

RBF Meshless Method of Lines for the Numerical Solution of Nonlinear Sine-Gordon Equation

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Abstract

In this paper, a meshless method of lines (MOL) is applied for the numerical solution of nonlinear sine-Gordon equations. This technique does not require a mesh in the problem domain, and only a set of scattered nodes provided by initial data is required for the solution of the problem using some radial basis functions (RBF). The scheme is tested for single solitons, collision of breathers and soliton doublets. The results obtained from the method are compared with the exact solutions and earlier work.

Keywords: MOL, RBF, sine-Gordon equation, single soliton, soliton doublets

Introduction

The study of solutions of nonlinear wave equations has gained much attention in the last decade. These wave equations have applications in many branches of applied mathematics and theoretical physics. Zabusky and Kruskal [1] observed that the KdV equation leads to solitary waves called solitons, localized entities which keep identity after interaction. John Scott Russell [2] noted solitary waves while observing the movement of a canal barge. The sine-Gordon equation also models a soliton wave, which occurs in several physical situations [3]. The exact solution of soliton type equations has been developed using Backlund transformation [4], the painleve method [5], the Lie group of methods [6], the tangent hyperbolic method [7] and the inverse scattering method [8,9]. The sine-Gordon equation has also been solved numerically by using the finite-difference method, finite element method, pseudo spectral method and Adomian decomposition method [10-18].

Recently, the theory of radial basis functions (RBFs) has enjoyed great success as a scattered data interpolating technique. A radial basis function, $\phi(x - x_j) = \phi(\|x - x_j\|)$ is a continuous spline which depends upon the separation distances of a subset of data centers, $X \subset \mathbb{R}^n$, $\{x_j \in X, j = 1, 2, \dots, N\}$. Due to spherical symmetry about the centers x_j , the RBFs are called radial. The distances, $\|x - x_j\|$, are usually taken to be the Euclidean metric.

The method of lines (MOL) [19] is a general procedure for the solution of time dependent partial differential equations (PDEs). The basic idea of the MOL is to replace the spatial (boundary value) derivatives in the PDEs with algebraic approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, only the initial value variable, typically time in a physical problem, remains. In other words, a system of ODEs that approximate the original PDE is obtained. Now any integration algorithm can be applied for initial value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well established, numerical methods for ODEs. Recently, G. Fasshauer [20] developed a meshless method of lines by connecting RBF method with Pseudo-spectral method. Recently, this approach has been applied for various types of PDEs [21-24].

RBF meshless method of lines has achieved a very high accuracy in solving partial differential equations with very easy implementation [25-27]. This method has an edge over finite element, finite difference and finite volume methods, because it does not require a mesh in the problem domain. Derivatives of the RBFs are used to approximate spatial derivatives using a set of nodes scattered in the problem domain. However, the interpolation matrix obtained in this technique becomes highly ill-conditioned when the number of nodes increases [28]. Several techniques have been used to overcome this drawback [29,30].

In this work, MOL is applied with RBFs approximation method, using the multiquadric radial basis function (MQ) $\{(r^2 + c^2)^{1/2}\}$, where c is a shape parameter, for the numerical solution of sine-Gordon equation given by;

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - \sin u(x,t), \quad x \in [a,b] \times [0,T] \quad (1)$$

with the initial and boundary conditions;

$$u(x,0) = u_1(x), \quad u_t(x,0) = u_2(x) \quad (2)$$

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t). \quad (3)$$

The sine-Gordon equation is transformed into coupled equations, given by;

$$\frac{\partial u(x,t)}{\partial t} = v(x,t), \quad \frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \sin u(x,t), \quad x \in [a,b] \times [0,T]. \quad (4)$$

The rest of the paper is organized as follows: in Section 2, the meshless method of lines using RBFs is discussed. Section 3 is devoted to the numerical tests of the method on the problems related to the sine-Gordon equation. In Section 4, the results are concluded.

