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Article

An Accurate Solution for the Steady Flow of Third-Grade Fluid in a Porous Half Space

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Abstract

In this paper, a simple and efficient analysis technique, Hankel-Padé method, is proposed to find the solution of a nonlinear ordinary differential equation that appears in the study of the steady flow of the third-grade fluid in a porous half space. The numerical solutions for some cases of the model's parameters are obtained by using the Hankel-Padé method. The convergence of the Hankel sequences is analyzed. Comparison to other available results for this problem reveals reliability and high accuracy of the proposed technique. Also the simple analytical expressions of the solutions of the governing non-linear boundary-layer problem are developed as rational approximation solutions. The effectivity and convergence of the rational approximation solutions are investigated by illustrative graphs and tables.

Keywords: Third-grade fluid, porous half space, analytical solution, Padé approximation, Hankel-Padé method

Introduction

In recent years, the interest in non-Newtonian fluids has increased due to their applications in industry and technology. Many materials such as polymer solutions or melts, certain oils and greases, and many other emulsions are classified as non-Newtonian fluids.

It is well known that the governing equations for problems dealing with non-Newtonian fluids are of higher order and nonlinearity than the Navier-Stokes equations [1]. Thus, finding analytic solutions of such equations is not an easy task.

Recently many authors have used semi-analytical methods for investigating some strong nonlinear ordinary or partial differential equations, for example see [2-4] for homotopy and variational methods. In these methods we can find a good approximation for missing value of the initial value or boundary condition.

Some time ago, Fernández *et al.* [5-7]. developed the Riccati-Padé method for the accurate calculation of resonances and bound states of the Schrödinger equation. The relatively new method consists of the expansion of a modified logarithmic derivative of the eigenfunction in a Taylor series about the origin. This method consists simply on the removal of any singularity at the origin so that the resulting function is analytic at that point. They constructed a sequence of roots of the Hankel determinants from the coefficients of that expansion, which converge towards the physical eigenvalues. This approach is also called the Hankel-Padé method [5-15]. Very recently, Abbasbandy and Hayat [16] obtained the unknown skin friction coefficient of the boundary layer Falkner-Skan equation for a wedge by using the Hankel-Padé method and they demonstrated the efficiency of this method in magnetohydrodynamics (MHD) flow problems.

The Hankel-Padé method has proved successful in treating some two-point nonlinear equations of physics [17-19]. In this work, we present an accurate solution of the nonlinear ordinary differential equation that appears in the study of the steady flow of the third-grade fluid in a porous half space using the Hankel-Padé method.

Materials and methods

The model problem

In [20,21], Hayat *et al.* generalized the relation for a second-grade fluid to the modified Darcy's Law, and modelled the flow of a third-grade fluid in a porous half space. Following the formulation given in [20-22], we consider the following modified Darcy's Law for a third-grade fluid

$$(\nabla p)_x = -\frac{\varphi}{k}[\mu u + \alpha_1 \frac{\partial u}{\partial t} + 2\,\beta_3 (\frac{\partial u}{\partial y})^2 u],$$

where u, p and μ denote the fluid velocity, the pressure and the dynamic viscosity, respectively. α_1 and β_3 are material constants and k and φ, respectively represent the permeability and porosity of the porous half space which occupies the region $y > 0$. Defining non dimensional fluid velocity f and the coordinate z

$$z = \frac{V_0}{\nu} y, \quad f(z) = \frac{u}{V_0},$$

where $V_0 = u(0)$ and $\nu = \mu/\rho$ (ρ is the fluid density). So, the boundary value problem modelling the steady state flow of a third-grade fluid in a porous half space becomes:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2} + b_1 (\frac{\mathrm{d}f}{\mathrm{d}\tau})^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2} - b_2 f (\frac{\mathrm{d}f}{\mathrm{d}\tau})^2 - cf = 0, \qquad (1)$$

with boundary conditions

$$f(0) = 1, \quad f(+\infty) = 0, \qquad (2)$$

where for parameters appearing in Eq. (1), we have

$$b_1 = \frac{6\beta_3 V_0^4}{\mu \nu^2}, \quad b_2 = \frac{2\beta_3 \varphi V_0^2}{k\,\mu} \quad \text{and} \quad c = \frac{\varphi \nu^2}{k V_0^2}$$

Note that the parameters are not independent, since $b_2 = \frac{b_1 c}{3}$.

