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Solving a Parabolic Inverse Source Problem by the Sinc-Galerkin Method

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Abstract

In this paper, the inverse problem of determining an unknown source term in a parabolic equation with Neumann boundary conditions and final measured data is considered. Initially, the unknown source term is estimated in the form of a combination of orthogonal functions. Since this inverse problem is ill-posed, the Tikhonov regularization technique is applied to find a stable solution. Then, a Sinc-Galerkin system is assembled to solve the direct problem. The approximate solution displays an exponential convergence rate. At the end of the paper, the proposed method is tested on 2 examples.

Keywords: Nonhomogeneous parabolic equation, inverse problem, unknown source term, Sinc-Galerkin method, Neumann boundary condition

Introduction

Inverse problems arise from many branches of science and engineering, which aim to detect some unknown parameters from some additional data related to those problems. These types of problems have been investigated in many recent papers [1-11]. In the present paper, we consider the following problem of determining $u(x,t)$ which satisfies the nonhomogeneous parabolic equation;

$$u_t - u_{xx} = s(x,t), \qquad 0 < x < 1,\ t > 0, \tag{1}$$

with the zero initial and boundary conditions, due to the fact that, it is more convenient for the numerical method that will be presented. Suppose that the initial and boundary conditions are as follows;

$$u(x,0) = u_0(x), \qquad 0 < x < 1, \tag{2}$$

$$u_x(0,t) = \psi(t), \quad u_x(1,t) = \varphi(t), \qquad t > 0, \tag{3}$$

where $u_0(x)$, $\psi(t)$ and $\varphi(t)$ are piecewise continuous functions in their domains and these functions satisfy the consistency conditions $u_0(0) = \psi(0)$, $u_0(1) = \varphi(0)$; then, we can apply the change of variables;

$$u(x,t) = v(x,t) + \left(\frac{\varphi(t) - \psi(t)}{2}\right)x^2 + \psi(t)x, \tag{4}$$

$$v(x,t) = w(x,t) + u_1(x)e^{-t}, \tag{5}$$

where $u_1(x)$ is the modified initial condition after using the first transformation. Thus, without loss of generality, we can consider a problem with zero initial and boundary conditions;

$$u_t - u_{xx} = s(x,t), \qquad 0 < x < 1,\ t > 0, \tag{6}$$

$$u(x,0) = 0, \qquad 0 < x < 1, \tag{7}$$

$$u_x(0,t) = 0,\ u_x(1,t) = 0, \qquad t > 0. \tag{8}$$

This problem is induced in the process of transportation, diffusion and conduction of natural materials [12,13].

In this paper, in addition to the function $u(x,t)$, the source term $s(x,t)$ is also unknown. This problem is called an inverse source problem [1]. Also for natural systems modeling, these types of problems play important roles. For example, in the modeling of air pollution phenomena, to determinate the unknown source term and the environmental protection, $s(x,t)$ is considered as an unknown source of pollutant [10,14].

The source term formation which is discussed in our inverse problem is;

$$s(x,t) = f(x)g(x,t) + h(x,t), \qquad 0 < x < 1, t > 0, \tag{9}$$

where $g(x,t)$ and $h(x,t)$ are known functions in their domains and $f(x)$ is unknown which remains to be determined. Such problems have been investigated by many researchers theoretically [3-11]. As mentioned in these references, an overspecified condition was also considered available at the time $t = T$. In this work, the additional condition is taken to be point evaluations, given as follows;

$$u(x_i, T) = u_i, \qquad i = 1, \ldots, N, \tag{10}$$

where $0 < x_i < 1,\ i = 1, \ldots, N$ are N points in the $(0,1)$. In the rest of this paper, by using an extra condition (10), a numerical algorithm is presented for solving this inverse problem based on the Sinc-Galerkin method.

The Sinc-Galerkin method was first presented by Stenger in [15]. Since then, the Sinc-Galerkin method has been applied to a variety of partial differential equations [11,15-20]. References [16,17] provide excellent overviews of existing methods based on Sinc functions. Fully Sinc-Galerkin techniques use a Sinc function basis in both space and time. This method exhibits an exponential order of convergence [16,17].

The paper is organized as follows:

In Section 2, the direct problem will be considered and by using the solution of this problem, a method will be introduced for estimating the unknown function $f(x)$. In section 3, some properties of the Sinc function and Sinc quadrature rule will be presented and Sinc-Galerkin system for solving the direct problem is constructed. Finally, some numerical examples are presented in section 4.

A regularization method for estimating the unknown source term

In order to solve the inverse problem (6) - (8) according to the condition (10), the unknown function $f(x)$ must firstly be identified. The following theorem shows unique solvability of the direct problem (6) - (8).