RBF meshless method of lines for sine-Gordon equation

Consider the transformed form of the sine-Gordon equation as coupled equations;

$$u_t(x,t) = v(x,t), \quad v_t(x,t) = u_{xx}(x,t) - \sin u(x,t), \quad (5)$$

with boundary conditions;

$$u(a,t) = f_1(t), \quad u(b,t) = f_2(t), \quad v(a,t) = g_1(t), \quad v(b,t) = g_2(t), \quad t > 0, \quad (6)$$

and initial conditions;

$$u(x,0) = f(x), \quad v(x,0) = g(x), \quad a \leq x \leq b. \quad (7)$$

For a given set of N collocation points $\{x_i\}_{i=1}^N$ in the domain $[a,b]$, the RBF approximation for u and v of Eq. (5) are given by;

$$U(x) = \sum_{j=1}^N \lambda_{1j} \phi_j(r), \quad V(x) = \sum_{j=1}^N \lambda_{2j} \phi_j(r), \quad (8)$$

where $\{\lambda_j\}_{j=1}^N$ are the unknown constants to be determined, $\phi_j(r) = \phi(\|x - x_j\|)$ can be any well known radial basis function and $r_j = \|x - x_j\|$ is the Euclidean norm between points x and x_j . Here the multiquadric $\phi(r) = \sqrt{r^2 + c^2}$ radial basis function is used. Now, for each node $x_i, i = 1, 2, 3, \dots, N$ in the domain $[a, b]$, Eq. (8) can be written as;

$$\mathbf{U} = \mathbf{A}\boldsymbol{\lambda}_1, \quad \mathbf{V} = \mathbf{A}\boldsymbol{\lambda}_2, \quad (9)$$

where

$$\mathbf{U} = [U(x_1), U(x_2), U(x_3), \dots, U(x_N)]^T, \quad \boldsymbol{\lambda}_i = [\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \dots, \lambda_{iN}]^T, \quad i = 1, 2,$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\Phi}^T(x_1) \\ \boldsymbol{\Phi}^T(x_2) \\ \dots \\ \boldsymbol{\Phi}^T(x_N) \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \dots & \phi_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \dots & \phi_N(x_N) \end{bmatrix},$$

$$\boldsymbol{\Phi}^T(x_i) = [\phi_1(x_i), \phi_2(x_i), \phi_3(x_i), \dots, \phi_N(x_i)], \quad i = 1, 2, 3, \dots, N.$$

Eq. (8) is written as;

$$U(x) = \boldsymbol{\Phi}^T(x) \mathbf{A}^{-1} \mathbf{U} = \mathbf{D}(x) \mathbf{U}, \quad V(x) = \boldsymbol{\Phi}^T(x) \mathbf{A}^{-1} \mathbf{V} = \mathbf{D}(x) \mathbf{V}, \quad (10)$$

where

$$\mathbf{D}(x) = \boldsymbol{\Phi}^T(x) \mathbf{A}^{-1} = [D_1(x), D_2(x), \dots, D_N(x)].$$

Using the approximation $U_i(t)$ and $V_i(t)$ of the solution $u(x_i, t)$ and $v(x_i, t)$ given in Eq. (9), Eq. (5) at each node $x_i, i = 1, 2, 3, \dots, N$ can be written as;

$$\frac{dU_i}{dt} = V_i, \quad \frac{dV_i}{dt} = D_{xx}(x_i)U - \sin(U_i) \quad (11)$$

where

$$\mathbf{D}_x(\mathbf{x}_i) = [D_{1x}(x_i), D_{2x}(x_i), \dots, D_{Nx}(x_i)], \quad D_{jx}(x_i) = \frac{\partial}{\partial x} D_j(x_i), \quad j = 1, 2, \dots, N,$$

$$\mathbf{D}_{xx}(\mathbf{x}_i) = [D_{1xx}(x_i), D_{2xx}(x_i), \dots, D_{Nxx}(x_i)], \quad D_{jxx}(x_i) = \frac{\partial^2}{\partial x^2} D_j(x_i), \quad j = 1, 2, \dots, N.$$

In a more compact form, Eq. (11) can be written as;

$$\frac{d\mathbf{U}}{dt} = \mathbf{V}, \quad \frac{d\mathbf{V}}{dt} = \mathbf{D}_{xx} \mathbf{U} - \sin(\mathbf{U}), \quad (12)$$

where

$$\mathbf{D}_x = \left[D_{jx}(x_i) \right]_{i,j=1}^N, \quad \mathbf{D}_{xx} = \left[D_{jxx}(x_i) \right]_{i,j=1}^N.$$

Eq. (12) is written as;

$$\frac{d\mathbf{U}}{dt} = F_1(\mathbf{U}, \mathbf{V}), \quad \frac{d\mathbf{V}}{dt} = F_2(\mathbf{U}, \mathbf{V}), \quad (13)$$

where

$$F_1(\mathbf{U}, \mathbf{V}) = \mathbf{V}, \quad F_2(\mathbf{U}, \mathbf{V}) = \mathbf{D}_{xx} \mathbf{U} - \sin(\mathbf{U}).$$