Now the interest is to find the solution of the two-point boundary value problem Eq. (1) with boundary condition Eq. (2). In [20], Hayat *et al.* employed the homotopy analysis method (HAM) [23,24] for solving Eq. (1) and obtained an analytical series solution for $f(\tau)$ in the following form:

$$\sum_{m=0}^{M} f_m(\tau) = \sum_{n=1}^{2M+1} e^{-n\tau} \sum_{m=n-1}^{2m+1-2n} a_{m,n}^k \tau^k .$$

Since $a_{m,n}^k$ itself is a series, the above solution contains many terms that are not easily computable for large value of M (see [22]). Recently, Ahmad [22] obtained a simple analytical solution for Eq. (1) in the following form:

$$f(\tau) = a_1 e^{-\sqrt{c}\,\tau} + a_3 e^{-3\sqrt{c}\,\tau} + a_5 e^{-5\sqrt{c}\,\tau}, \qquad (3)$$

where a_1, a_3 and a_5 all depend on the problem's parameters, b_1, b_2 and c, or in general just on b_2.

Hankel-Padé method

A direct solution of Eq. (1) with boundary conditions Eq. (2) can be obtained by a shooting method using the Runge-Kutta algorithm or other iterative numerical methods, see for example [25]. The problem is that we have to make an initial guess for the value of $f'(0)$ to start the shooting method or other methods and this guess is very important to obtain a good solution. The Hankel-Padé method can be used in calculating $f'(0)$. For this purpose, based on the Hankel-Padé approach, we expand the solution $f(\tau)$ of the nonlinear ordinary differential equation Eq. (1) as a Taylor's series about $\tau = 0$:

$$f(\tau) = \sum_{j=0}^{\infty} f_j \tau^j, \qquad (4)$$

where the every coefficient f_j depends on the only unknown one $f_1 = f'(0)$. On substitution of the series Eq. (4) into Eq. (1) we can calculate the coefficients f_j. The first three terms of f_j are:

$$f_0 = 1, \quad f_2 = \frac{1}{2}\frac{c + b_2 f_1^2}{1 + b_1 f_1^2},$$
$$f_3 = -\frac{1}{6}\frac{f_1\left(-c - 2c\,b_1 f_1^2 - c b_1^2 f_1^4 - 2b_2 c + 2b_1 c b_2 f_1^2 - 2b_2^2 f_1^2 - b_2 f_1^2 - 2b_2 f_1^4 b_1 - b_2 f_1^6 b_1^2 + 2b_1 c^2\right)}{1 + 3\,b_1 f_1^2 + 3\,b_1^2 f_1^4 + b_1^3 f_1^6}. \tag{5}$$

The Hankel-Padé method is based on the transformation of the power series Eq. (4) into a rational function or Padé approximant

$$[S/N](\tau) = \frac{\sum_{j=0}^{S} a_j\,\tau^j}{\sum_{j=0}^{N} b_j\,\tau^j}, \tag{6}$$

where $b_0 = 1$, and in order to have the correct limit at infinity according to the boundary conditions Eq. (2), one would expect that $N > S$. However, in order to obtain an accurate value of f_1, it is more convenient to choose $S = N + d$, $d = 1,2,\ldots$ (for more details see [14-16,26]), so with this assumption the rational function Eq. (6) has $2N + d + 1$ coefficients that we may choose. If $[S/N](\tau)$ is exactly a Padé approximant of a Taylor's series $f(\tau)$, then $f(\tau) - [S/N](\tau) = O(\tau^{2N+d+1})$, but in this special case, since every coefficient f_j depends on f_1, the coefficient f_1 remains undetermined. If we require that $f(\tau) - [S/N](\tau) = O(\tau^{2N+d+2})$, we have another equation from which f_1 can be obtained.

