Boundary Layer Flow and Heat Transfer for a Third Grade Fluid over a Nonlinear Radially Stretching Sheet

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Abstract

The steady axisymmetric flow and heat transfer of a thermodynamically compatible third grade fluid over an isothermal radially stretching sheet is investigated. A nonlinear stretching sheet is considered. The governing boundary layer equations for velocity and temperature fields are reduced to a system of ordinary differential equations by using appropriate similarity transformations. The resulting equations are then solved analytically by the homotopy analysis method (HAM). The developed analytical expressions for the velocity and temperature fields are graphically presented and influence of the pertinent parameters on the velocity and thermal boundary layers are discussed in detail. In addition, the skin friction coefficient and local Nusselt number are tabulated for several influential parameters. Increases in viscoelastic and third grade parameter increase the boundary layer thickness, whereas the cross-viscous parameter decreases the boundary layer thickness. Discrete squared residual has been plotted for the different order of approximations and the results have been plotted with the maximum accuracy.

Keywords: Heat transfer, third grade fluid, radially stretching sheet, boundary layer, stretching sheet

Introduction

Non-Newtonian fluids are extensive in their areas of applications owing to their behavior being encountered in almost all the chemical and allied processing industries. Non-Newtonian flows are not only dominantly interested by researchers but also by process and chemical engineers, who deal with the complex materials (such as foams, slurries, emulsions, polymer melts and solutions, etc.). Various fluids models have been proposed in the literature, which explore the different behaviors of non-Newtonian fluids. These models include the differential type, rate type, integral type and power-law models. The differential type fluids are important as they explore the features such as normal stress effects, shear thinning and shear thickening effects. The second grade fluid, which only explains the normal stress effects is the simplest subclass of the differential type fluids. However, the third grade fluid, in addition, exhibits the properties of shear thinning and shear thickening. The visco-elastic materials such as Sorbothane is a synthetic polymer used as a shock absorber and vibration damper.

Some investigations dealing with the flow and heat transfer of a third grade fluid are reported in [1-4]. The boundary layer flow over a stretching sheet has significant technological applications in chemical and metallurgical industries such as; continuous stretching of plastic films, drawing of copper wires, polymer extrusion, hot rolling, metal extrusion, glass fibers etc. Investigation of heat transfer phenomena over a stretching surface is of great importance in several engineering applications. Such applications include heated materials travelling between a feed roll and a wind up roll or on a conveyer belt and the materials manufactured by extrusion processes. Crane [5] initiated the study of viscous flow over a stretching surface by considering steady two-dimensional boundary-layer flow over a uniformly stretching sheet and found the closed form solution. He also reported the heat transfer analysis for this

problem. Later on this problem attracted various researchers and several extensions have been made by invoking different interesting features [6,7]. Some very recent studies regarding the boundary layer flow and heat transfer of a third grade fluid over stretching sheet are provided in [8-13].

A literature survey indicates that a lot of work has been done on boundary layer flow and heat transfer over a nonlinear planner stretching sheet. However, rare investigations exist regarding axisymmetric flow and heat transfer over a nonlinear radially stretching sheet. To the best of our knowledge not a single attempt is made so far, which deals with the boundary layer flow and heat transfer of a third grade fluid over a nonlinear radially stretching sheet. The present investigation attempts to fill this gap and extends the literature to the boundary layer flow and heat transfer in a third grade fluid due to a radially moving sheet with nonlinear velocity.

Materials and methods
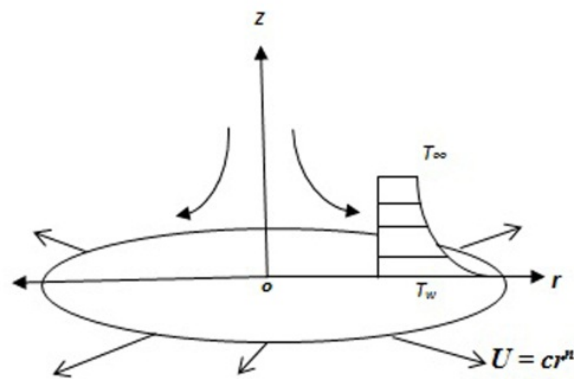
Mathematical modeling

Figure 1 Physical model and coordinate system.