Now the system of ODEs (13) is solved by using Fourth-order Runge-Kutta scheme;

$$\begin{aligned} K_1^n &= F_1(U^n, V^n), \quad J_1^n = F_2(U^n, V^n), \\ K_2^n &= F_1(U^n + (\delta t/2)K_1^n, V^n + (\delta t/2)J_1^n), \quad J_2^n = F_2(U^n + (\delta t/2)K_1^n, V^n + (\delta t/2)J_1^n), \\ K_3^n &= F_1(U^n + (\delta t/2)K_2^n, V^n + (\delta t/2)J_2^n), \quad J_3^n = F_2(U^n + (\delta t/2)K_2^n, V^n + (\delta t/2)J_2^n), \\ K_4^n &= F_1(U^n + \delta t K_3^n, V^n + \delta t J_3^n), \quad J_4^n = F_2(U^n + \delta t K_3^n, V^n + \delta t J_3^n), \\ U^{n+1} &= U^n + \frac{\delta t}{6} (K_1^n + 2K_2^n + 2K_3^n + K_4^n), \quad V^{n+1} = V^n + \frac{\delta t}{6} (J_1^n + 2J_2^n + 2J_3^n + J_4^n). \end{aligned}$$

Numerical examples

In this section, the proposed method for the numerical solution of sine-Gordon equations is applied. The accuracy of the meshless method of lines is tested in terms of L_2 , L_∞ error norms and the conservation of energy of the sine-Gordon equation. These error norms and energy are defined as;

$$L_2 \|u - U\|_2 = \left[\delta x \sum_{j=1}^N (u - U)^2 \right]^{1/2}, \quad (14)$$

$$L_\infty \|u - U\|_\infty = \max_j |u - U|, \quad (15)$$

$$E = 1/2 \int_{-\infty}^{\infty} \left[(u_t)^2 + (u_x)^2 + 2(1 - \cos u) \right] dx, \quad (16)$$

where U and u denote the numerical and exact solution respectively. The test problems are given below.

Problem 1. Single soliton:

Sine-Gordon Eq. (1) is considered as a system of 2 equations;

$$u_t = v, \quad v_t = u_{xx} - \sin u. \quad (17)$$

The exact solutions of the Eq. (17) are given as;

$$u(x, t) = 4 \tan^{-1} \left(\exp[\gamma(x - Ct) + \beta] \right), \quad v(x, t) = \frac{-4\gamma C \exp[\gamma(x - Ct) + \beta]}{1 + \exp[\gamma(x - Ct) + \beta]^2}, \quad (18)$$

where $\gamma = (1 - C^2)^{-1/2}$. The problem is solved in the spatial interval $-2 \leq x \leq 58$, by the present method using MQ radial basis functions along with RK4 time integration scheme, at times $t = 0, 9, 36, 108$, when time step size $\delta t = 0.001$, number of collocation points $N = 121$ and MQ shape parameter $c = 1$, $C = 0.5$, $\beta = 0$. The L_∞ , L_2 error norms and the conserved quantity $E(t)$ are given in **Table 1**. In **Figure 1**, the numerical solutions, along with exact solutions at times $t = 0, 9, 36, 108$, are shown. It is observed that the energy $E(t)$ is almost constant, which confirms the accuracy of the meshless method. The results are compared with the results in [10]. The results when MQ is used have a good agreement with the exact solution and is better than in earlier work [10]. It is observed that the solution converges when $0.2 < c < 4.4$.

Table 1 Error norms and energy constant for single soliton using MQ when $\delta t = 0.001$, $N = 121$, $c = 2$, $C = 0.5$, $\beta = 0$, $E(0) = 9.147397$ in $[-2, 58]$ corresponding to Problem 1.

t	L_∞ (RK4)	$ E(t) - E(0) $ (RK4)	L_∞ [10]	$ E(t) - E(0) $ [10]
9	6.296E-004	0.000159	4.518E-001	0.000321
36	8.109E-004	0.000162	2.293E+000	0.000297
108	2.568E-003	0.000199	5.120E+000	0.000280

