,t) = (3\mathrm{i})^2 e^{ix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$,

p^3: $\quad u_3(x,t) = -(\mathrm{i})^3\, e^{ix}[63 - \frac{18\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$,

⋮.

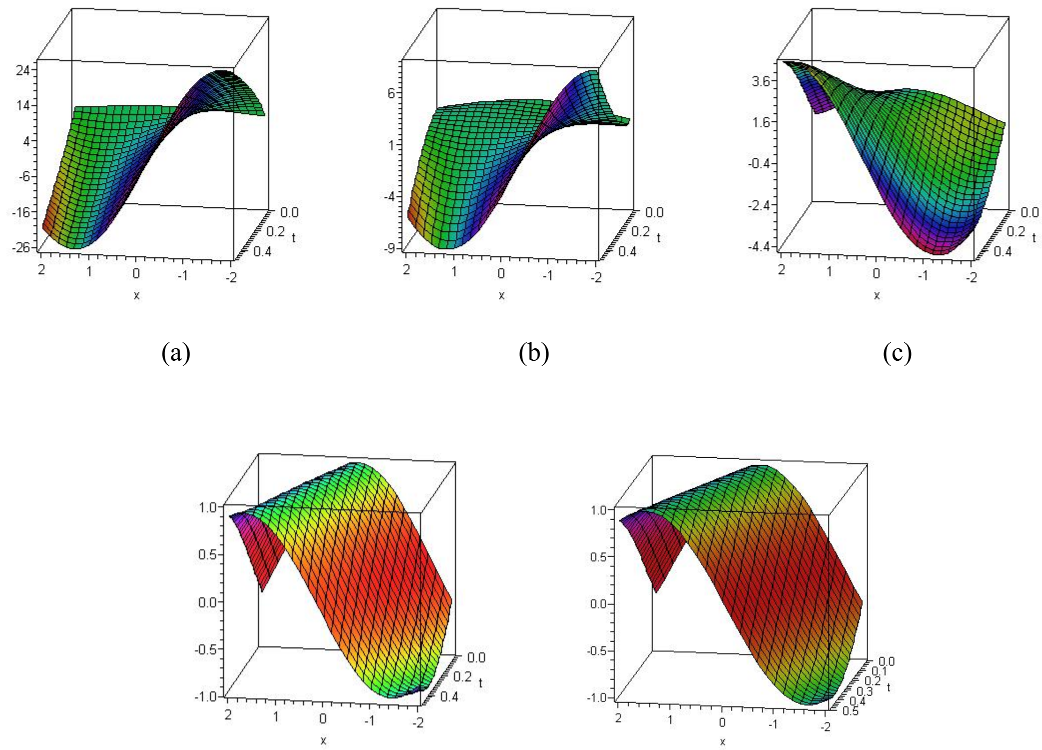
The solution in the series form is given by;

$$u = u_0 + u_1p^1 + u_2p^2 + u_3p^3 + \cdots$$
$$u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$u(x,t) = e^{ix}(1 - 3\mathrm{i}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (3\,\mathrm{i})^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (\mathrm{i})^3\left[63 - \frac{18\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots). \tag{4.11}$$

For the special case $\alpha = 1$, the form Eq. (4.11) is obtained.

$$u(x,t) = e^{\iota(x-3t)} \tag{4.12}$$

which is the exact solution of the Schrödinger equation. Graphs for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in **Figure 4**.



(a) (b) (c)

(d) (e)

Figure 4 The surface shows solution $u(x,t)$ when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution Eq. (4.12).

Conclusions

HPM has been implemented to find appropriate solutions of fractional Schrödinger partial differential equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm. Moreover, it has been observed that the method is easier to implement as compared to other techniques and can be used to solve nonlinear problems of a complex physical nature.

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