Consider the quiescent incompressible third grade fluid in which steady, laminar and axisymmetric flow is induced due to stretching of the sheet along the radial direction with a nonlinear velocity given by $U(r) = cr^n$ with *n* as a positive real number and $c > 0$. We assume that the sheet is isothermal with a uniform temperature T_w while T_∞ is the ambient fluid temperature with $T_w > T_\infty$ as shown in **Figure 1**. The equations governing the steady flow of an incompressible fluid in the absence of body forces are;

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \boldsymbol{\tau}, \tag{2}$$

where ρ is the fluid density, $\mathbf{V}$ the velocity field and $\boldsymbol{\tau}$ the Cauchy stress tensor which in a third grade fluid is given by [14];

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1. \tag{3}$$

Here p is the pressure, $\mathbf{I}$ the identity metric, μ dynamic viscosity, $\alpha_i \left(i = 1,2\right)$, $\beta_i \left(i = 1,2,3\right)$ the

material constants, and $\mathbf{A}_i \left(i = 1, 2, 3 \right)$ the Rivlin-Erickson tensors given by;

$$\mathbf{A}_1 = \left(\operatorname{grad} \mathbf{V} \right) + \left(\operatorname{grad} \mathbf{V} \right)^T, \tag{4}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1} \left(\operatorname{grad} \mathbf{V} \right) + \left(\operatorname{grad} \mathbf{V} \right)^T \mathbf{A}_{n-1}, \; n = 2, 3. \tag{5}$$

Fosdick and Rajagopal [15] have shown that the third grade fluid is consistent with thermodynamics if material constants satisfy;

$$\mu \geqslant 0, \; \alpha_1 \geqslant 0, \; \left| \alpha_1 + \alpha_2 \right| \leqslant \sqrt{24 \mu \beta_3}, \; \beta_1 = \beta_2 = 0, \; \beta_3 \geqslant 0. \tag{6}$$

In view of conditions Eq. (6) the Cauchy stress tensor Eq. (3) reduces to;

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(tr\mathbf{A}_1^2 \right)\mathbf{A}_1. \tag{7}$$

For the velocity field $\mathbf{V} = [u(r,z), 0, w(r,z)]$ and Cauchy stress tensor Eq. (7), the governing equations for the steady flow are given by;

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{8}$$

$$\rho \left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} \right) = \frac{\partial T_{rr}}{\partial r} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}, \tag{9}$$

$$\rho \left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} \right) = \frac{1}{r}\frac{\partial}{\partial r}\left(rT_{rz} \right) + \frac{\partial T_{zz}}{\partial z}, \tag{10}$$

where

$$T_{rr} = -p + 2\mu\frac{\partial u}{\partial r} + 2\alpha_1\left[u\frac{\partial^2 u}{\partial r^2} + w\frac{\partial^2 u}{\partial r\partial z} + 2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \right]$$

$$+\alpha_2\left[4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 \right] \quad +4\beta_3\frac{\partial u}{\partial r}\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + 2\left\{\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \right], \tag{11}$$

$$T_{\theta\theta} = -p + 2\mu\frac{u}{r} + 2\alpha_1\left(\frac{u}{r}\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial z} + \frac{u^2}{r^2} \right) + 4\alpha_2\frac{u^2}{r^2}$$

$$+4\beta_3\frac{u}{r}\left[2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2 + 2\left\{\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \right], \tag{12}$$

$$T_{zz} = -p + 2\mu\frac{\partial w}{\partial z} + 2\alpha_1\left[u\frac{\partial^2 w}{\partial r\partial z} + w\frac{\partial^2 w}{\partial z^2} + 2\left(\frac{\partial w}{\partial z}\right)^2 + \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \right]$$

$$+\alpha_2\left[4\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right]+4\beta_3\left(\frac{\partial w}{\partial z}\right)\left[2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2+2\left\{\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}\right],\tag{13}$$

$$T_{rz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+\alpha_1\left[\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)+\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right.$$

$$\left.+3\left(\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right)\right]+2\alpha_2\left(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)$$

$$+2\beta_3\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)\left[2\frac{u^2}{r^2}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2+2\left\{\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}\right].\tag{14}$$