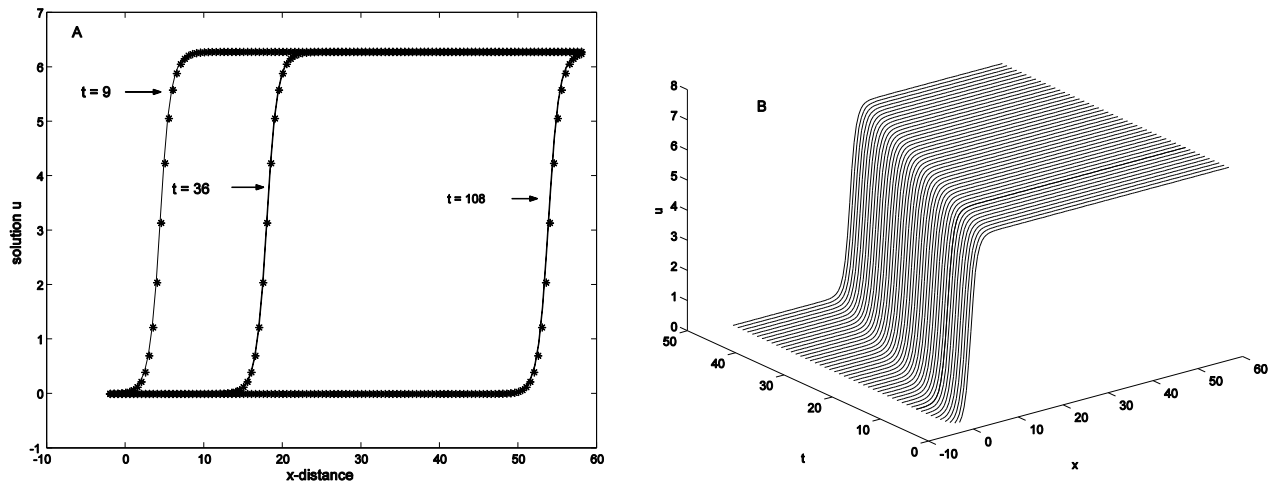


Figure 1 Single soliton: (A) the dot curves represent the exact solution, while the full curve shows the numerical solution at times $t = 9, 36, 108$ and (B) the numerical solution is shown over the time interval $[1, 48]$, when $C = 0.5$, $\beta = 0$, $\delta t = 0.001$, $c = 2$, $N = 121$ corresponding to Problem 1.

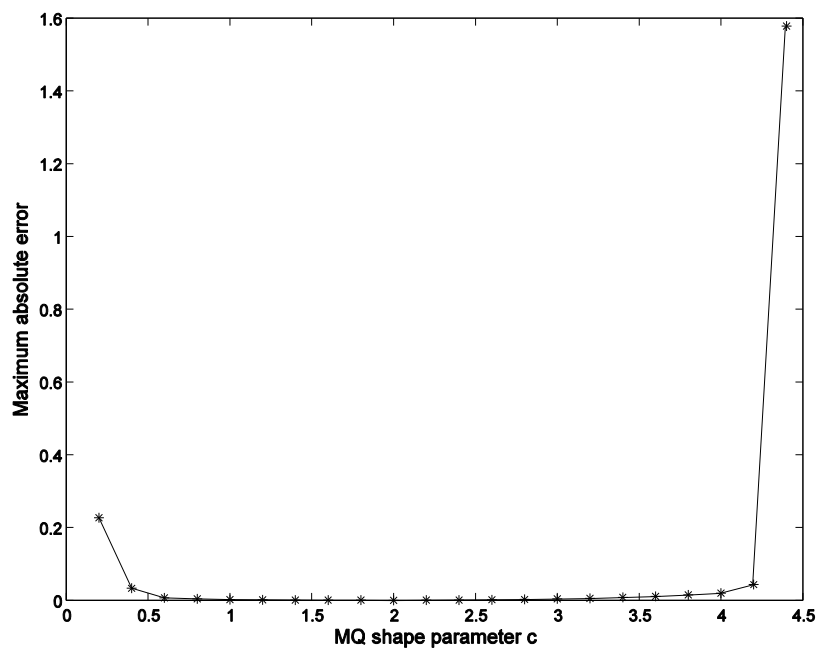


Figure 2 Single soliton: plots show maximum absolute error versus MQ shape parameter C , at time $t = 1$, when $C = 0.5$, $\beta = 0$, $N = 121$ corresponding to Problem 1.