Under such conditions, the coefficients a_j and b_j satisfy:

$$\sum_{i=0}^{j} b_i f_{j-i} = a_j, \quad j = 0,\ldots,N+d, \tag{7}$$

$$\sum_{i=0}^{j} b_i f_{j-i} = 0, \quad j = N+d+1,\ldots,2N+d+1, \tag{8}$$

where $b_j = 0$ if $j > N$. The coefficients $b_0, b_1,\ldots, b_N$ cannot satisfy the $N + 1$ linear homogeneous equations Eq. (8) unless

$$|f_{i+j+d+1}|_{i,j=0,\ldots,N} = \det\begin{pmatrix} f_{d+1} & f_{d+2} & \cdots & f_{d+N+1} \\ f_{d+2} & f_{d+3} & \cdots & f_{d+N+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{d+N+1} & f_{d+N+2} & \cdots & f_{d+2N+1} \end{pmatrix} = 0.$$

The above matrix is a Hankel matrix. In this paper we show the above Hankel determinant by H_D^d, so

$$H_D^d = |f_{i+j+d+1}|_{i,j=0,\ldots,N} = 0, \tag{9}$$

where $D = N + 1$ is the dimension of the Hankel matrix. Each Hankel determinant Eq. (9) is a rational function of f_1 and we expect that there is a sequence of roots $f_1^{[D,d]}$, D = 2, 3,…, that converges towards the actual value of $f'(0)$. In our case, since f_2 is the first nonzero coefficient that depends on f_1, we choose Hankel sequences with $d \geq 1$. The Hankel determinant exhibits many roots and their number increases with D. If we compare the roots of two successive sequences, we can identify the sequence of roots $f_1^{[D,d]}$ that converges towards the actual value of f_1. This identification is not always easy, due to the accumulation of several approximate roots for large D, and in this case other numerical methods may help in choosing the best root. For this purpose, we find it convenient to solve Hankel determinant $H_D^d = 0$ analytically for very small values of D (say D = 2) by using a symbolic software's program such as Maple, then we can use proper initial guesses for a search for the root of the $H_{D+1}^{d}=0$ by means of a straightforward Newton-Raphson algorithm.

Results and discussion

In this section, we obtain the analytical solution of the problem Eq. (1) with some typical values of parameters. For this purpose, in the first case, we consider the problem Eq. (1) with b_1 = 0.6, b_2 = 0.1 and c = 0.5, and also in the second case, we consider the problem Eq. (1) with b_1 = 0.2, b_2 = 0.1 and c = 1.5. Because, our purpose is to present a simple method which gives results more accurate than obtained results by other elaborate methods for this problem, so for comparison we should choose the same model parameters.

It is evident that the main problem in the two cases is to obtain the value of the $f'(0)$, because if we have this unknown value in the problem, then we can resort to any numerical integration routine to obtain the solution of the problem. For this purpose, we employ the Hankel-Padé method as presented in the previous section.