By using the usual boundary layer approximations, the governing boundary layer equation for a third grade fluid is;

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=\nu\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 u}{\partial z^3}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\right)$$

$$+\frac{\alpha_2}{\rho}\left(2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\right)+6\frac{\beta_3}{\rho}\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2},\tag{15}$$

where ν is the kinematic viscosity. The relevant velocity boundary conditions are;

$$u=U(r)=cr^n,\quad w=0\quad at\quad z=0,\tag{16}$$

$$u\to 0\quad as\quad z\to\infty.\tag{17}$$

The introduction of the following similarity transformations;

$$\psi(r,z)=-r^2 U\,\mathrm{Re}^{-1/2}\,f(\eta),\ \text{and}\ \eta=\frac{z}{r}\mathrm{Re}^{1/2},\tag{18}$$

where η is the dimensionless similarity variable, $\mathrm{Re}=\frac{Ur}{\nu}$ the Reynolds number and $\psi(r,z)$ the Stokes stream function defined by $u=-\frac{1}{r}\frac{\partial\psi}{\partial z}$ and $w=\frac{1}{r}\frac{\partial\psi}{\partial r}$ results in;

$$u=Uf'(\eta)\quad\text{and}\quad w=-U\,\mathrm{Re}^{-1/2}\left[\frac{n+3}{2}f(\eta)+\frac{n-1}{2}\eta f'(\eta)\right].\tag{19}$$

Using the above similarity relations, the continuity Eq. (8) is identically satisfied and Eqs. (15) - (17) lead to;

$$f''' + \frac{n+3}{2}ff'' - n(f')^2 + \alpha_1\left\{3(n-1)f'f'' + 3(n-1)\left(f''\right)^2 - \frac{n+3}{2}ff^{iv} + 2(n-1)\eta f''f'''\right\}$$

$$+\alpha_2\left\{\frac{3n-5}{2}\left(f''\right)^2 - 2f'f'' + (n-1)\eta f''f'''\right\} + \beta\left(f''\right)^2 f''' = 0, \tag{20}$$

$$f(0) = 0, \ f'(0) = 1 \text{ and } f'(\infty) = 0. \tag{21}$$

In the above equation $\alpha_1^* = \frac{\alpha_1 U}{r\mu}$ and $\alpha_2^* = \frac{\alpha_2 U}{r\mu}$ are the dimensionless measures of the second grade fluid parameters α_1 and α_2, respectively and $\beta = \frac{\beta_3 U^3}{r^2\nu^3\rho}$ is the dimensionless measure of the third grade fluid parameter when dropping the asterisks for simplicity.

The solution of the boundary layer equations is often used to determine the technically important wall shear stress $\tau_{rz}\big|_{z=0}$. For the dimensionless wall shear stress we introduce the local skin friction coefficient C_f given by;

$$C_f = \frac{\tau_{rz}\big|_{z=0}}{\frac{1}{2}\rho U^2}. \tag{22}$$

Utilization of the above expression for τ_{rz} and the above introduced similarity transformations, the local skin friction coefficient becomes;

$$\frac{1}{2}\text{Re}^{1/2} C_f = f''(0) + \alpha_1\frac{7n-3}{2}f''(0) - 2\alpha_2 f''(0) + 2\beta\left\{f''(0)\right\}^3. \tag{23}$$

Heat transfer analysis

Heat transfer takes place from an isothermal radially stretching sheet to the ambient fluid at temperatures T_w and T_∞, respectively. Assuming that the radiant heating is negligible and the viscous dissipation and heat generation are absent, the governing energy equation for temperature field $T = T(r,z)$ is given by;

$$\rho c_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right), \tag{24}$$

where c_p is specific heat at constant pressure and κ the thermal conductivity of the fluid.

Use of the usual thermal boundary layer approximations, the above energy equation reduces to;

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial z^2}. \tag{25}$$

The temperature boundary conditions for the above equations are;

$$T = T_w \quad \text{at} \quad z = 0, \tag{26}$$

$$T \to T_\infty \quad \text{as } z \to \infty. \tag{27}$$

Defining the dimensionless temperature;

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{28}$$

and utilizing the similarity transformations Eq. (18) results in;

$$\theta'' + \Pr \frac{n+3}{2} f\theta' = 0, \tag{29}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0, \tag{30}$$

where $\Pr = \frac{\mu c_p}{\kappa}$ is the Prandtl number.

