

## A New Analytical Technique to Solve Some Nonlinear Equations Involving Trigonometric Nonlinearities

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### Abstract

An analytical approximate method, namely the Homotopy Analysis Method (HAM), is employed to propose to solve some nonlinear partial differential equations with a trigonometric nonlinearity typically arising in heat transfer. Two different cases of equations are studied to illustrate the validity and efficiency of the method. The obtained results reveal that the Homotopy Analysis Method is very effective, straightforward and simple.

**Keywords:** Homotopy analysis method, symbolic computation, approximate series solution, heat transfer

### Introduction

Partial differential equations can be used to describe a wide variety of phenomenon in engineering applications. Due to this fact, in recent years there has been a growing interest towards the application of the numerical and analytical methods for solving such equations. For instance the finite difference method [1], finite element method [2], Adomian's decomposition method [3] and Homotopy Perturbation Method (HPM) [4], and the small parameter method [5] have been applied in the literature.

In this paper, we apply the HAM for solving the following type of nonlinear parabolic partial differential equation

$$\frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( f(w) \frac{\partial w}{\partial x} \right), \quad (x, t) \in \Omega, \quad (1)$$

subject to a known initial condition

$$w(x, 0) = g(x).$$

Eq. (1) covers a large class of equations that frequently arise in the study of heat and mass transfer problems and the theory of flows through

porous media [6]. Here we consider a trigonometric form for  $f(w)$ .

A number of studies have been reported in the literature focusing on the problem of heat and mass transfer in porous media [6]. In [7-9] some approaches to heat transfer problems were developed by using some new proposed approximate-analytical methods.

In 1992, Liao [10] proposed an approximation technique for nonlinear problems, namely the Homotopy Analysis Method. By means of this method, the original nonlinear problem is transformed into a sequence of linear problems that are easily solved by symbolic computational software. This method has been successfully applied to solve many types of nonlinear problems in engineering and science (see, e.g., [7-8,10-12]).

To the best of our knowledge no HAM-based solution has been reported for problem (1). The rest of the present contribution is constructed as follows. In section 2, we will briefly describe the basic idea of HAM. Then it is applied to the two special kinds of problem (1) in section 3. And finally conclusions are drawn in section 4.

**The HAM approach**

Let us assume the following nonlinear differential equation in the form of:

$$N[\omega(x, t)] = 0. \tag{2}$$

Liao [10], using  $p \in [0, 1]$  as an embedding parameter, constructed the so-called zero-order deformation equation

$$(1-p)L[\omega(x, t; p) - w_0(x, t)] = p\hbar N[\omega(x, t; p)], \tag{3}$$

where  $L$  is a linear operator,  $\hbar \neq 0$  is a convergence-control parameter, and  $w_0(x, t)$  is an initial guess of the unknown solution  $\omega(x, t; p)$ . From Eq. (3), when  $p = 0$  and  $p = 1$ ,

$$\omega(x, t; 0) = w_0(x, t) = w(x, 0)$$

and

$$\omega(x, t; 1) = w(x, t),$$

both hold. Therefore, as  $p$  increases from zero to one, the solution  $\omega(x, t; p)$  varies from the initial guess  $w_0(x, t)$  to the solution  $w(x, t)$ . Expanding  $\omega(x, t; p)$  in the Taylor series with respect to  $p$ , one has

$$\omega(x, t; p) = w_0(x, t) + \sum_{m=1}^{\infty} w_m(x, t) p^m, \tag{4}$$

where

$$w_k(x, t) = \frac{1}{m!} \left. \frac{\partial^m \omega(x, t; p)}{\partial p^m} \right|_{p=0}. \tag{5}$$

As the convergence of the series Eq. (4) depends on the parameter  $\hbar$ , assume that  $\hbar$  is chosen so properly so that the series Eq. (4) is convergent at  $p = 1$ . So, we have the solution series

$$w(x, t) = \omega(x, t; 1) = w_0(x, t) + \sum_{k=1}^{\infty} w_k(x, t).$$

must be one of the solutions of the original problem (1), as proved by Liao [10].

Now we define the vector

$$\vec{w}_s(x, t) = \{w_0(x, t), \dots, w_s(x, t)\}.$$

According to the definition in Eq. (5), the governing equation and the corresponding initial condition of  $w_m(x, t) (m \geq 1)$  can be deduced from Eq. (3).