For the first case, we compute the coefficients f_j from Eq. (5), then calculate the roots of the Hankel determinant Eq. (9) for d = 1 and d = 2. Comparing the roots of H_{D}^{d} with those of H_{D-1}^{d}, we identify the sequence $f_1^{[D,d]}$. In **Table 1** the values of Hankel sequences with d = 1 and d = 2 for the first case are presented. For this case, the value of $f'(0)$, correct to six decimal positions, obtained by the shooting method is $f'(0)=-0.678301$ [22]. Also Ahmad [22] computed $f'(0)=-0.681835$ by an approximate expression to that presented in Eq.(3), and is correct only in first decimal digit. Note that Hayat *et al.* [20] only showed their obtained results through figures, therefore their result for $f'(0)$ is not available for comparison. From **Table 1** it is evident that the Hankel sequences converge rapidly towards the accurate result (compared with the result obtained by the shooting method). In **Figure 1** the values of $\Delta=\log|f_1^{[D,d]}-f_1^{[D-1,d]}|$, for $D=4,\ldots,40$ and $d=1,2$ are plotted that shows convergence of the sequence of $f_1^{[D,d]}$. From the results of the Hankel sequence for $D\leq 40$ presented in **Table 1** and **Figure 1**, we believe that the results in the last row of **Table 1** for $f'(0)$ are accurate at least to nine decimal places.

Table 1 Numerical values of $f'(0)$ for $b_1 = 0.6$, $b_2 = 0.1$ and $c = 0.5$.

D	d = 1	d = 2
4	−0.649973739	−0.71787347
6	−0.672114711	−0.688180815
8	−0.676701313	−0.682386744
10	−0.677873693	−0.682566968
12	−0.678200374	−0.676563957
14	−0.678279559	−0.741649743
16	−0.678296946	−0.678411311
18	−0.678300929	−0.678311994
20	−0.678301503	−0.678304025
22	−0.678301619	−0.678302367
24	−0.678301613	−0.678301875
26	−0.678301603	−0.678301705
28	−0.678301604	−0.67830165
30	−0.678301609	−0.678301633
32	−0.678301615	−0.678301616
34	−0.678301618	−0.678301623
36	−0.67830162	−0.678301621
38	−0.67830162	−0.67830162
40	−0.67830162	−0.678301619

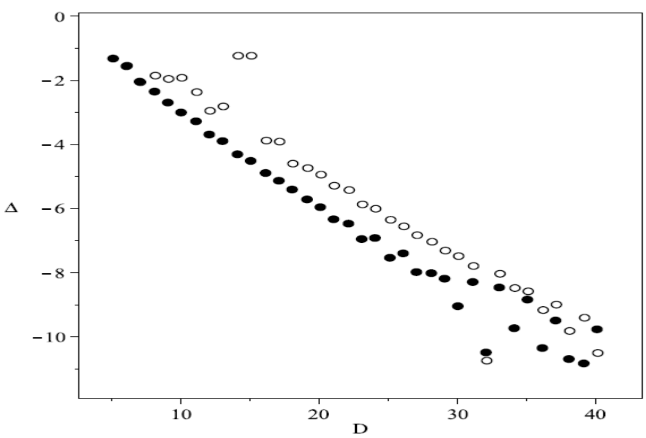

Figure 1 $\Delta = \log|f_1^{[D,d]} - f_1^{[D-1,d]}|$; filled circles for $d = 1$, and circles for $d = 2$, for the first case with $b_1 = 0.6$, $b_2 = 0.1$ and $c = 0.5$.

Now, we consider the second case. Similar to the first case, we construct the sequences $f_1^{[D,d]}$ for the presented problem. In **Table 2** the values of Hankel sequences with d = 1 and d = 2 are presented. In **Figure 2** the values of $\Delta = \log |f_1^{[D,d]} - f_1^{[D-1,d]}|$, for D = 5,…,40 and d = 1,2 are plotted that shows convergence of the sequence of $f_1^{[D,d]}$. We are not aware of any result of such accuracy in the literature for this case with which we can compare our results. But from the results of the Hankel sequence presented in **Table 2** and **Figure 2**, we believe that the results in last row of **Table 2** of $f'(0)$ are accurate at least to eight decimal digits.

Table 2 Numerical values of $f'(0)$ for b_1 = 0.2, b_2 = 0.1 and c = 1.5.