For the dimensionless wall heat flux $q_w = -\kappa \left(\frac{\partial T}{\partial z} \right)\big|_{z=0}$, we introduce the local Nusselt number Nu_r given by;

$$Nu_r = \frac{-r \frac{\partial T}{\partial z}\big|_{z=0}}{T_w - T_\infty}. \tag{31}$$

Using the above similarity transformations, the local Nusselt number becomes;

$$\text{Re}^{-1/2} Nu_r = -\theta'(0). \tag{32}$$

Solution of the problem

The system of nonlinear ordinary differential equations Eqs. (20) and (29) are solved analytically with the boundary conditions Eqs. (21) and (30). The analytical solutions are obtained by means of the homotopy analysis method (HAM). For the HAM solutions we choose the initial guesses $f_0(\eta)$ and $\theta_0(\eta)$ as;

$$f_0(\eta) = 1 - \exp(-\eta), \ \theta_0(\eta) = \exp(-\eta), \tag{33}$$

and the auxiliary linear operators as;

$$\mathcal{L}_f(\phi) = \frac{d^3\phi}{d\eta^3} - \frac{d\phi}{d\eta}, \quad \mathcal{L}_\theta(\phi) = \frac{d^2\phi}{d\eta^2} - \frac{d\phi}{d\eta}. \tag{34}$$

Now by letting $\hbar_f$ and $\hbar_\theta$ as the non-zero convergence control parameters we can construct the zero and higher order deformation problems in view of Refs. [16,17]. The parameters $\hbar_f$ and $\hbar_\theta$ adjust and control the convergence of the series solutions and the optimal values of these parameters are chosen by minimizing the discrete squared residuals (see Ref. [18] for more detail) which are defined by;

$$E_{f,m} = \frac{1}{N+1}\sum_{j=0}^{N}\left[N_f\left(\sum_{i=0}^{m}F_j\left(i\triangle\eta\right)\right)\right]^2,$$ (35)

$$E_{\theta,m} = \frac{1}{N+1}\sum_{j=0}^{N}\left[N_\theta\left(\sum_{i=0}^{m}G_j\left(i\triangle\eta\right)\right)\right]^2,$$ (36)

where $N+1$ are the number of points chosen over the domain in which the nonlinear equations $N_f\left(f\left(\eta\right)\right) = 0$ and $N_\theta\left(\theta\left(\eta\right)\right) = 0$ are stated.

Results and discussion

The objective of the present analysis is to examine the flow and heat transfer characteristics of an incompressible third grade fluid over an isothermal radially nonlinear stretching sheet. The influence of the stretching parameters n, viscoelastic parameter α_1 cross-viscous parameter α_2, third grade parameter β and Prandtl number Pr on flow and heat transfer is the main interest of study. Further, the skin friction coefficient and local Nusselt number are also analyzed for different values of these parameters.

The velocity at the boundary is dependent upon the stretching parameter n. The stretching parameter n greatly affects the velocity and temperature profiles of the fluid and is illustrated in **Figure 2**. From **Figure 2(a)**, it is noted that the boundary layer thickness decreases as the value of n is enhanced. It is due to the fact that the greater velocity at sheet along the inertial forces tends to reduce the effect of viscous forces in the boundary layer. As a result the radial component of velocity asymptotically reaches the free stream value rapidly. **Figure 2(b)** shows that n strongly affects the temperature profile. The temperature profile and the thermal boundary layer decrease by enhancing the value of n. The increased velocity at the wall forces the cooler ambient to rush towards the hot sheet thus reducing the temperature within boundary layer.
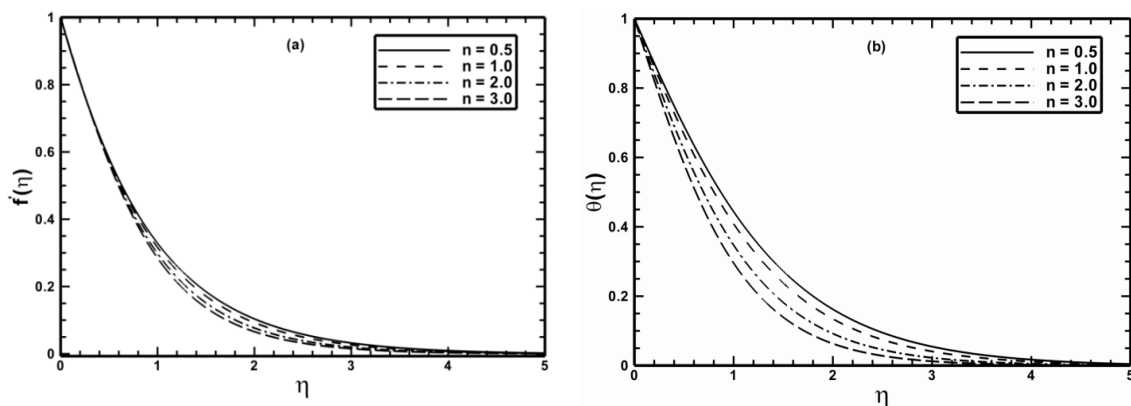