Theorem 1. Consider the problem (6) - (8). If the source function $s(x,t)$ is bounded over its domain and is uniformly Holder continuous on each compact subset of this domain, this problem has a unique bounded solution.

Proof. Refer to [13].

By using the separation of variables, the solution mentioned at this theorem may be expressed as follows;

$$u(x,t) = 2\sum_{i=0}^{\infty}\left(\int_0^t\int_0^1 s(\xi,\tau)\cos(i\pi\xi)e^{-(i\pi)^2(t-\tau)}d\xi d\tau\right)\cos(i\pi x). \tag{11}$$

For obtaining an estimation of $f(x)$ in $C(0,1)$, we can consider a finite dimensional approximation based on the independent functions $x^{j-1}, j=1,...,N$ by the assumptions on the source function in the Theorem 1, where N is the number of points in (10). Assume;

$$f(x) \sim \sum_{j=1}^{N} c_j x^{j-1}. \tag{12}$$

The real coefficients c_j are determined by substituting (12) in the exact solution (11) and by putting $x = x_i, i = 1,...,N,$ from the additional condition (10) as;

$$u_i = u(x_i,T) = 2\sum_{i=1}^{\infty}\left(\int_0^T\int_0^1 s(\xi,\tau)\cos(i\pi\xi)e^{-(i\pi)^2(T-\tau)}d\xi d\tau\right)\cos(i\pi x_i). \tag{13}$$

This leads to a system of N equations to obtain N unknown coefficients c_j.

Suppose that $X=(c_i)$, $U=(u_i)$, $D=(d_i)$, $i=1,...,N$ where;

$$d_i = 2\sum_{k=1}^{\infty}\Big(\int_0^T\int_0^1 h(\xi,\tau)\cos(k\pi\xi)e^{-(k\pi)^2(T-\tau)}d\xi d\tau\Big)\cos(k\pi x_i), \tag{14}$$

and $A=(a_{ij})$, $i,j=1,...,N$ where;

$$a_{ij} = 2\sum_{k=1}^{\infty}\Big(\int_0^T\int_0^1 \xi^{j-1}g(\xi,\tau)\cos(k\pi\xi)e^{-(k\pi)^2(T-\tau)}d\xi d\tau\Big)\cos(k\pi x_i). \tag{15}$$

Also, let's consider $B=U-D$, then the above system of equations will be in the form of;

$$AX = B. \tag{16}$$

The Tikhonov regularization method is applied to find the solution of system (16). Since the extra condition (10) comes from practical measurements that are contaminated with random noise, numerical reconstruction of the solution of system (16) is very difficult. By this technique, we have a minimization problem [21] as;

$$\min_{X\in R^N} \| AX-B\|^2 +\alpha \| X\|^2, \tag{17}$$

in which $\alpha > 0$ is a regularization parameter which controls the trade-off between fidelity to the data and smoothness of the solution. Different methods have been applied for determining the regularization parameter. The method which we apply is the *L*-curve method. The *L*-curve is a plot of the squared estimate norm of the regularized solution $\| X \|$ against the squared norm of the regularized residual $\|AX\text{-}B\|$ for a range of values of regularization parameters [22-24].

Solving the direct problem

The Sinc function is defined on the whole real line by;

$$Sinc(x)=\begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x\neq 0,\\ 1, & x=0. \end{cases} \tag{18}$$

All Sinc methods are based on the use of the Cardinal function $C(f,h)$ which is a Sinc expansion of f, defined by;

$$C(f,h)(x)=\sum_{k\in Z} f(kh)Sinc\left\{\frac{x}{h}-k\right\},\ x\in R, \tag{19}$$

for $h>0$.

To construct approximations using the Sinc functions on the intervals $(0,1)$ and $(0,\infty)$, we use the composition of these functions with the conformal maps $\phi(x)=ln(\frac{x}{1-x})$ and $\gamma(t)=ln(t)$. Thus, we get the translated Sinc functions with evenly spaced nodes as;

$$S_i(x)=Sinc\left(\frac{\phi(x)-ih}{h}\right),\ S'_j(t)=Sinc\left(\frac{\gamma(t)-jk}{k}\right), \tag{20}$$

where $h>0$ and $k>0$ are real numbers, i and j are integers.

The Sinc basis functions tend to zero as *t* approaches to ∞. Also, in the case of Neumann boundary conditions, the x partial derivatives of these functions are undefined at $x=0$ and $x=1$. To remedy these, we first modify the Sinc basis functions and add some additional basis functions [17]. According to this, suppose that;

$$\xi_i(x)=\begin{cases} (2x+1)(x-1)^2 & i=-M_x-1,\\ \dfrac{S_i(x)}{\phi'(x)}, & i=-M_x,\dots,N_x,\\ (-2x+3)x^2, & i=N_x+1, \end{cases} \tag{21}$$

and

$$\xi_j(t)=\begin{cases} S_j(t), & j=-M_t,\dots,N_t, \\ \dfrac{t}{t+1}, & j=N_t+1, \end{cases} \tag{22}$$

where M_x, M_t, N_x and N_t are positive integers.