Problem 2. Soliton-antisoliton breather:

Sine-Gordon Eq. (1) is considered as a system of 2 equations;

$$u_t = v, \quad v_t = u_{xx} - \sin u. \quad (19)$$

The exact solutions of a breather (or bion), which is a bound kink-antikink pair, and the energy, are given by;

$$u(x, t) = 4 \tan^{-1} \left[C^{-1} \sin(\bar{\gamma} Ct) \operatorname{sech}(\bar{\gamma} x) \right],$$

$$v(x, t) = \frac{4 \bar{\gamma} \cos(\bar{\gamma} Ct) \operatorname{sech}(\bar{\gamma} x)}{1 + \left[C^{-1} \sin(\bar{\gamma} Ct) \operatorname{sech}(\bar{\gamma} x) \right]^2}, \quad (20)$$

where $\bar{\gamma} = (1 + C^2)^{-1/2}$, $E = 16\bar{\gamma}$. This problem is solved in the spatial interval $[-10, 10]$, using RK4 scheme, when time step size $\delta t = 0.001$, number of collocation points $N = 201$, MQ shape parameter $c = 0.5$, and parameter $C = 0.5$, $\beta = 0$ are used. In **Figure 3(a) - 3(c)**, the numerical solutions, along with exact solutions at times $t = 1, 10, 30$. are shown. In **Figure 3(d)**, the numerical solution over the time interval $[1, 30]$ is shown. It is observed that the energy $E(t)$ is almost constant, showing the accuracy of the meshless method. The results are compared with the results in [10], which have a good agreement with the exact solution and the earlier work by [10], as shown in **Table 2**.

Table 2 Error norms and energy constant for single soliton when $\delta t = 0.0001$, $N = 201$, $c = 0.5$, $C = 0.5$, $\beta = 0$ in $[-10, 10]$ corresponding to Problem 2.

t	L_∞ (RK4)	$ E(t) - E(0) $ (RK4)	L_∞ [10]	$ E(t) - E(0) $ [10]
1	2.485E-008	0.000001	0.988E-003	0.000369
10	3.145E-007	0.000004	0.1622E-002	0.003498
20	1.875E-006	0.000001	0.103E-002	0.007407

