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Three-Step Iterative Methods with Sixth-Order Convergence for Solving Nonlinear Equations

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Abstract

In this paper, we develop new families of sixth-order methods for solving simple zeros of non-linear equations. These methods are constructed such that the convergence is of order six. Each member of the families requires two evaluations of the given function and two of its derivative per iteration. These methods have more advantages than Newton's method and other methods with the same convergence order, as shown in the illustration examples.

Keywords: Iterative methods, Simple-zero of nonlinear equations, Newton's method

Introduction

Newton's method for the calculation of the simple root α i.e., $f(\alpha)=0$ and $f'(\alpha)\neq 0,$ of a nonlinear equation $f(x)=0$ given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{1}$$

is probably the most widely used iterative method of quadratic order, where $f:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable real valued function.

During the last several years numerous papers [1-7], devoted to iterative schemes of different order of convergence, have appeared in various journals. In this paper, we consider the iterative methods to find a simple zero in such a way that new approximations to a zero of $f(x)=0$ is calculated by calculating $f(x)=0$, and possibly its derivatives for a number of values of the independent variable, at each step.

In the literature, some higher order methods have been developed by considering three-step methods in which we use three different points per iteration. Consider the following fifth-order iterative scheme defined in [2], as

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} -\frac{5f'^2(x_n)+3f'^2(y_n)}{f'^2(x_n)+7f'^2(y_n)}.\frac{f(y_n)}{f'(x_n)}, \tag{2}$$

where

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}. \tag{3}$$

Chun and Ham [3] proposed a sixth-order family as follows

$$x_{n+1}=z_n -\left(1+2\frac{f(y_n)}{f(x_n)}+\mu\frac{f(y_n)}{f(x_n)}+\gamma\frac{f^3(z_n)}{f^3(x_n)}\right)\frac{f(z_n)}{f'(x_n)}, \tag{4}$$

where

$$z_n=y_n-\frac{f(x_n)}{f(x_n)-2f(y_n)}.\frac{f(y_n)}{f'(x_n)}, \tag{5}$$

and $\mu,\gamma\in\mathbb{R}.$ Throughout the rest of this paper y_n is defined by Eq. (3), and z_n by Eq. (5).

Grau and Barrero [4] developed a sixth-order method which is given by

$$x_{n+1} = z_n - \frac{f(z_n) + (\beta - 2)f(y_n)}{f(z_n) + \beta f(y_n)} \cdot \frac{f(z_n)}{f'(x_n)}, \tag{6}$$

and $\beta \in \mathbb{R}$.

Kou and Li [5] presented a sixth-order method as follows

$$x_{n+1} = z_n - \left(1 + L_f(x_n) + \frac{3f(z_n)}{f'(x_n)(x_n - z_n)}\right)\frac{f(z_n)}{f'(x_n)}, \tag{7}$$

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{[f'(x_n)]^2}. \tag{8}$$

Another sixth-order family of modified Ostrowski's method was considered by Sharma and Guha [6]

$$x_{n+1} = z_n - \frac{f(z_n) + f(y_n)}{f(z_n) + 3f(y_n)} \cdot \frac{f(z_n)}{f'(x_n)}. \tag{9}$$

Weihong *et al*. [7] developed a one-parameter family of eighth-order method, which is

$$x_{n+1} = z_n - \frac{f(x_n) + (2+\alpha)f(t_n)}{f(x_n) + \alpha f(t_n)} \cdot \frac{f(w_n)}{F(x_n)},$$

where

$$t_n = y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \cdot \frac{f(y_n)}{f'(x_n)},$$

$$w_n = z_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \cdot \frac{f(z_n)}{F(x_n)},$$

and

$$F(x_n) = f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n).$$

A sixth-order method developed by Parhi and Gupta [8], is given by:

$$y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{10}$$

$$u_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)},$$

$$x_{n+1} = u_n - \left[\frac{f'(y_n) + f'(x_n)}{-f'(x_n) + 3f'(y_n)}\right]\frac{f(u_n)}{f'(x_n)}.$$

Some more recent methods are also used in the literature [9-13].