The approximate solution for $u(x,t)$ in (1) is defined by the expansion;

$$u_{m_x,m_t}(x,t)=\sum_{i=-M_x-1}^{N_x+1}\sum_{j=-M_t}^{N_t+1} u_{ij}\xi_i(x)\xi_j(t), \tag{23}$$

where $m_x=M_x+N_x+3$, $m_t=M_t+N_t+2$. The unknown coefficients u_{ij} are determined by;

$$\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-f(x)g(x,t)-h(x,t)\ ,\ S_i(x)S_j'(t)\right)=0. \tag{24}$$

The inner product in (24) is defined by;

$$(p,q)=\int_0^\infty\int_0^1 p(x,t)q(x,t)v(x)w(t)dxdt. \tag{25}$$

For approximating this inner product, we use the Sinc quadrature rule for double integrals [20].

Suppose that the weight function in the inner product (25) is;

$$v(x)w(t)=\frac{\sqrt{\gamma'(t)}}{\phi'(x)}. \tag{26}$$

A complete discussion on the choice of the weight function can be found in [17]. Substituting (23) in (24) and applying the Sinc quadrature rule, results in a system of $M_x \times M_t$ equations for unknown coefficients u_{ij}, $i=-M_x-1,\dots,N_x+1,\ j=-M_t,\dots,N_t+1$. The following theorem shows the exponential convergence of the Sinc-Galerkin method for solving the mentioned problem.

Theorem 2. Consider the maps φ and γ defined in Theorem 1. For the weight function (26), assume that $F/\phi' \in B(D_E)$ and that $uH \in B(D_E)$, where;

$$H=(1/\phi')'',(\phi''/\phi'),\phi'. \tag{27}$$

Also, assume that $F\sqrt{\gamma'} \in B(D_W)$, and that $uH \in B(D_W)$, where;

$$H=\sqrt{\gamma'},(\gamma')^{3/2},\gamma''/\sqrt{\gamma'}. \tag{28}$$

Further, assume;

$$|u(x,t)|\leqslant Cx^{\alpha_x}(1-x)^{\beta_x}t^{\alpha_t+1/2}t^{-\beta_t+1/2},\qquad (x,t)\in(0,1)\times(0,\infty), \tag{29}$$

making the selections;

$$N_x = \left[\left|\frac{\alpha_x}{\beta_x} M_x + 1\right|\right], \quad M_t = \left[\left|\frac{\alpha_x}{\alpha_t} M_x + 1\right|\right], \quad N_t = \left[\left|\frac{\alpha_x}{\beta_t} M_x + 1\right|\right], \tag{30}$$

where $h = h_x = h_t$, and;

$$h = \sqrt{\pi d / (\alpha_x M_x)}, \tag{31}$$

results in;

$$\left\| u - u_{m_x, m_t} \right\|_\infty \leqslant K M_x^2 \exp(-(\pi d \alpha_x M_x)^{1/2}). \tag{32}$$

Proof. [17].

The r'th derivative of $S_i(x)$ with respect to φ, at the nodal point $x = x_j = j\, h_x,\ j = -M_x, ..., N_x$ is [17];

$$\delta_{ij}^{(r)} = h^r \frac{d^r}{d\phi^r}[S(i, h_x) o \phi(x)]|_{x=x_j},\ r = 0,1,2,... \tag{33}$$

Thus, for $r = 0,1,2$, we have;

$$\delta_{ij}^{(0)} \equiv [S(i, h_x) o \phi(x)]|_{x=x_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{34}$$

$$\delta_{ij}^{(1)} \equiv h \frac{d}{d\phi}[S(i, h_x) o \phi(x)]|_{x=x_j} = \begin{cases} 0, & i = j, \\ \dfrac{(-1)^{j-i}}{j-i}, & i \neq j, \end{cases} \tag{35}$$

$$\delta_{ij}^{(2)} \equiv h^2 \frac{d^2}{d\phi^2}[S(i, h_x) o \phi(x)]|_{x=x_j} = \begin{cases} \dfrac{-\pi^2}{3}, & i = j, \\ \dfrac{-2(-1)^{j-i}}{(j-i)^2}, & i \neq j, \end{cases} \tag{36}$$

Similarly, these definitions can be introduced for $S_j'(t)$ as well.