D	d = 1	d = 2
4	−1.125787539	−1.243393323
6	−1.164136828	−1.191964136
8	−1.172081055	−1.181928511
10	−1.174111677	−1.182240669
12	−1.174677505	−1.171843148
14	−1.174814658	−1.284575037
16	−1.174844773	−1.17504286
18	−1.174851325	−1.174870837
20	−1.174852666	−1.174857034
22	−1.174852867	−1.174854163
24	−1.174852857	−1.17485331
26	−1.174852839	−1.174853016
28	−1.174852841	−1.174852921
30	−1.17485285	−1.174852891
32	−1.174852859	−1.17485288
34	−1.174852865	−1.174852874
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38	−1.174852868	−1.174852868
40	−1.174852868	−1.174852868

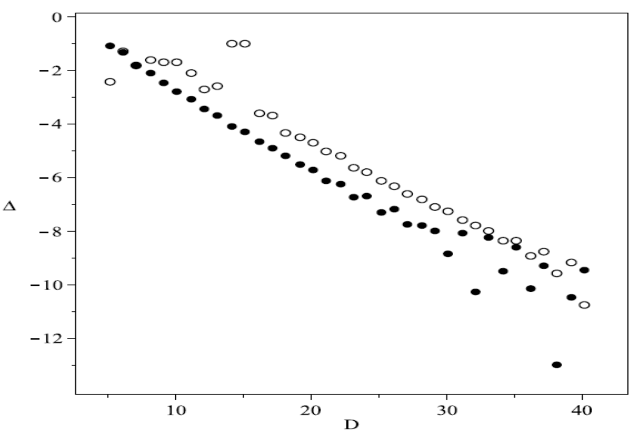

Figure 2 $\Delta = \log|f_1^{[D,d]} - f_1^{[D-1,d]}|$; filled circles for $d = 1$, and circles for $d = 2$, for the first case with $b_1 = 0.2$, $b_2 = 0.1$ and $c = 1.5$.

In this paper, the main purpose is to compute the unknown value $f'(0)$ by a simple method with high accuracy, so after computing the unknown value of $f'(0)$, we can easily obtain a numerical solution for $f(\tau)$, by a shooting method using Runge-Kutta algorithm or other iterative numerical methods. After computing $f'(0)$, we can obtain a simple and straightforward analytical expression for solution of the governing equation from rational approximation Eq. (6), that have a suitable behavior at infinity. In **Figures 3** and **4**, we plot the analytical solutions for $f(\tau)$ obtained from [3/8] Padé approximant against τ for different values of parameters b_1, b_2 and c. From these figures it is evident that the Padé approximant have a correct behavior at infinity according to the boundary condition.

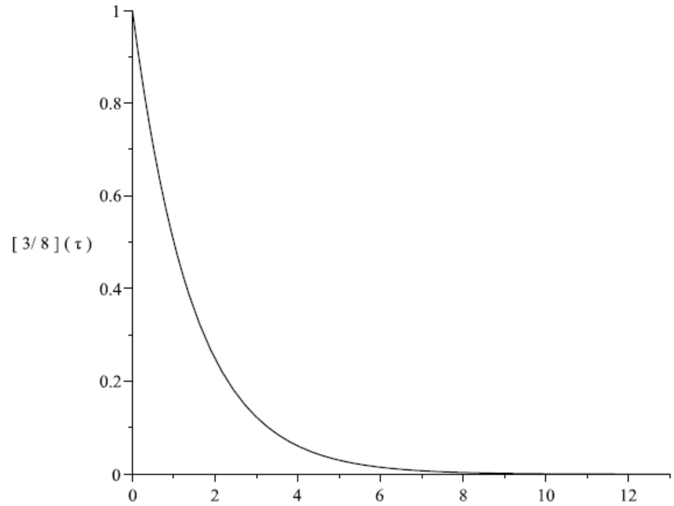

Figure 3 Variation of the velocity distribution for the $b_1 = 0.6$, $b_2 = 0.1$ and $c = 0.5$, with Padé approximation [3/8].

