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On Classical Quasi-Primary Radical of Submodules and Classical Quasi-Primary Radical Formula of Submodules

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Abstract

In this paper we characterize the classical quasi-primary radical of submodules and classical quasi-primary radical formula of modules over commutative rings with identity. These are extended from radical, radical primary, and radical formula of submodules, respectively. Finally, we obtain necessary and sufficient conditions of a submodule in order to be a top classical quasi-primary radical formula of submodules.

Keywords: Classical quasi-primary submodule, classical quasi-primary radical of submodule, classical quasi-primary radical formula of submodule

Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. We recall that a proper ideal P of R is called a *primary ideal* if $ab \in P,$ where $a,b \in R,$ implies that either $a \in P$ or $b^n \in P,$ for some positive integer n [1]. The notion of primary ideal was generalized by Fuchs [1] through defining an ideal P of a ring R to be it *quasi primary* if its radical is a prime ideal, i.e., if $ab \in P,$ where $a,b \in R,$ then either $a^n \in P$ or $b^n \in P,$ for some positive integer n (see also [2,3]). A subset N of the R-module M is a submodule, expressed $N \leq M$, if it is a subgroup that is closed under the R-action: For all $r \in R$ and $n \in N$, we have $rn \in N.$ We say that a submodule N of M is proper if $N \neq M.$ A proper submodule N of an R-module M is a *primary submodule* of M if for $m \in M$ and $r \in R$ such that $rm \in N,$ then $m \in N$ or $r \in \sqrt{(N:M)} = \{a \in R \mid a^n M \subseteq N,$ for some positive integer $n\}$. An R-module M is a *primary module* if every proper submodule N of M is a primary submodule of M (see, for example, [4-8]). A *classical primary submodule* in M as a proper submodule N of M such that if $abK \subseteq N$, where $a,b \in R$ and $K \leq M$, then either $aK \subseteq N$ or $b^n K \subseteq N$ for some positive integer $n.$ Clearly, in case $M = R,$ where R is any commutative ring, classical primary submodules coincide with primary ideals [9-11]. A *classical quasi-primary submodule* in M as a proper submodule N of M such that if $abK \subseteq N$, where $a,b \in R$ and $K \leq M,$ then either $a^n K \subseteq N$ or $b^n K \subseteq N$ for some positive integer $n.$ The idea of decomposition of submodules into classical primary submodules was introduced by Baziar and Behboodi in [9]. The *primary radical* of N in M, denoted by $prad_M(N),$ is defined to be the intersection of all primary submodules containing $N.$ Should there be a primary submodule of M containing $N,$ then we put $prad_M(N) = M.$ Radicals have been investigated in a number of papers, for example [1,2,12-14]. A *classical quasi-primary radical* of N in $M,$ denoted by $cprad_M(N),$ is defined to be the intersection of all classical quasi-primary submodules containing $N.$ Should there be a classical quasi-primary submodule of M containing $N,$ then we put $cprad_M(N) = M.$

In this note, we shall need the notion of the envelope of a submodule introduced by McCasland and Moore [15]. For a submodule N of an R-module $M,$ the envelope of N in M, denoted by $E_M(N),$ is defined to be the subset $\{rm : r \in R$ and $m \in M$ such that $r^k m \in N$ for some positive integer $n\}$ of M [15-17]. Note that, in general, $E_M(N)$ is not an R-module. With the help of envelopes, the notion of the classical primary radical formula is defined as follows: A submodule N of an R-module M is said to satisfy the classical quasi-radical formula in M, if $\langle E_M(N)\rangle = cprad_M(N).$ In this paper we introduce the notion of a classical quasi-primary radical of submodules and classical quasi-primary radical formula of modules over a commutative ring with identity. Finally, we obtain necessary and sufficient conditions of a submodule in order to be a top classical quasi-primary radical formula of submodules.

Basic results

Let $R = \prod_{i=1}^{n} R_i$ where each R_i is a commutative ring with identity. Then an ideal $I = \prod_{i=1}^{n} I_i$ of P is prime if and only if I_i is equal to the corresponding ring R_i and the other is prime. Moreover, any R-module M can be uniquely decomposed into a direct product of modules, i.e. $M = \prod_{i=1}^{n} M_i,$ where $M_i = (0,0,0,\ldots,0,1,0,\ldots 0)M$ is an R_i-module with action;

$$(r_1, r_2,\ldots,r_n)(m_1,m_2,\ldots,m_n) = (r_1m_1, r_2m_2,\ldots,r_nm_n), \tag{1}$$

where $r_i \in R_i$ and $m_i \in M_i$ [7].