Let;

$$I_x^{(l)} \equiv [\delta_{ij}^{(l)}]_{m_x \times m_x},\ l = 0,1,2, \tag{37}$$

$$I_t^{(l)} \equiv [\delta_{ij}^{(l)}]_{m_t \times m_t},\ l = 0,1. \tag{38}$$

Also, let $D(e(x))$ denote the diagonal matrix with;

$$D(e(x))_{ij} = \begin{cases} e(x_i), & i = j, \\ 0 & i \neq j \end{cases}. \tag{39}$$

Consider the $(m_x - 2) \times (m_x - 2)$ matrix;

$$A_0 = -\frac{1}{h_x^2} I_x^{(2)} + \frac{1}{h_x} I_x^{(1)} D\left(\frac{\phi''(x)}{(\phi'(x))^2}\right) + D\left(\frac{1}{\phi'(x)}\left(\frac{1}{\phi'(x)}\right)''\right), \tag{40}$$

and the $(m_t - 1) \times (m_t - 1)$ matrix;

$$B_0 = \left[-\frac{1}{h_t} I_t^{(1)} + D\left(\frac{1}{2}\right)\right] D\left(\sqrt{\gamma'(t)}\right). \tag{41}$$

In this notation, the system of equations for unknown coefficients u_{ij} in (23), from the Sinc-Galerkin approach, takes the matrix form;

$$ACD_t + D_s CB^T = F. \tag{42}$$

The $m_x \times m_x$ matrices A and D_s are given by;

$$A = \left[\vec{a}_{-M_x-1} \,|\, \overline{A_0} \,|\, \vec{a}_{N_x+1}\right], \tag{43}$$

$$D_s = \left[\vec{d}_{-M_x-1} \,|\, D\left(\frac{1}{(\phi')^3}\right) |\, \vec{d}_{N_x+1}\right], \tag{44}$$

and the column vectors $\vec{a}_{-M_x-1}$, $\vec{a}_{N_x+1}$, $\vec{d}_{-M_x-1}$ and $\vec{d}_{N_x+1}$ have the components;

$$(a_{-M_x-1})_j = \left(\frac{w_a''}{(\phi')^2}\right)(x_j),\ (a_{N_x+1})_j = \left(\frac{w_b''}{(\phi')^2}\right)(x_j), \tag{45}$$

$$(d_{-M_x-1})_j = \left(\frac{w_a}{(\phi')^2}\right)(x_j),\ (d_{N_x+1})_j = \left(\frac{w_b}{(\phi')^2}\right)(x_j), \tag{46}$$

for $j = -M_x - 1, ..., N_x + 1$.
Also the $m_t \times m_t$ matrices B and D_s are given by;

$$B=\left[\overline{B_0}\,|\,\vec{b}_{N_t+1}\right],\ D=\left(\frac{1}{\sqrt{\gamma'(t)}}\right),\ A=\left[\begin{array}{c|c} D\left(\frac{1}{\sqrt{\gamma'(t)}}\right) & \vec{0} \\ \hline \multicolumn{2}{c}{\vec{d}^{\,T}_{N_t+1}} \end{array}\right]. \tag{47}$$

The components of the column vectors $\vec{b}_{N_t+1}$ and $\vec{d}_{N_t+1}$ are as;

$$(b_{N_t+1})_k=\left(\frac{-w'_\infty}{\sqrt{\gamma'}}\right)(t_k),\ (d_{N_t+1})_k=\left(\frac{-w_\infty}{\sqrt{\gamma'}}\right)(t_k), \tag{48}$$

for $k=-M_t,...,N_t+1$.

The matrix $\overline{A_0}$ is a $m_x\times(m_x-2)$ copy of A_0, and $\overline{B_0}$ is a $m_t\times(m_t-1)$ copy of B_0. Also the $(m_x+2)\times(m_t+1)$ matrix F contains the evaluation of $s(x,t)$ at the points (x_i,t_j) for $i=-M_x-1,...,N_x+1$ and $j=-M_t,...,N_t+1$.

Results and discussion

In this section, 2 examples are presented to verify the numerical approach. First, the values of h, M_x, N_x, M_t and N_t are chosen by theorem 2. Also the parameters α_x, β_x, α_t and β_t are determined for a given problem with a known solution, by applying the relation (29). Note that, in practice, we place $\alpha_x=\beta_x=\alpha_t=\beta_t=1$ and $d=\pi/2$. Afterward from the relationship (30), we take $M_x=N_x=M_t=N_t=M$ and $\frac{\pi}{\sqrt{2M}}$.

To generate the noisy data, the relationship;

$$U^\delta=(u_i^\delta)=(u_i)+\delta.randn(N), \tag{49}$$

will be used, where $U=(u_i),\ i=1,...,N$ are additional data, $randn(.)$ is a normal distribution function with zero mean and unit standard deviation and δ indicates the level of noise.