Figure 2 Effects of the stretching parameter n on the velocity and temperature profiles when $\alpha_1 = \alpha_2 = \beta = 0.2$ and $Pr = 0.7$ are fixed.

Figure 3 elucidates the effects of the viscoelastic parameter α_1 on the flow and heat transfer in a third grade fluid. This figure reveals that α_1 impacts the velocity boundary layer strongly as compared to the thermal boundary layer. **Figure 3** shows that the velocity and boundary layer thickness amplify due to retarded diffusion in momentum transfer but opposite effects are observed for the thermal boundary layer. The effect of the cross-viscous parameter α_2 on the flow and heat transfer is presented through **Figure 4**. The figure shows that the behavior of velocity and temperature profiles are contrary to the effect of viscoelastic parameter.

The influence of third grade parameter β on the velocity and thermal boundary layers is demonstrated in **Figure 5**. The augmentation in the non-dimensionless third grade parameter β results in thickening of the boundary layer due shear thickening effects of the fluid (**Figure 5(a)**). But, β affects the thermal boundary layer oppositely by decreasing the temperature profile, as the shear thickening retards the heat transfer (**Figure 5(b)**).
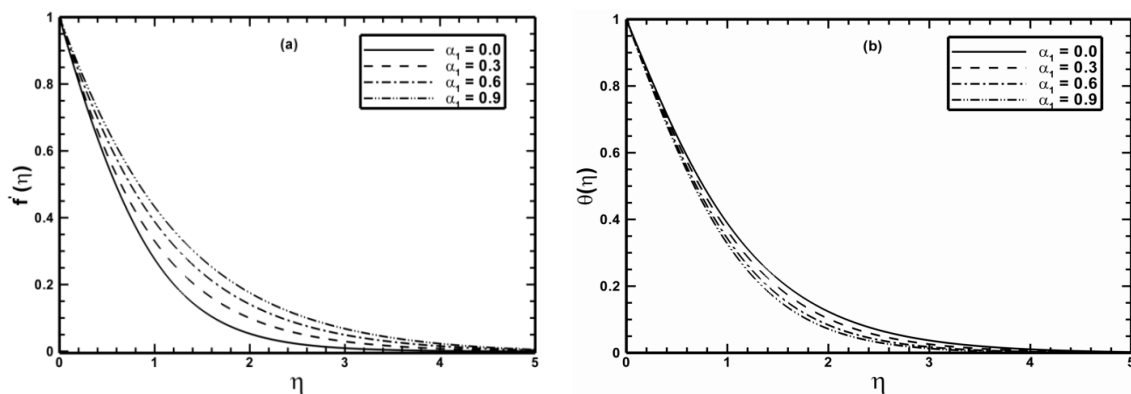