Lemma 2.1 [17] Let $N = N_1 \times N_2$ be a submodule of M . Then;

$$\langle E_M(N)\rangle = \langle E_{M_1}(N_1)\rangle \times \langle E_{M_2}(N_2)\rangle. \tag{2}$$

Proof See [17].

Corollary 2.2 [17] Let $N = \prod_{i=1}^{n} N_i$ be a submodule of M . Then;

$$\langle E_M(N) \rangle = \prod_{i=1}^{n} \langle E_{M_i}(N_i) \rangle. \tag{3}$$

Proof See [17].

Lemma 2.3 [16] If N is a weakly prime submodule, then $\langle E_M(N) \rangle = N$. (4)

Proof. See [17].

Lemma 2.4 [16] Let N be a semiprime submodule of an R -module M . Then;

$$\langle E_M(N) \rangle = N. \tag{5}$$

Proof See [16].

Some basic properties of the classical quasi-primary submodules

The results of the following lemmas seem to be at the heart of the theory of classical quasi-primary submodules; these facts will be used so frequently that normally we shall make no reference to this lemma.

Definition 3.1 A proper submodule N of an R -module M is said to be a classical quasi-primary submodule of M if $abK \subseteq N,$ where $a,b \in R$ and $K \leq M,$ then either $a^n K \subseteq N$ or $b^n K \subseteq N$ for some positive integer $n.$

Lemma 3.2 Let $M = M_1 \times M_2,$ where M_i is an R_i-module. A submodule $N_1 \times M_2$ is a classical quasi-primary submodule of M if and only if N_1 is a classical quasi-primary submodule of $M_1.$

Proof Suppose that $N_1 \times M_2$ is a classical quasi-primary submodule of R -module $M.$ We will show that N_1 is a classical quasi-primary submodule of $M_1.$ Clearly, N_1 is a proper submodule of R_1-module $M_1.$ To see the classical quasi-primary property of $N_1,$ let K be a submodule of R_1-module M_1 and $a,b \in R_1$ such that $abK \subseteq N_1$. Then;

$$(a,0)(b,0)(K \times \{0\}) = abK \times \{0\} \subseteq N_1 \times M_2. \tag{6}$$

Since $N_1 \times M_2$ is a classical quasi-primary submodule of R -module M , it follows that;

$$(a^n K \times \{0\}) = (a,0)^n (K \times \{0\}) \subseteq N_1 \times M_2 \tag{7}$$

or

$$(b^n K \times \{0\}) = (b,0)^n (K \times \{0\}) \subseteq N_1 \times M_2 \tag{8}$$

for some positive integer n, that is $a^n K \subseteq N_1$ or $b^n K \subseteq N_1$, for some positive integer n. Therefore N_1 is a classical quasi-primary submodule of R_1-module M_1. Conversely, suppose that N_1 is a classical quasi-primary submodule of R_1-module M_1. We will show that $N_1 \times M_2$ is a classical quasi-primary submodule of R-module M. Clearly, $N_1 \times M_2$ is a proper submodule of R-module M. To see the classical quasi-primary property of $N_1 \times M_2$, let $K \times L$ be a submodule of R-module M and $(a_1,a_2),(b_1,b_2) \in R$ such that;

$$a_1b_1K \times a_2b_2L = (a_1,a_2)(b_1,b_2)(K \times L) \subseteq N_1 \times M_2. \tag{9}$$

Since N_1 is a classical quasi-primary submodule of R_1-module M_1 and $a_1b_1K \subseteq N_1$, we have $a_1^n K \subseteq N_1$ or $b_1^n K \subseteq N_1$, for some positive integer n. Therefore;

$$(a_1,a_2)^n (K \times L) = a_1^n K \times a_2^n L \subseteq N_1 \times M_2 \tag{10}$$

or

$$(b_1,b_2)^n (K \times L) = b_1^n K \times b_2^n L \subseteq N_1 \times M_2, \tag{11}$$

for some positive integer n, and hence $N_1 \times M_2$ is a classical quasi-primary submodule of R-module M.

Corollary 3.2 Let $M = M_1 \times M_2$, where M_i is an R_i-module. A submodule $M_1 \times N_2$ is a classical quasi-primary submodule of R-module M if and only if N_2 is a classical quasi-primary submodule of R_2-module M_2.
Proof This follows from Lemma 3.2.
Corollary 3.3 Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. A submodule;

$$M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n \tag{12}$$

is a classical quasi-primary submodule of R-module M if and only if N_j is a classical quasi-primary submodule of R_j-module M_j.
Proof This follows from Lemma 3.2 and Corollary 3.3.