Example 1. In the first example, we consider the Eq. (1) with $g(x,t)=1$, and $h(x,t)=0$. By these assumptions, the true source function and the exact solution will be $f(x)=\cos(\pi x)$ and $u(x,t)=\frac{(1-e^{-\pi^2 t})cos(\pi x)}{\pi^2}$, respectively. We consider the additional data in the eight equidistant points $(j\,h,1),\ j=1,...,8,$ with $h=\frac{1}{9}$. The $L-curve$ for determining the regularization parameter is shown in **Figure 1** at various noise levels. The exact source function $f(x)$ and its approximations are shown in **Figure 2**, and the absolute errors are listed in **Table 1**. Also the Sinc-Galerkin method is applied to provide a solution of the direct problem. Observe **Figure 3** and **Table 2** to compare the exact solution and the approximate solution of the problem.

Table 1 The absolute errors between the exact and approximate values of the source function in example 1 for the noise levels $\delta = 10^{-3}$ and $\delta = 10^{-4}$.

x	$\delta = 10^{-3}$	$\delta = 10^{-4}$
0.1	9.03619×10^{-3}	2.75252×10^{-3}
0.2	3.65715×10^{-2}	3.23481×10^{-3}
0.3	3.27526×10^{-2}	1.05494×10^{-3}
0.4	3.04045×10^{-3}	1.61805×10^{-3}
0.5	2.09633×10^{-2}	3.49844×10^{-3}
0.6	3.56338×10^{-2}	2.67991×10^{-3}
0.7	1.32249×10^{-2}	9.82808×10^{-3}
0.8	2.83532×10^{-2}	1.03352×10^{-3}
0.9	4.38587×10^{-2}	3.48541×10^{-3}

Table 2 The approximate solution and its absolute errors in example 1 at various *x* and *t*.

x	*t*	Approximate solution	Exact solution	Absolute error
	1	0.0974443	0.0976357	1.08711×10^{-3}
0.1	2	0.0971346	0.0963622	7.72399×10^{-4}
	3	0.0968573	0.0963622	4.95173×10^{-4}
	4	0.0959462	0.0963622	4.15932×10^{-4}
	1	0.0599614	0.0595520	4.09414×10^{-4}
0.3	2	0.0595740	0.0595551	1.88805×10^{-5}
	3	0.0592736	0.0595551	2.81515×10^{-4}
	4	0.0583558	0.0595551	1.19928×10^{-3}
	1	-0.0306047	-0.0313083	7.03611×10^{-4}
0.6	2	-0.0311836	-0.0313100	1.26336×10^{-4}
	3	-0.0315407	-0.0313100	2.30728×10^{-4}
	4	-0.0324753	-0.0313100	1.16528×10^{-3}
	1	-0.0968924	-0.0963572	5.35192×10^{-4}
0.9	2	-0.0976088	-0.0963622	1.24662×10^{-3}
	3	-0.0980068	-0.0963622	1.64463×10^{-3}
	3	-0.0989532	-0.0963622	2.59102×10^{-3}

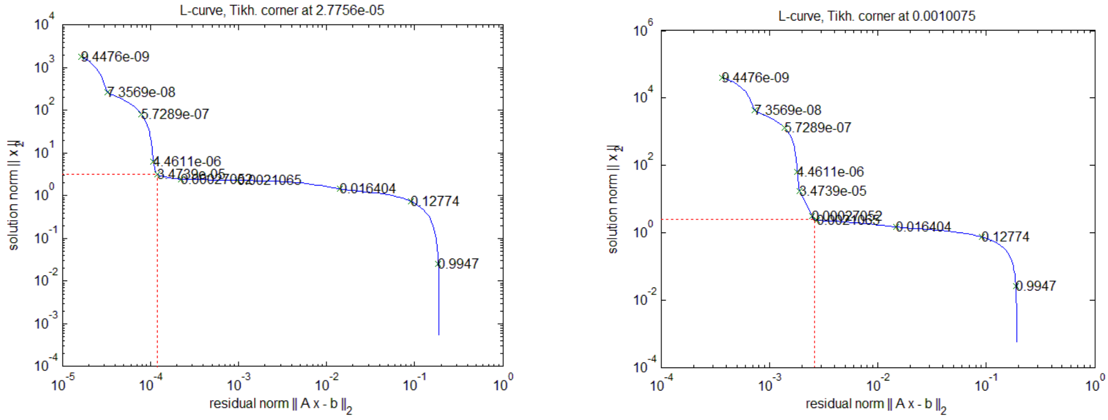

Figure 1 The L-curve plots with the noise levels, Left: $\delta = 10^{-4}$ and Right: $\delta = 10^{-3}$ for example 1.