Differentiating the zeroth order deformation Eq. (3)  $m$  times with respect to  $p$ , then setting

$p = 0$ , and finally dividing them by  $m!$ , we obtain the  $m^{\text{th}}$  order deformation equation

$$L[w_m(x, t) - \chi_m w_{m-1}(x, t)] = \hbar R_m(\vec{w}_{m-1}(x, \xi)),$$

and its initial conditions

$$w_m(x, 0) = 0, \tag{6}$$

where

$$R_m(\vec{w}(x, t)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\omega(x, t; p)]}{\partial p^{m-1}} \right|_{p=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Note that the coefficients  $c_m$ 's by applying the inverse operator  $L^{-1}(\cdot)$  to both sides of the  $m^{\text{th}}$  order deformation Eq. (6), as

$$w_m(x, t) = \chi_m w_{m-1}(x, t) + \hbar L^{-1}(R_m(\vec{w}_{m-1}(x, t))) + c_m.$$

subject to initial condition

$$w_m(x, 0) = 0,$$

So, in this way  $w_m(x, t)$ 's can be recursively determined, and the  $N^{\text{th}}$  order of the approximation to the problem can be generally expressed by

$$\vec{w}_N(x, t) \approx \sum_{m=0}^N w_m(x, t).$$

Such an equation is a family of solutions expressed in terms of the parameter  $\hbar$ . In order to obtain a convergent series solution to the nonlinear problem (1), a valid region of  $\hbar$  can be obtained via the  $\hbar$ -curve [10].

**Applications**

In this part, we present a selection of examples to illustrate the efficiency of the HAM.

**Example 1** Let us first consider Eq. (1) with as

$$f(w) = \alpha \cos^2(\beta w)$$

$$\frac{\partial w}{\partial t} = \alpha \frac{\partial}{\partial x} \left( \cos^2(\beta w) \frac{\partial w}{\partial x} \right), (x, t) \in \Omega. \tag{7}$$

whose exact solution can be written as [12]:

$$w(x, t) = \frac{1}{\beta} \arcsin \left( \frac{x + C_1}{\sqrt{2\alpha t + C_2}} \right),$$

where  $C_1$  and  $C_2$  are arbitrary constants.

For simplicity, let us take  $\alpha = 1, \beta = 2$  and  $C_1 = 0, C_2 = 1$  in Eq. (7). Thus, the exact solution is

$$w(x, t) = \frac{1}{2} \arcsin\left(\frac{x}{\sqrt{2t+1}}\right). \tag{8}$$

Now Eq. (8) is solved with the consideration of the initial condition

$$w(x, 0) = \frac{1}{2} \arcsin(x).$$

In the view of HAM for Eq. (1), it is straightforward to use the nonlinear operator  $N$  given by

$$N[\omega(x, t; p)] = \frac{\partial \omega}{\partial t} - \frac{\partial}{\partial x} \left( \cos^2(2x) \frac{\partial \omega}{\partial x} \right).$$

Now with the aid of the governing Eq. (1), we consider the initial approximation as

$$w_0(x, t) = w(x, 0) = \frac{1}{2} \arcsin(x),$$

and the auxiliary linear operator

$$L[\omega(x, t)] = (1+t) \frac{\partial \omega(x, t)}{\partial t} + \omega(x, t).$$

In view of such a linear operator, the solution  $w(x, t)$  can be expressed by a set of base functions  $\{(1+t)^{-m} \mid m = 0, 1, 2, \dots\}$

in the form

$$w(x, t) = \sum_{m=0}^{+\infty} \frac{d_m(x)}{(1+t)^m},$$

where  $d_m$ 's for  $m \geq 1$ , are functions to be determined later.