Lemma 3.5 Let $M = M_1 \times M_2$, where M_i is an R_i-module. If $N_1 \times \{0\}$ is a classical quasi-primary submodule of M, then N_1 is a classical quasi-primary submodule of M_1.

Proof Suppose that $N_1 \times \{0\}$ is a classical quasi-primary submodule of R-module M. We will show that N_1 is a classical quasi-primary submodule of M_1. Clearly, N_1 is a proper submodule of R_1-module M_1. To see the classical quasi-primary property of N_1, let K be a submodule of R_1-module M_1 and $a,b \in R_1$ such that $abK \subseteq N_1$. Then;

$$(a,0)(b,0)(K \times \{0\}) = abK \times \{0\} \subseteq N_1 \times \{0\}. \tag{13}$$

Since $N_1 \times M_2$ is a classical quasi-primary submodule of R-module M, it follows that;

$$(a^n K \times \{0\}) = (a,0)^n (K \times \{0\}) \subseteq N_1 \times \{0\} \tag{14}$$

or

$$(b^n K \times \{0\}) = (b,0)^n (K \times \{0\}) \subseteq N_1 \times \{0\}, \tag{15}$$

for some positive integer n, that is $a^n K \subseteq N_1$ or $b^n K \subseteq N_1$, for some positive integer n. Therefore N_1 is a classical quasi-primary submodule of R_1-module M_1.

Corollary 3.6 Let $M = M_1 \times M_2$, where M_i is an R_i-module. If $\{0\} \times N_2$ is a classical quasi-primary submodule of R-module M, then N_2 is a classical quasi-primary submodule of R_2-module M_2.

Proof This follows from Lemma 3.5.

Corollary 3.7 Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If $\{0\} \times \{0\} \times \ldots \times N_j \times \ldots \times \{0\}$ is a classical quasi-primary submodule of R-module M, then N_j is a classical quasi-primary submodule of R_j-module M_j.

Proof This follows from Lemma 3.5 and Corollary 3.6.

Some basic properties of the classical quasi-primary radical formula of submodules

Our starting point is the following lemma:

Lemma 4.1 Let $M = M_1 \times M_2$, where M_i is an R_i-module. If W is a classical quasi-primary submodule of R-module M, then $P = M_1$ or P is a classical quasi-primary submodule of R_1-module M_1, where $P = \{x \in M_1 : (x,0) \in W\}$.

Proof Suppose that $P \neq M_1$. We will show that P is a classical quasi-primary submodule of R_1-module M_1. It is clear that, P is a proper submodule of R_1-module M_1. To see the classical quasi-primary property of P, let $a,b \in R_1$ and K be submodule of M_1 such that $abK \subseteq P$. Let $k \in K$.

Then $abk \in P$ so that $(a,0)(b,0)(k,0) = (abk,0) \in W$. Thus $(a,0)(b,0)(K \times \{0\}) \subseteq W$. Since W is a classical quasi-primary submodule of M, we have $(a,0)^n (K \times \{0\}) \subseteq W$ or;

$$(b,0)^n (K \times \{0\}) \subseteq W \tag{16}$$

for some positive integer n. Thus $(a^n k,0) = (a,0)^n (k,0) \in W$ or $(b^n k,0) = (b,0)^n (k,0) \in W$. It follows that $a^n k \in P$ or $b^n k \in P$. Therefore $a^n K \in P$ or $b^n K \in P$ and hence P is a classical quasi-primary submodule of M_1.

Corollary 4.2 Let $M = M_1 \times M_2$, where M_i is an R_i-module. If W is a classical quasi-primary submodule of R-module M, then $P = M_2$ or P is a classical quasi-primary submodule of R_2-module M_2, where $P = \{x \in M_2 : (0,x) \in W\}$.
Proof This follows from Lemma 4.1.

Corollary 4.3 Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module and let W be a classical quasi-primary submodule of R-module M. If $P = \{x \in M_j : (0,0,\ldots,x,0\ldots,0) \in W\}$, then $P = M_j$ or P is a classical quasi-primary submodule of R_j-module M_j.
Proof This follows from Lemma 4.1 and Corollary 4.2.

Definition 4.4 A classical quasi-primary radical of N in M, denoted by $cprad_M(N)$, is defined to be the intersection of all classical quasi-primary submodules containing N. Should there be a classical quasi-primary submodule of M containing N, then we put $cprad_M(N) = M$.