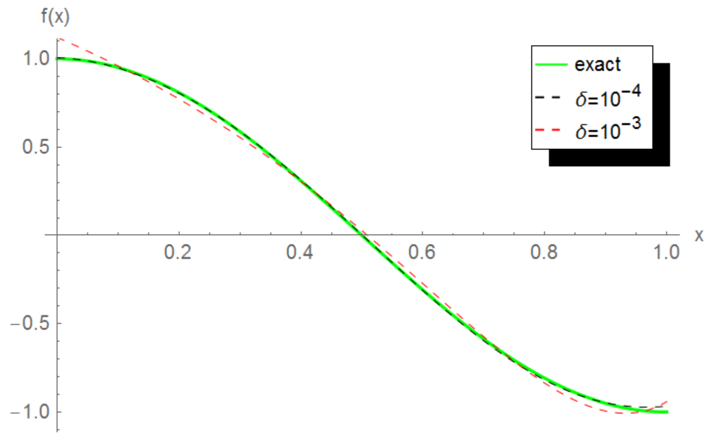

Figure 2 Exact source function (Green) and its approximations in example 1 for the noise level $\delta = 10^{-3}$ (Red) and $\delta = 10^{-4}$ (Black).

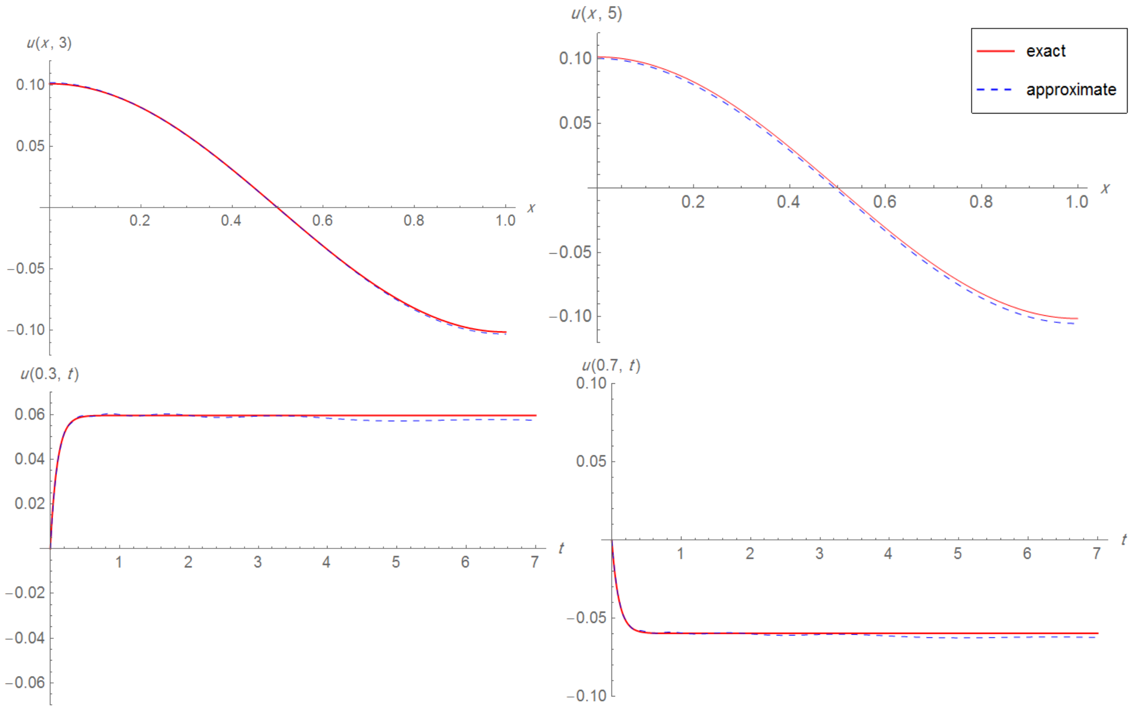

Figure 3 The exact solution (Red) and the approximate solutions (Blue) of example 1 with M = 16 and $\delta = 10^{-3}$ for various values of *t* and *x*.

Example 2. In this example, we assume that $g(x,t) = \mathrm{e}^{-t}((-1+x)^2 x^2 - 2t(1+x(-4+x(1+x)^2)))$ and $h(x,t) = 0$. By this assumption, we have $u(x,t) = tx^2(1-x)^2 e^{x-t}$ and $f(x) = e^x$. The additional data is considered in the 10 equidistant points with $h = \frac{1}{11}$. A random noise $\delta.randn(10)$ is added to this data. The exact source function and its approximations are shown in **Figure 4**. **Figure 5** demonstrates the $L-curve$ of the regularization parameter determination for various noise levels. **Figure 6** shows the exact and approximate solutions of the problem. Finally, **Table 3** displays the absolute error of this approximate solution for various values of x when $t = 4$ and **Figure 7** exhibits the error function of the approximate solution in the domain $\{(x,t) \,|\, 0 < x < 1, 0 < t < 10\}$.