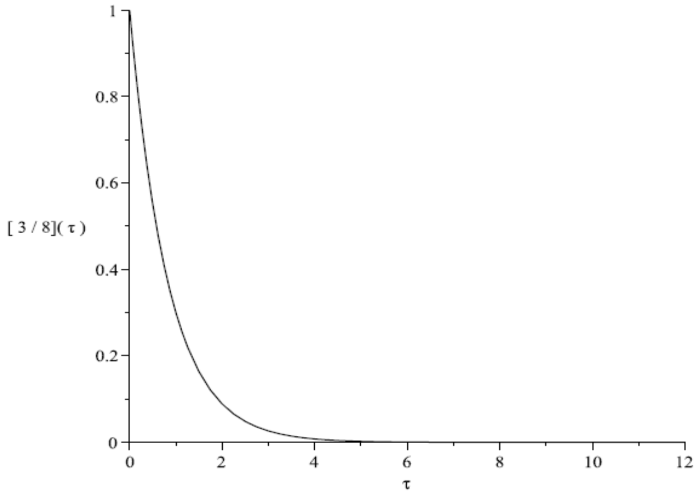

Figure 4 Variation of the velocity distribution for the $b_1 = 0.2$, $b_2 = 0.1$ and $c = 1.5$, with Padé approximation [3/8].

The Padé approximate does not necessarily require any information about the radius of convergence [27,28]. But a Padé approximant can only indicate a singularity by the poles that are the zeros of its denominator. So when a Padé approximant is used for a problem it is important to determine the nature and the location of the singularities. We check that, when we use the obtained $[S/S+d]$ Padé approximate as a solution for the problem Eq. (4), the Padé approximate don't have a positive real pole for many cases of the values *S* and *d*. Therefore this analytical solution describes the solution accurately over the whole domain [0,∞). For this purpose some results are presented in **Table 3**. In this table the values of poles and also the maximum value of residual for some cases are presented. Baker [28] for some similar cases proved that when *S* goes to infinity the Padé approximant converges uniformly to the solution $f(\tau)$. To show this convergence we define the residual function for the problem as follows:

$$RES(f) = \frac{d^2 f}{d\tau^2} + b_1 \left(\frac{df}{d\tau}\right)^2 \frac{d^2 f}{d\tau^2} - b_2 f \left(\frac{df}{d\tau}\right)^2 - cf,$$

and also we define the maximum value of the residual function as,

$$\| RES(f(\tau)) \|_\infty = \{\max | RES(f(\tau)) | : \tau \in [0,\infty)\}.$$

Table 3 The values of poles and the maximum norms of residual function for different types of the Padé rational approximations.

	Case 1 $b_1 = 0.6$, $b_2 = 0.1$ and $c = 0.5$		**Case 2** $b_1 = 0.2$, $b_2 = 0.1$ and $c = 1.5$	
$[L/N]$	Real-Poles	$\Vert RES([S/N])\Vert_\infty$	Real-Poles	$\Vert RES([S/N])\Vert_\infty$
[3/8]	–	0.00029	–	0.00087
[4/9]	–3.8716	0.00019	–2.2348	0.0006
[5/11]	–5.8923	0.00011	–3.4023	0.0003
[6/11]	–6.1647	0.00007	–3.5586	0.00021
[10/15]	–7.2569	0.000045	–4.1845	0.00013
[15/22]	–	0.000006	–	0.000075
[20/30]	–	0.00000019	–	0.000012

In **Figures 5** and **6** we plot the residual function for some cases of model parameters. From these figures and **Table 3** it is evident that when the values of S and d increase the maximum value of the residual function decreases rapidly, and this shows that the Padé approximant converges uniformly to the exact solution. The theory of Padé approximants and also it's convergence analysis can be found in [27,28].