Lemma 4.5 Let $M = M_1 \times M_2$, where M_i is an R_i-module and let N be a submodule of R_1-module M_1. Then $m \in cprad_{M_1}(N)$ if and only if $(m,0) \in cprad_M(N \times \{0\})$.
Proof Suppose that $M = M_1 \times M_2$, where M_i is an R_i-module. Let N be a submodule of R_1-module M_1 and let $m \in cprad_{M_1}(N)$. If there is no classical quasi-primary submodule containing $N \times \{0\}$, then $cprad_M(N \times \{0\}) = M$. Therefore $(m,0) \in M = cprad_M(N \times \{0\})$. There exists a classical quasi-primary submodules containing $N \times \{0\}$, then there exists a classical quasi-primary submodule W, with $N \times \{0\} \subseteq W$. By Lemma 4.1 and $P = \{x \in M_1 : (x,0) \in W\}$, we have $P = M_1$ or P is a classical quasi-primary submodule of R-module M_1.

Case 1: $P = M_1$. Since $m \in cprad_{M_1}(N)$ we have $m \in P$. Then $(m,0) \in W$. Therefore if W is a classical quasi-primary submodule containing $N \times \{0\}$, then $(m,0) \in W$.

Case 2: $P \neq M_1$. Since $P \neq M_1$, we have P is a classical quasi-primary submodule of R-module M_1. Let $x \in N$. Then $(x,0) \in N \times \{0\}$ so that $x \in P$. It follows that $N \subseteq P$. We have

$$\begin{aligned} cprad_{M_1}(N) &\subseteq cprad_{M_1}(P) \\ &= P \end{aligned} \tag{17}$$

so that $m \in P$. Therefore if W is a classical quasi-primary submodule containing $N \times \{0\}$, then $(m,0) \in W$ and hence $(m,0) \in M = cprad_M(N \times \{0\})$.

Corollary 4.6 Let $M = M_1 \times M_2$, where M_i is an R_i-module and let N be a submodule of R_2-module M_2. Then $m \in cprad_{M_2}(N)$ if and only if $(0,m) \in cprad_M(\{0\} \times N)$.
Proof This follows from Lemma 4.5.

Corollary 4.7 Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module and let N be a submodule of R_j-module M_j. Then $m \in cprad_{M_j}(N)$ if and only if;

$$(0,0,\ldots,m,0,\ldots,0) \in cprad_M(\{0\} \times \{0\} \times \ldots \times N \times \{0\} \times \ldots \times \{0\}). \tag{18}$$

Proof This follows from Lemma 4.5 and Corollary 4.6.

Lemma 4.8 Let $M = M_1 \times M_2$, where M_i is an R_i-module. If N_i is a submodule of R_i-module M_i, then $cprad_{M_1}(N_1) \times cprad_{M_2}(N_2) \subseteq cprad_M(N_1 \times N_2)$.
Proof Suppose that $M = M_1 \times M_2$, where M_i is an R_i-module. Let N_i be a submodule of R_i-module M_i. We will show that $cprad_{M_1}(N_1) \times cprad_{M_2}(N_2) \subseteq cprad_M(N_1 \times N_2)$. Let;

$$(x,y) \in cprad_{M_1}(N_1) \times cprad_{M_2}(N_2). \tag{19}$$

Then $x \in cprad_{M_1}(N_1)$ and $y \in cprad_{M_2}(N_1)$. By Lemma 4.4 and Lemma 4.5, we have;

$$(x,0) \in cprad_M(N_1 \times \{0\}) \subseteq cprad_M(N_1 \times N_2) \tag{20}$$

and

$$(0, y) \in cprad_M(\{0\} \times N_2) \subseteq cprad_M(N_1 \times N_2). \tag{21}$$

Then;

$$(x, y) = (x, 0) + (0, y) \in cprad_M(N_1 \times N_2) \tag{22}$$

and hence $cprad_{M_1}(N_1) \times cprad_{M_2}(N_2) \subseteq cprad_M(N_1 \times N_2)$.

Corollary 4.9 Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If N_i be a submodule of R_i-module M_i, then $\prod_{i=1}^{n} cprad_{M_i}(N_i) \subseteq cprad_M(\prod_{i=1}^{n} N_i)$.

Proof This follows from Lemma 4.8.