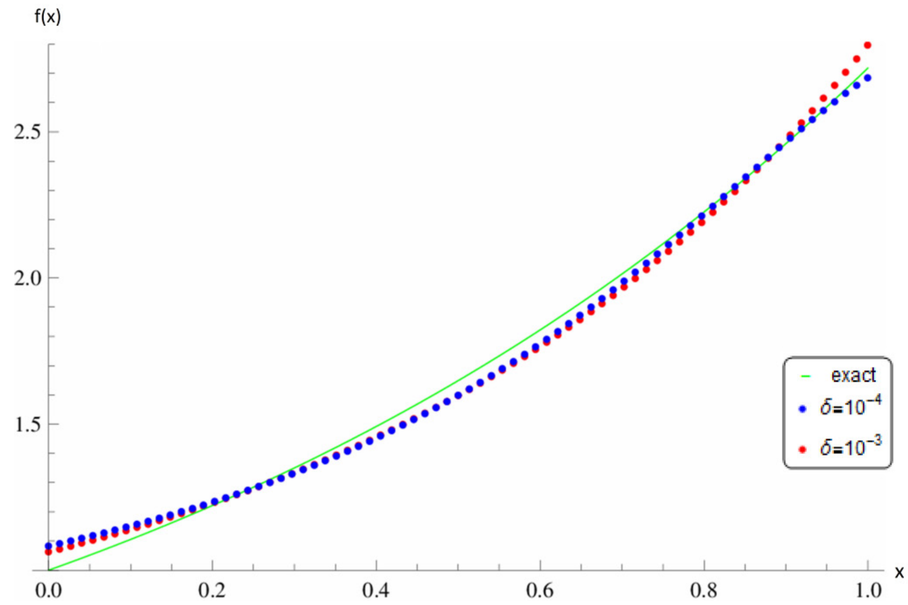

Figure 4 Exact source function (Green) and its approximations in example 2 for the noise levels $\delta = 10^{-3}$ (Red) and $\delta = 10^{-4}$ (Blue).

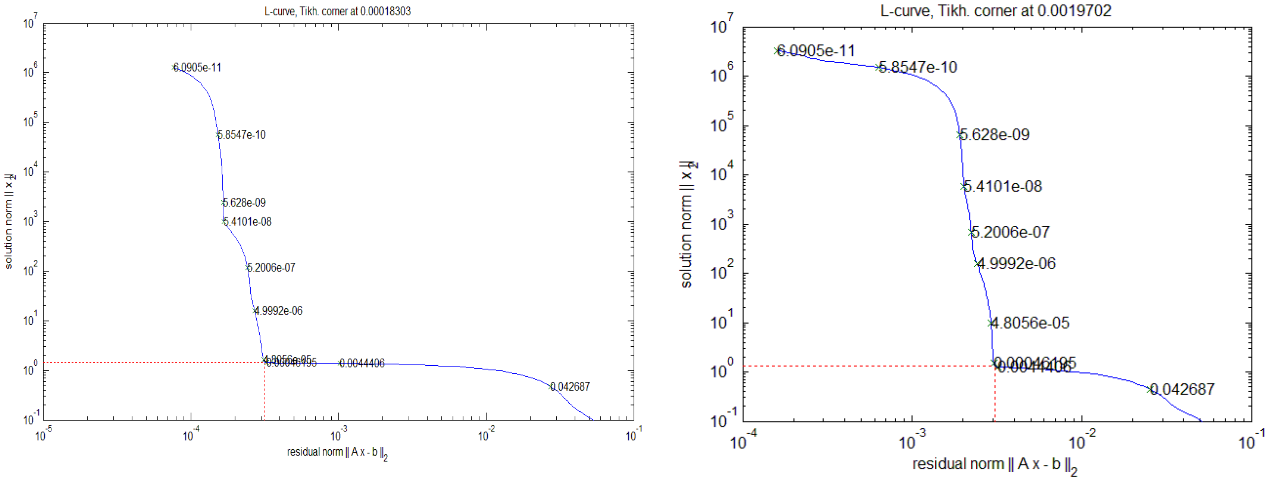

Figure 5 The L-curve plots of example 2 for data with the noise levels $\delta = 10^{-4}$ (Left) and $\delta = 10^{-3}$ (Right).