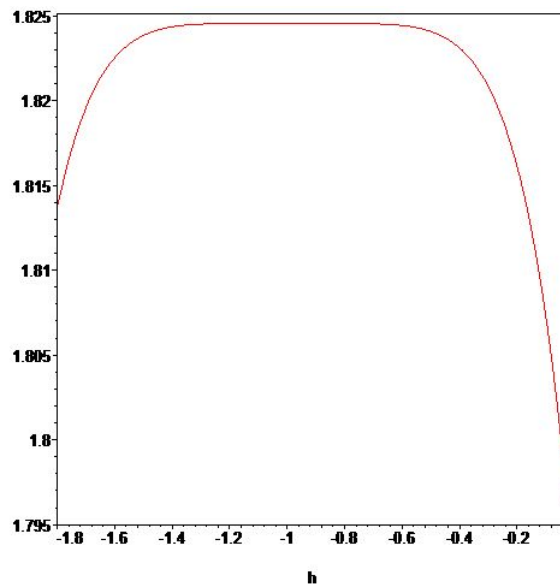
To obtain higher order terms  $w_m(x, t)$ , the  $m^{\text{th}}$  order deformation equation and its boundary conditions are calculated:

$$w_m(x, t) = \mathcal{L}_m w_{m-1}(x, t) + \frac{\hbar}{1+t} \left( \int_0^t R_m(\bar{w}_{m-1}(x, \xi)) d\xi \right) + \frac{c_m}{1+t},$$

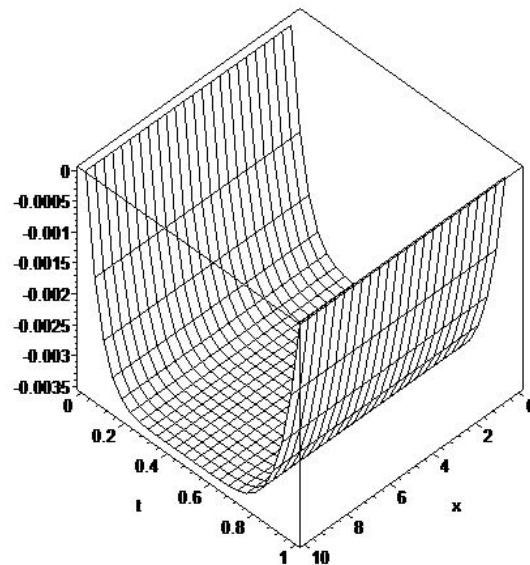
where  $c_m$ 's are determined by considering  $w_m(x, 0) = 0$ .

To find the valid region of  $\hbar$ , the  $\hbar$ -curve given by the 5<sup>th</sup> order approximation is drawn in **Figure 1**, which clearly indicates that the valid region of is about  $-1.5 < \hbar < -0.6$ .

When  $\hbar = -1.1$ , we obtain an approximate solution which is in a good agreement with the exact solution as shown in **Figure 2**, where the absolute errors of the shooting method approximation, the 5<sup>th</sup> order HAM approximations are depicted.



**Figure 1**  $\hbar$ -curve for the 5<sup>th</sup> order of HAM approximation ( $\bar{w}_5(1,2)$  versus  $\hbar$ ) for example 1.



**Figure 2** Absolute errors of 5<sup>th</sup> order of HAM approximations with  $\hbar = -1.1$  for example 1.

**Example 2** Now consider Eq. (1) by letting  $f(w) = \alpha \cos^2(\beta x)$  as

$$\frac{\partial w}{\partial t} = \alpha \frac{\partial}{\partial x} \left( \sinh^2(\beta x) \frac{\partial w}{\partial x} \right), (x, t) \in \Omega. \tag{9}$$

In [12] the exact solution of Eq. (9) is obtained as follows

$$w(x, t) = \frac{1}{\beta} \arccos h \left( \frac{x + C_1}{\sqrt{C_2 - 2\alpha t}} \right),$$

where  $C_1$  and  $C_2$  are arbitrary constants.

For numerical comparisons, we let  $\alpha = -1, \beta = 2$  and  $C_1 = 0, C_2 = 1$ . So, the exact solution is

$$w(x, t) = \frac{1}{2} \arccos h \left( \frac{x}{\sqrt{1 + 2t}} \right).$$

and the initial condition should be considered as

$$w(x, 0) = \frac{1}{2} \arccos h(x).$$

In the view of HAM for Eq. (1), the nonlinear operator  $N$  is given by

$$N[\omega(x, t; p)] = \frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x} \left( \sinh^2(2x) \frac{\partial \omega}{\partial x} \right).$$