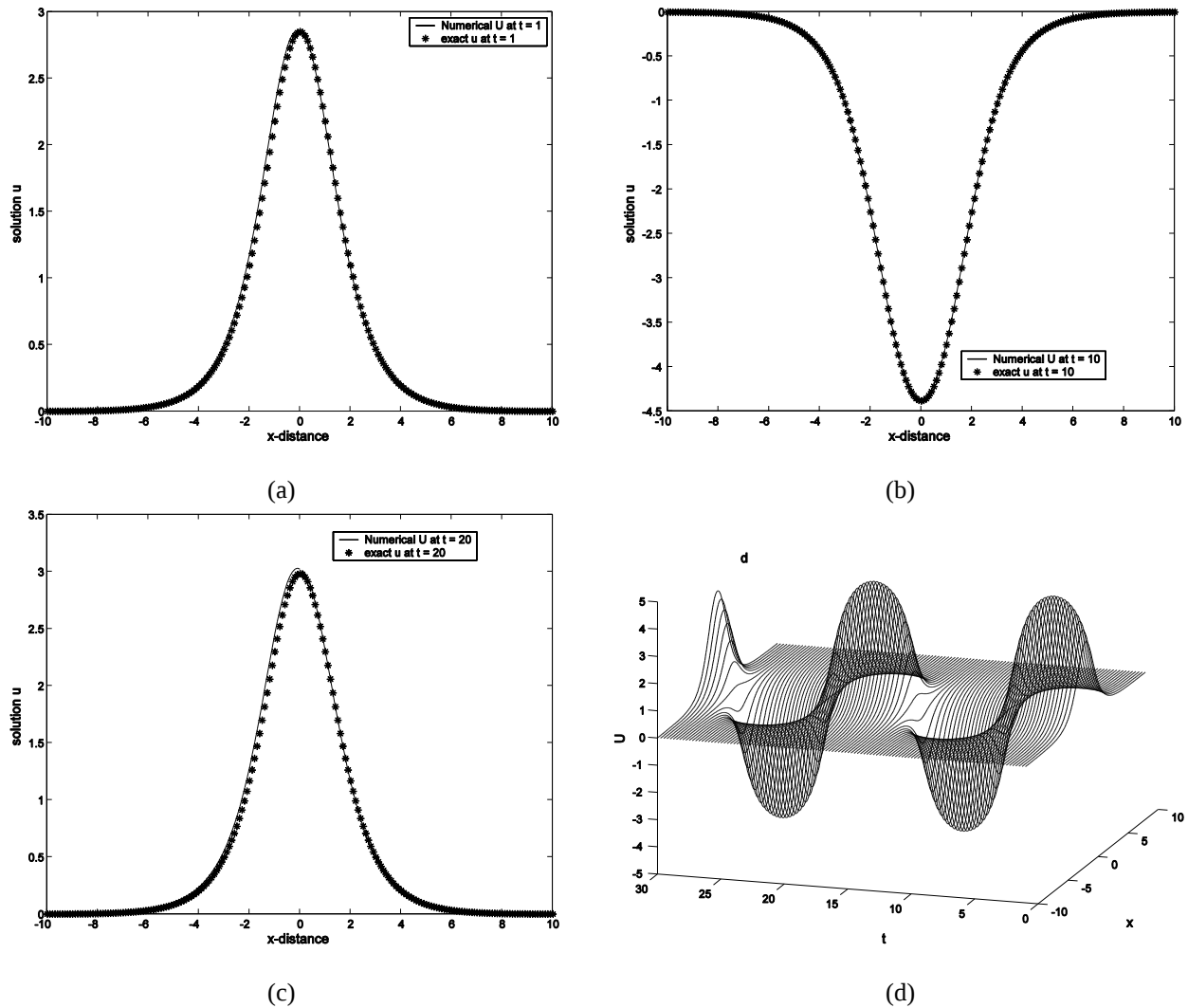


Figure 3 Soliton-antisoliton breather: (a), (b), (c) the dot curves represent the exact solution, while the full curve represents the numerical solution at times $t = 1, 10, 20$ and (d) the numerical solution is shown over the time interval $[1, 30]$ for $C = 0.5$, $\beta = 0$, $\delta t = 0.001$, $N = 201$, corresponding to Problem 2.

Problem 3. Soliton-soliton doublets:

Sine-Gordon Eq. (1) is considered as a system of 2 equations;

$$u_t = v, \quad v_t = u_{xx} - \sin u. \quad (21)$$

The exact solutions, and the energy, are given by;

$$\begin{aligned} u(x, t) &= 4 \tan^{-1} [C \operatorname{sech}(\gamma Ct) \sinh(\gamma x)], \\ v(x, t) &= \frac{-4C^2 \gamma \operatorname{sech}(\gamma Ct) \tanh(\gamma Ct) \sinh(\gamma x)}{1 + [C \operatorname{sech}(\gamma Ct) \sinh(\gamma x)]^2}, \end{aligned} \quad (22)$$

where $\gamma = (1 - C^2)^{-1/2}$, $E = 16\gamma$. The problem is solved in $[-20, 20]$. The L_∞ , L_2 error norms and the conserved quantity $E(t)$ using MQ, when time step size $\delta t = 0.001$, number of collocation points $N = 121$, MQ shape parameter $c = 1$, and parameter $C = 0.5$, $\beta = 0$ in the spatial interval $[-20, 20]$, are shown in **Table 3**. In **Figure 4 (A) - 4(C)**, the numerical solutions, along with exact solutions at times $t = 2, 10, 20$, are shown. In **Figure 3(C)**, the numerical solution over the time interval $[1, 30]$ is shown. It is clear that the energy $E(t)$ is almost constant, showing the accuracy of the meshless method of lines. The results are compared with the results in [10]. It is noted that the results of the present method are in good agreement with the exact solution and the earlier work by [10], as shown in **Table 3**.

Table 3 Error norms and energy constant for single soliton when $\delta t = 0.0001$, $N = 201$, $c = 0.5$, $C = 0.5$, $\beta = 0$ in $[-20, 20]$ corresponding to problem 3.

t	L_∞ (RK4)	$ E(t) - E(0) $ (RK4)	L_∞ [10]	$ E(t) - E(0) $ [10]
2	4.006E-006	0.000003	0.127E-003	0.000344
10	2.152E-005	0.000005	0.191E-003	0
20	4.352E-005	0.000005	0.251E-003	0