Theorem 4.10 Let $M = M_1 \times M_2$, where M_i is an R_i-module. If N is a submodule of R_1-module M_1, then $cprad_{M_1}(N) \times cprad_{M_2}(M_2) = cprad_M(N_1 \times M_2)$. (23)

Proof Suppose that $M = M_1 \times M_2$, where M_i is an R_i-module. Let N be a submodule of R_1-module M_1. By Lemma 4.8, we have $cprad_{M_1}(N) \times cprad_{M_2}(M_2) \subseteq cprad_M(N \times M_2)$. We will show that;

$$cprad_{M_1}(N) \times cprad_{M_2}(M_2) \supseteq cprad_M(N \times M_2). \tag{24}$$

If there is no classical quasi-primary submodule containing N, then $cprad_{M_1}(N) = M_1$. Then;

$$cprad_M(N \times M_2) \subseteq cprad_{M_1}(N_1) \times cprad_{M_2}(M_2). \tag{25}$$

If there are classical quasi-primary submodules containing N, then there exists W a classical quasi-primary submodule of M_1 containing N. Then $W \times M_2$ is a classical quasi-primary submodule of M containing $N \times M_2$. Let P be a classical quasi-primary submodule of M containing $N \times M_2$. Then;

$$N \times M_2 \subseteq cprad_{M_1}\left(N_1\right) \times M_2 = cprad_{M_1}\left(N\right) \times cprad_{M_2}\left(M_2\right). \tag{26}$$

Therefore;

$$cprad_M(N_1 \times N_2) \subseteq cprad_{M_1}(N) \times cprad_{M_2}(M_2) \tag{27}$$

and hence $cprad_{M_1}(N) \times cprad_{M_2}(M_2) = cprad_M(N_1 \times M_2)$.

Corollary 4.11 Let $M=M_1\times M_2$, where M_i is an R_i-module. If N is a submodule of R_2-module M_2, then;

$$cprad_M\left(M_2\times N\right)=cprad_{M_1}\left(M_2\right)\times cprad_{M_2}\left(N\right). \tag{28}$$

Proof This follows from Lemma 4.10.

Corollary 4.12 Let $M=\prod_{i=1}^{n}M_i$, where M_i is an R_i-module. If N_i be a submodule of R_i-module M_i, then;

$$\prod_{i=1}^{n}cprad_{M_i}(N_i)=cprad_M(\prod_{i=1}^{n}N_i). \tag{29}$$

Proof This follows from Lemma 4.10 and Corollary 4.11.

Theorem 4.13 Let $M=M_1\times M_2$, where M_i is an R_i-module. If N_1 is a classical quasi-primary submodule of M_1, then N_1 is to satisfy the classical quasi-primary radical formula in M_1 if and only if $N_1\times M_2$ is to satisfy the classical quasi-primary radical formula in M.
Proof Suppose that N_1 is a classical quasi-primary submodule of M_1 and N_1 is to satisfy the classical quasi-primary radical formula in M_1. We will show that $N_1\times M_2$ is to satisfy the classical quasi-primary radical formula in M. Since N_1 is a classical primary submodule of M_1, it follows that;

$$\begin{aligned} cprad_M(N_1\times M_2) &= cprad_{M_1}(N_1)\times cprad_{M_2}(M_2)\\ &= \left\langle E_{M_1}(N_1)\right\rangle\times M_2\\ &= \left\langle E_M(N_1\times M_2)\right\rangle. \end{aligned} \tag{30}$$

Therefore $N_1\times M_2$ to satisfy the classical quasi-primary radical formula in M. Conversely, suppose that N_1 is a classical quasi-primary submodule of M_1 and $N_1\times M_2$ to satisfy the classical quasi-primary radical formula in M. We will show that N_1 is to satisfy the classical quasi-primary radical formula in M_1. Since $N_1\times M_2$ is a classical quasi-primary prime submodule of M, it follows that;

$$\begin{aligned} \left\langle E_{M_1}(N_1)\right\rangle\times M_2 &= \left\langle E_M(N_1\times M_2)\right\rangle\\ &= cprad_{M_1}(N_1)\times cprad_{M_2}(M_2). \end{aligned} \tag{31}$$

Then $cprad_{M_1}(N_1) = \langle E_{M_1}(N_1) \rangle$ and hence N_1 is to satisfy the classical quasi-primary radical formula in M_1.

Theorem 4.14 Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If N_j is a classical quasi-primary submodule of M_j, then N_j satisfies the classical quasi-primary radical formula in M_j if and only if $M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$ to satisfy the classical quasi-primary radical formula in M.

Proof This follows from Theorem 4.13.

Conclusions

Many new classes of classical quasi-primary radical formula of modules over commutative rings with identity have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates the classical quasi-primary submodule, classical quasi-primary radical of submodule, classical quasi-primary radical formula of submodule of modules over commutative rings with identity. Some characterizations of classical quasi-primary radical of submodule and classical quasi-primary radical formula of submodule are obtained. Finally, we obtain necessary and sufficient conditions of a submodule in order to be a classical quasi-primary radical formula of submodules.

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