Figure 3 Effect of the viscoelastic parameter α_1 on the velocity and temperature profiles when $n = 1.5, \alpha_2 = \beta = 0.2$ and $Pr = 0.7$ are fixed.
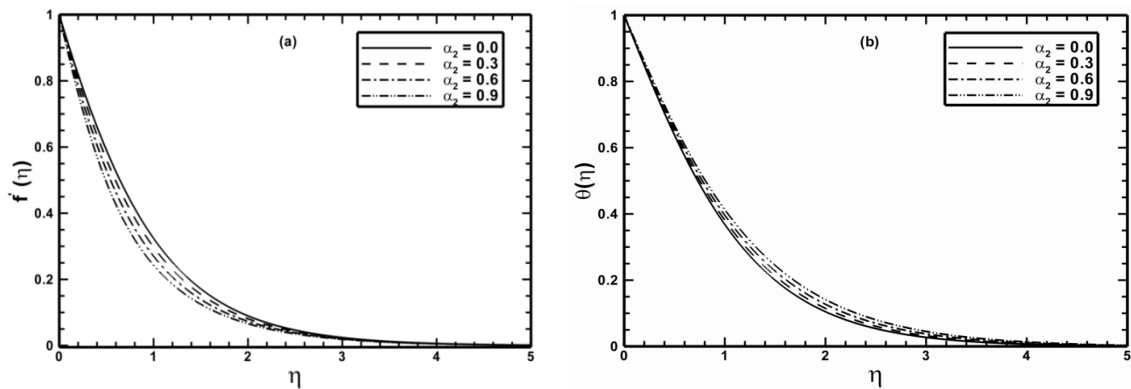

Figure 4 Effects of the cross-viscous parameter α_2 on the velocity and temperature profiles when $n = 1.5, \alpha_1 = \beta = 0.2$ and $Pr = 0.7$ are fixed.
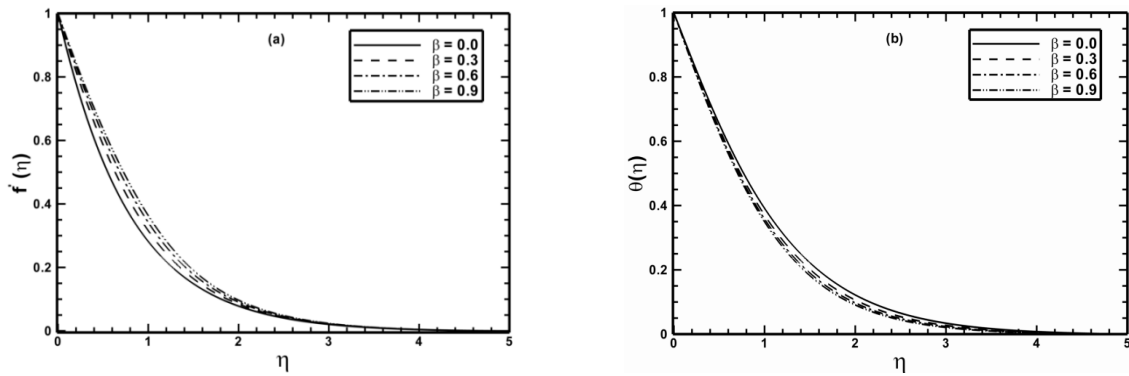
Figure 5 Effects of the third grade parameter β on the velocity and temperature profiles when $n = 1.5, \alpha_1 = \alpha_2 = 0.2$ and $Pr = 0.7$ are fixed.

The Prandtl number, Pr significantly controls the thermal boundary layer. Graphs in **Figure 6** elucidate the behavior of the thermal boundary layer thickness while Pr is enhanced. It is quite obvious from this figure that for the reduction in the thermal boundary layer is more due to the smaller heat diffusivity in the shear thickening fluid.
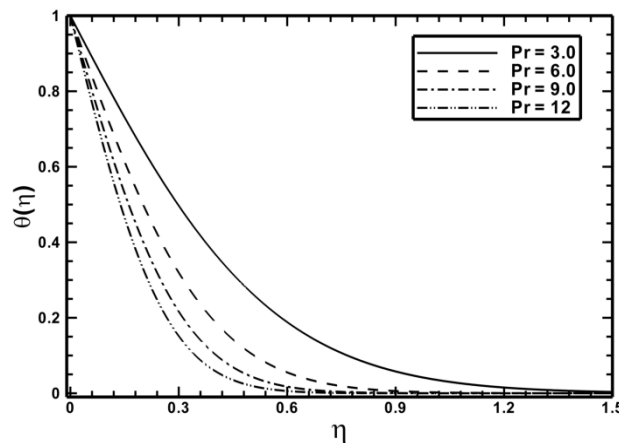

Figure 6 Effect of Prandtl number Pr on temperature profile when $n = 1.5, \alpha_1 = \alpha_2 = 0.2, \beta = 0.2$ are fixed.