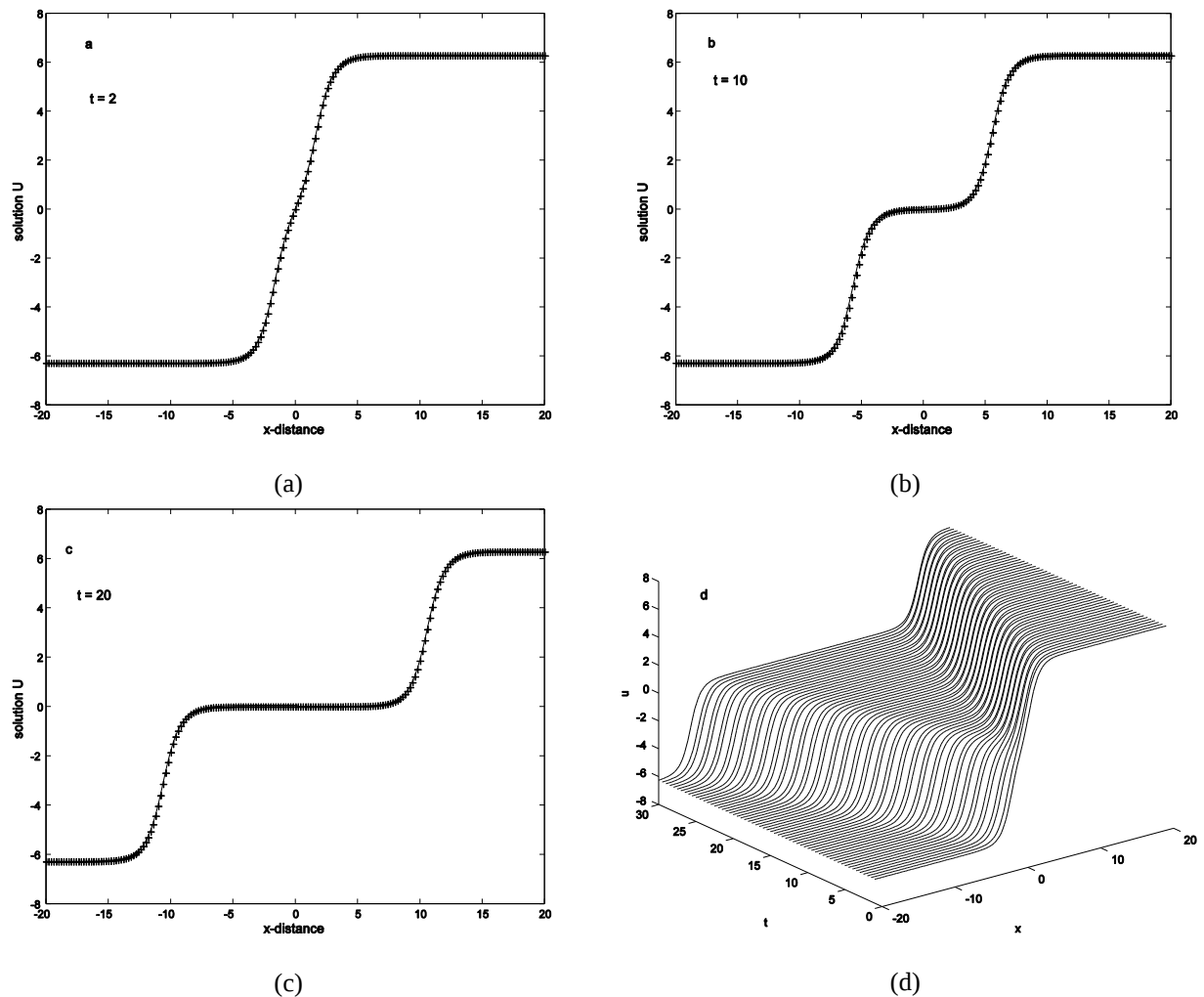


Figure 4 Soliton-soliton doublets: (a), (b), (c) the dot curves represent the exact solution, while the full curve represents the numerical solution at times $t = 1, 10, 20$ and (d) the numerical solution is shown over the time interval $[1, 30]$ when $C = 0.5$, $\beta = 0$, $\delta t = 0.001$, $N = 201$, corresponding to Problem 3.

Concluding remarks

In this paper, a meshless method of lines using radial basis functions is applied for the numerical solution of sine-Gordon equations. The results of this method are in good agreement with the exact solution and with earlier work [10]. As a whole, the present method produces better results with an ease of implementation. The technique used in this paper provides an efficient alternative for the solution of time-dependent nonlinear partial differential equations. From application viewpoints, the implementation of this method is very simple and straightforward.

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