Now with the aid of the governing Eq. (1), we consider the initial approximation as

$$w_0(x, t) = w(x, 0) = \frac{1}{2} \arccos h(x),$$

and the auxiliary linear operator

$$L[\omega(x, t)] = \frac{\partial \omega(x, t)}{\partial t} + \omega(x, t).$$

In view of such a linear operator, the solution  $w(x, t)$  can be expressed by a set of base functions  $\{\exp(-mt) | m \geq 0\}$

Indeed, our solution can be expressed in the following form

$$w(x, t) = \sum_{m=0}^{+\infty} c_m(x) \exp(-mt),$$

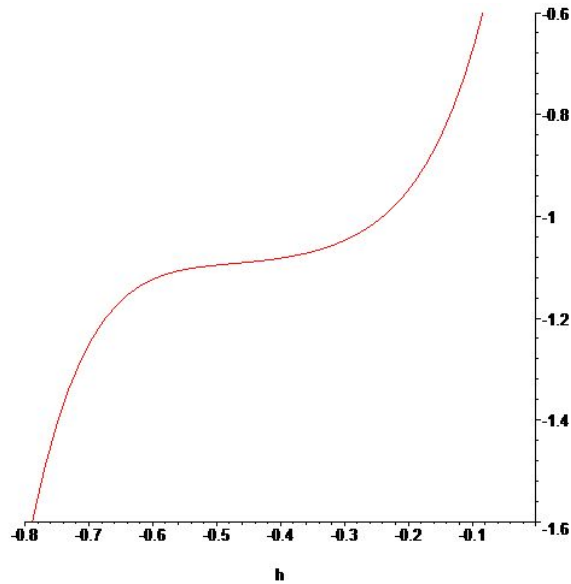
To obtain higher order terms  $w_m(x, t)$ , the  $m^{\text{th}}$  order deformation equation and its boundary conditions are calculated:

$$\begin{aligned} w_m(x, t) = & \mathcal{H}_m w_{m-1}(x, t) \\ & + \hbar \exp(-t) \left( \int_0^t \exp(-\xi) R_m(\bar{w}_{m-1}(x, \xi)) d\xi \right) \\ & + c_m \exp(-t). \end{aligned}$$

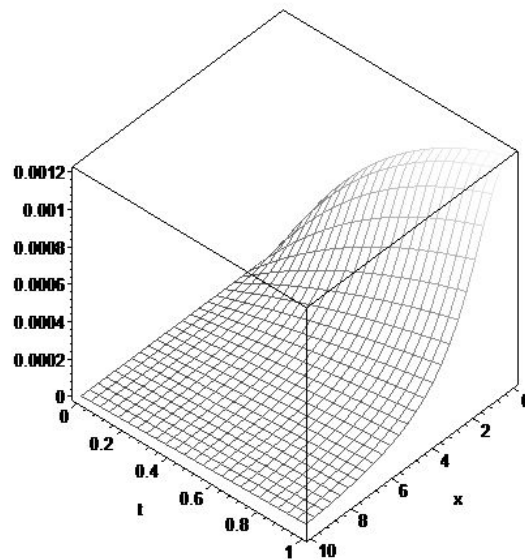
where  $c_m$ 's are obtained by imposing the boundary conditions  $w_m(x, 0) = 0$ .

From the  $\hbar$ -curve for the 5<sup>th</sup> order of HAM approximation depicted in **Figure 3**, we obtain  $[-0.7, -0.3]$  as the proper interval for  $\hbar$ . Also, in

**Figure 4** we plot the comparisons made between the computed errors of the 5<sup>th</sup> order approximation of HAM with  $\hbar = -0.48$  for Example 2.



**Figure 3**  $\hbar$ -curve for the 5<sup>th</sup> order of HAM approximation ( $\tilde{w}_5(1,2)$  versus  $\hbar$ ) for example 2.



**Figure 4** Absolute errors of 5<sup>th</sup> order of HAM approximations at  $\hbar = -0.48$  for example 2.

### Conclusion

In this paper, we proposed the HAM series solution of some problems in heat transfer. The efficiency and accuracy of the proposed method was demonstrated by some test problems. It can be seen that the series solution of the problem by using HAM is very close to the exact solution of the problem. The success of this approach lies in the fact that the method provides a convenient way to adjust and control the convergence region and rate of the series solutions obtained.

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