Motivated by the recent activities in this direction, in this paper we propose a class of sixth-order methods to find a simple zero. The rest of this paper is organized as follows. In Section 2, we consider a general iterative scheme, analyze it to present a family of fifth-order methods then several known special cases of this family are listed. Section 3, the formulae are tested and their performance is compared with some known methods. Finally, conclusions are drawn in the last section.

Main result

In this section, by the idea of constructing iteration methods with unknown functions, we have generalized and analyzed an iterative scheme in three steps in the following form

$$y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$t_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \tag{11}$$

$$x_{n+1} = u_n - H\left(f'(x_n), f'(y_n)\right)\frac{f(t_n)}{f'(x_n)},$$

In Theorem 1, we will impose the condition on $H(u,v)$ in such a way that the iterative methods defined by Eq. (11) have a sixth-order of convergence, but first let us point out the definition of order of convergence for fixed point iteration method as follows

***Definition of order of convergence*:** The sequence produced by the iteration function F, as

$x_{n+1} = F(x_n)$

is said to be convergent to the fixed point α of F, i.e., $F(\alpha) = \alpha$, of order p if and only if we have

$x_{n+1} = F(x_n) = \alpha + C_p e_n^p$,

where $C_p \neq 0$ and $e_n = x_n - \alpha$.

Theorem 1. Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}$ on an open interval which contains x_0 as a close initial approximation to α. If $H(u,v)$ satisfies the conditions

$H(1,1) = 1,$

$H_u(1,1) = 1,$

$H_{uu}(1,1) = -H_v(1,1),$

then the method defined by Eq. (11), is of at least sixth-order.

Proof. Let α be a simple zero of f. Consider the fixed-point iteration function F to be defined as

$$F(x) = x - \frac{f(x)}{f'(x)} - H(f'(x), f'(y(x)))\frac{f(t(x))}{f'(x)},$$

Where

$$y(x) = x - \frac{f(x)}{f'(x)},\ t(x) = x - \frac{2f(x)}{f'(x) + f'(y(x))}.$$

Here, due to tedious evaluation of derivatives of F, we employ the symbolic computation of the Maple package to compute the Taylor expansion of $F(x_n)$ around $x = \alpha$. We find that

$$x_{n+1} = F(x_n) = \alpha + K_3 e_n^3 + K_4 e_n^4 + K_5 e_n^5 + O(e_n^6),$$

Where

$$K_3 = -\frac{1}{12}\left(-1 + H(1,1)\right)\left(12c_2^2 + 6c_3\right).$$

It can be easily verified that if we take

$$H(1,1) = 1, \tag{12}$$

then we have $K_3 = 0$.

Considering Eq. (12), K_4 leads to

$$K_4 = 2\left(-H_u(1,1) + 1\right)c_2^3 + \left(-H_u(1,1) + 1\right)c_2 c_3.$$

This also can be vanished, by letting

$$H_u(1,1) = 1, \tag{13}$$

Substituting Eq. (12) and Eq. (13) into K_5, we get

$$K_5 = -2\left(H_{uu}(1,1) + H_v(1,1)\right)c_2^4 - \left(H_{uu}(1,1) + H_v(1,1)\right)c_2^2 c_3.$$

So the proof is complete if we have

$$H_{uu}(1,1) = -H_v(1,1).$$

It is seen that, Eq. (11) with $H(u,v) = \frac{u + v}{-u + 3v}$, gives a sixth-order method which has been introduced by Parhi and Gupta [8] and Eq. (10). Moreover some new sixth-order methods can be introduced by certain choices for $H(u,v)$ in Eq. (11). For example, if we take $H(u,v) = \frac{7u^2 - 8uv + 3v^2}{2u^2}$ in Eq. (11), we obtain the following sixth-order method

$$\begin{aligned} y_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ u_n &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \\ x_{n+1} &= u_n - \left[\frac{3}{2}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2 - 4\frac{f'(y_n)}{f'(x_n)} + \frac{7}{2}\right]\frac{f(u_n)}{f'(x_n)}. \end{aligned} \tag{14}$$