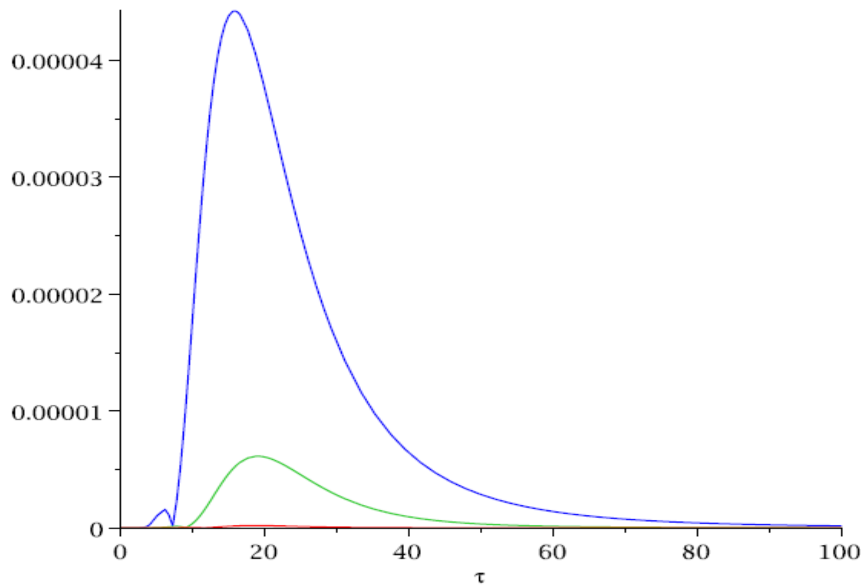

Figure 5 Absolute residual function for the $b_1 = 0.6$, $b_2 = 0.1$ and $c = 0.5$, with Padé approximation [10/15] (blue line), [15/22] (green line), [20/30] (red line).

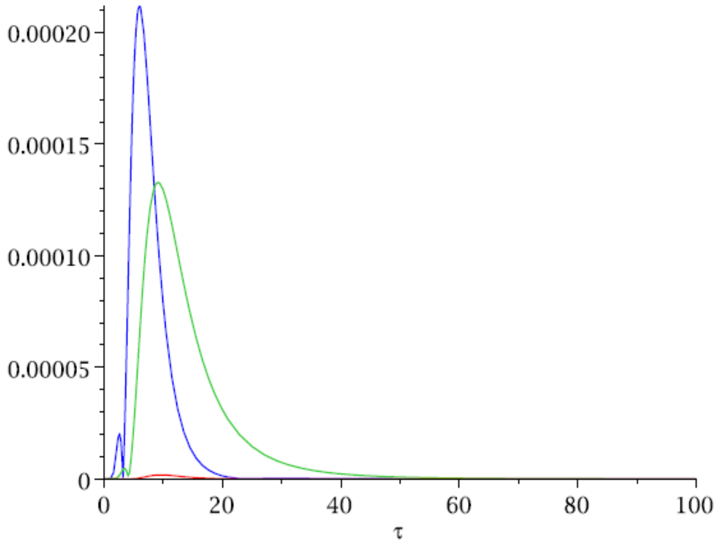

Figure 6 Absolute residual function for the $b_1 = 0.2$, $b_2 = 0.1$ and $c = 1.5$, with Padé approximation [6/11] (blue line), [10/15] (green line), [20/30] (red line).

Conclusion

In this work, a non-linear problem that appears in a model for the steady flow of the third-grade fluid in a porous half space is analyzed. The numerical solutions are given for some typical values of problem's parameters by using the Hankel-Padé method. The obtained results show that the Hankel-Padé method is a simple, efficient and powerful approach which can obtain a value for the unknown parameter appearing in the nonlinear equation for a two-point boundary value problem more accurately than some elaborate methods and can be a suitable alternative for the treatment of the magnetohydrodynamics viscous flow models. Also, a simple accurate analytical solution of the governing problem is derived by the Padé approximant that discusses the physics of the problem very well. Finally, the effectivity and convergence of the rational approximation solutions are investigated.

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