In order to check the validity and accuracy of the present HAM results, we have plotted the discrete squared residual for different values of stretching parameter *n* as shown in **Figure 7**. The discrete squared residual is kept less than 10^{-5}. It is observed that the smaller values of *n* converges rapidly as compared to the larger values.

Table 1 summarizes the effect of the stretching parameters *n*, viscoelastic parameter α_1, cross-viscosity parameter α_2, third grade parameter β and Prandtl number *Pr* on skin friction coefficient and local Nusselt number. A detail examination of this table reveals the following overall trends: the skin friction coefficient and local Nusselt number increase, when *n* , α_1 , β and *Pr* are increased. A decrease in

the skin friction is observed with an increase in α_2. This table also shows that an increase in *Pr* greatly boosts the heat transfer at the wall.
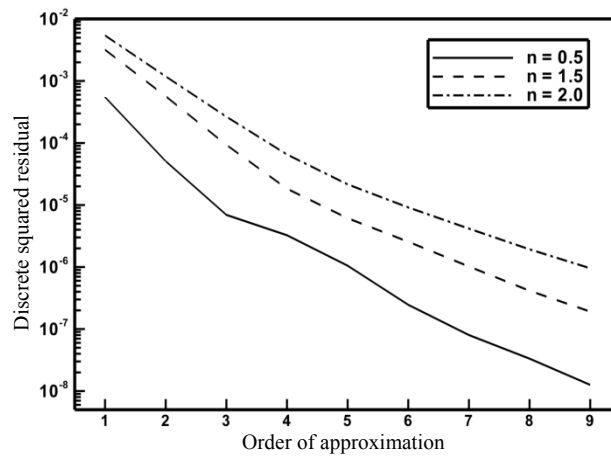

Figure 7 Variation of discrete squared residual with order of approximation.

Table 1 Numerical values of the skin friction coefficient and local Nusselt number.

n	α_1	α_2	β	*Pr*	$\frac{1}{2}\mathrm{Re}^{1/2}\,C_f$	$\mathrm{Re}^{-1/2}\,Nu_r$
1	0.2	0.2	0.2	0.7	-1.44852	-0.70260
2	0.2	0.2	0.2	0.7	-2.11553	-0.80551
3	0.2	0.2	0.2	0.7	-2.77500	-0.90224
1.5	0.3	0.2	0.2	0.7	-2.02300	-0.76842
1.5	0.6	0.2	0.2	0.7	-2.65442	-0.79956
1.5	0.9	0.2	0.2	0.7	-3.19523	-0.82091
1.5	0.2	0.0	0.2	0.7	-2.02145	-0.76645
1.5	0.2	0.3	0.2	0.7	-1.65775	-0.74859
1.5	0.2	0.6	0.2	0.7	-1.25058	-0.72804
1.5	0.2	0.2	0.2	0.7	-1.78330	-0.75483
1.5	0.2	0.2	0.4	0.7	-1.87524	-0.76946
.5	0.2	0.2	0.6	0.7	-1.94893	-0.78035
1.5	0.2	0.2	0.2	3.0		-1.83724
1.5	0.2	0.2	0.2	6.0		-2.70159
1.5	0.2	0.2	0.2	9.0		-3.36318

Conclusions

In this paper, we have analyzed the flow and heat transfer in a third grade fluid over a nonlinear radially stretching sheet. The analytical solutions are obtained with great accuracy. The computations have indicated that;

- An increase in the stretching parameter *n* significantly enhances the convective heat transfer at the wall.

- A rise in the viscoelastic parameter α_1 thickens the velocity boundary layer and reduces the thermal boundary layer thickness.

- A variation in the cross-viscous parameter α_2 reduces the skin friction at the wall and local Nusselt number.

- The third grade parameter β increases the momentum boundary layer thickness and heat transfer at the wall.

Acknowledgements

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