The main point that should be mentioned here is that the order of Newton's method is improved four units with additional evaluations of the two functions. Indeed per iteration in the methods defined by Eq. (11) requires two functions and two first derivative evaluations. If we consider the definition of the efficiency index [14] as $\sqrt[r]{p}$, where p is the order of the method and r is the number of functional evaluations per iteration required by the method, thus the iteration scheme defined by Eq. (14) has the efficiency index equal to $\sqrt[4]{6} \approx 1.565$, which is better than that of Newton's method $\sqrt{2} \approx 1.4142$. So the order of convergence and computational efficiency of Newton's method are greatly improved.

Numerical examples

In this section, Method 1 (BGM), defined by Eq. (14), is employed to solve some nonlinear equations and compared with Newton's method (NM), Eq. (1), Kou and Li's method [15] (KM), Sharma and Guha's method defined by Eq. (9) (SM) for solving the following test functions.

$$\begin{aligned} f_1(x) &= x^3 + 4x^2 - 10, \\ f_2(x) &= \sin^2(x) - x^2 + 1, \\ f_3(x) &= (x-1)^3 - 1, \\ f_4(x) &= e^{x^2 + 7x - 30} - 1, \\ f_5(x) &= x^{10} - 1, \\ f_6(x) &= x^2 - e^x - 3x + 2, \\ f_7(x) &= (x-2)^{23} - 1. \end{aligned}$$

Tables 1 and **2** respectively show the number of needed iterations and function computations for all methods carried out in the Maple with 128 digit floating point required such that $|f(x_n)| < 10^{-50}$.

Table 1 Comparisons of number of iteration of various sixth-order convergent iterative methods.

	NM	KM	SM	BGM
$f_1, x_0 = 1$	5	2	2	2
$f_1, x_0 = 2$	5	2	2	2
$f_2, x_0 = 1$	6	2	2	2
$f_2, x_0 = 3$	6	3	3	2
$f_3, x_0 = 2.5$	6	2	2	2
$f_3, x_0 = 3.5$	7	3	3	3
$f_4, x_0 = 3.25$	8	4	4	3
$f_4, x_0 = 0.5$	12	5	4	4
$f_5, x_0 = 0.8$	Div	18	12	10
$f_5, x_0 = 2$	10	3	3	3
$f_6, x_0 = 2$	5	4	4	3
$f_6, x_0 = 3$	6	3	3	3
$f_7, x_0 = 4.5$	Div	6	6	7
$f_7, x_0 = 3.5$	14	15	4	4

Table 2 Comparisons of number of functional computations needed of various sixth-order convergent iterative methods.

	NM	KM	SM	BGM
$f_1, x_0 = 1$	10	8	8	8
$f_1, x_0 = 2$	10	8	8	8
$f_2, x_0 = 1$	12	8	8	8
$f_2, x_0 = 3$	12	12	12	8
$f_3, x_0 = 2.5$	12	6	8	8
$f_3, x_0 = 3.5$	14	12	12	12
$f_4, x_0 = 3.25$	16	16	16	12
$f_4, x_0 = 0.5$	24	20	16	16
$f_5, x_0 = 0.8$	-	72	48	40
$f_5, x_0 = 2$	20	12	12	12
$f_6, x_0 = 2$	10	16	16	12
$f_6, x_0 = 3$	12	12	12	12
$f_7, x_0 = 4.5$	-	24	24	28
$f_7, x_0 = 3.5$	28	60	16	16

The results in **Tables 1** and **2** show that the new method requires a lower number of functional computations than other methods, especially Newton's method. Therefore, it is of practical interest and can work better than them.

Conclusions

We have obtained a class of three-step methods of order sixth, which produces many sixth-order iterative schemes. Per iteration the present method requires three evaluations of the function and one evaluation of its first. It can be seen that the numerical results, agree well with the convergence analysis developed in Theorem 1.

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