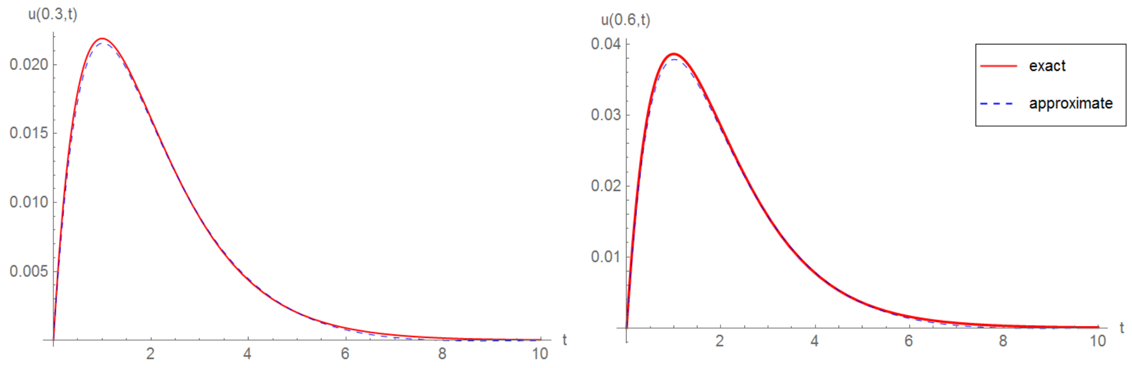

Figure 6 The exact solution (Red) and the approximate solution (Blue) of example 2 with M = 20 and $\delta = 10^{-3}$.

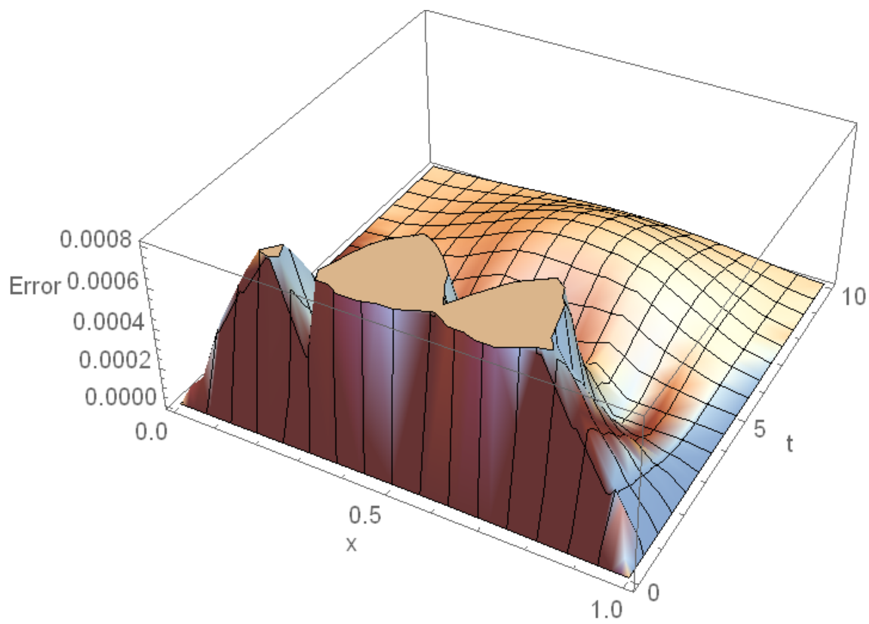

Figure 7 The error function of the approximate solution of example 2 for the domain $\{(x,t)\,|\,0 < x < 1, 0 < t < 10\}$.

Table 3 The approximate solution and its absolute errors in example 2 when $t = 4$.

x	Exact solution	Approximate solution	Absolute error
0.1	0.00065584	0.00067081	1.49755×10^{-5}
0.2	0.00229077	0.00220283	8.79386×10^{-5}
0.3	0.00436123	0.00448550	1.24272×10^{-4}
0.4	0.00629539	0.00618306	1.12326×10^{-4}
0.5	0.00754935	0.00758069	3.13452×10^{-5}
0.6	0.00768920	0.00786601	1.76806×10^{-4}
0.7	0.00650619	0.00641814	8.80493×10^{-5}
0.8	0.00417405	0.00426625	9.22019×10^{-5}
0.9	0.00145959	0.00137995	7.96435×10^{-5}

Conclusions

In this paper, a numerical method for solving an inverse source problem was proposed. First, the unknown space-dependent source term was approximated as a finite dimensional combination of orthogonal functions. This approach led to a system of equations. Since this system was ill-posed, the Tikhonov regularization technique was applied to find a stable solution. Then, the solution of the problem was obtained by using a numerical algorithm based on the Galerkin method with the Sinc basis functions in both space and time domains. Finally, some numerical test examples were presented to verify the applicability and efficiency of the method.

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