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Non-negative Solutions of the Nonlinear Diophantine Equation $(8^n)^x + p^y = z^2$ for Some Prime Number p

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Abstract

In this paper, we explain all non-negative integer solutions for the nonlinear Diophantine equation of type $8^x + p^y = z^2$ when p is an arbitrary odd prime number and incongruent with 1 modulo 8. Then, we apply the result to describe all non-negative integer solutions for the equation $(8^n)^x + p^y = z^2$ when $n \ge 2$. The results presented in this paper generalize and extend many results announced by other authors.

Keywords: Catalan's conjecture, Mersenne prime number, Nonlinear Diophantine equation, Non-negative solution

Introduction

Diophantine analysis is a branch of the theory of numbers studying polynomial equations in 2 or more variables which are to be solved in integers. Traditionally, concerning a Diophantine equation, the following problems arise: (i) Is the equation solvable? (ii) Are there finitely or infinitely many solutions? (iii) Can all solutions be solved? Diophantine equations can be applied in real life and are used extensively in many fields. For example, it is used to solve the chemical equations [1] and used in other areas like public-key cryptography [2,3], algebraic curves [4] and projective curves [5].

Nonlinear Diophantine equation is first introduced by Fermat in the form of $x^n + y^n = z^n$. For n = 2, all of the solutions are called Pythagorean triple and the equation has no positive integer solution when n > 2. This result has come to be known as Fermat's last theorem [6] and it is the foundation of the birth of theories about nonlinear Diophantine equations. In recent years, the nonlinear Diophantine equation of the form $a^x + b^y = z^2$ has been extensively studied when a and b are fixed integers. For example, in 2007, Acu [7] showed that the Diophantine equation $2^x + 5^y = z^2$ had 2 non-negative integer solutions (3, 0, 3) and (2, 1, 3). In 2011, Suvarnamani *et al.* [8] showed that the 2 Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ had no solution in the set of non-negative integers. In the same year, Suvarnamani showed in [9] Theorem 2.2 that (3, 0, 3) was the solution for the equation of $2^x + p^y = z^2$ for any prime number p and (4, 2, 5) was also the solution when p = 3. Moreover, when p was a prime number of the form $p = 1 + 2^{k+1}$ where $k \ge 0$, he also showed that $(2k, 1, 1+2^k)$ was the non-negative solutions of the equation. Recently, in 2019, Laipaporn *et al.* [10] studied the equation $3^x + p5^y = z^2$. They showed that the equation had no solution when p was congruent to 5 or 7 modulo 24.

Many authors also studied some particular cases of the Diophantine equation;

 $8^x + p^y = z^2$

(1)

when p is a fixed prime number.

In 2012, Peker and Cenberci [11] suggested that the Diophantine equation $8^x + 19^y = z^2$ had no nonnegative integer solution. Later, in the same year. Sroysang [12] opposed this result by showing that (1, 0, 3) was the only non-negative solution of the equation. In the end of this paper, he also posed an open problem: What is the set of all non-negative integers' solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$? This question was answered in the following year by Rabago [13] that $\{(1, 0, 3), (1, 1, 5), (2, 1, 9), (3, 1, 23)\}$ was the set of all non-negative solutions of the equation. In other papers, Sroysang [14-16] also showed that (1, 0, 3) was the only non-negative solution of the Eq. (1) for p = 7,13 and 59. In 2015, Qi and Li [17] described all positive solutions of (1) when $p \equiv \pm 3 \pmod{8}$ and $p \equiv 7 \pmod{8}$. In particular, when $p \equiv 1 \pmod{8}$ and $p \neq 17$, they demonstrated that the equation had at most 2 positive integer solutions. In 2017, Asthana and Singh [18] studied a case when p = 113, they indicated that the Diophantine equation $8^x + 113^y = z^2$ had 3 non-negative integer solutions: (1, 0, 3), (1, 1, 11) and (3, 1, 25). Recently, in 2019 Makate *et al.* [19] revealed that (1, 0, 3) was the only non-negative solution of the Eq. (1) for p = 61 and 67.

The above research results have sparked inquiries to find solutions of the nonlinear Diophantine Eq. (1) for other odd prime numbers p. In this paper, we are primarily interested in non-negative solutions of the equation (1) when p is an odd prime number and incongruent with 1 modulo 8. After that, by using the result, we give conditions for the existence and uniqueness of non-negative integer solutions of the equations $(8^n)^x + p^y = z^2$ when $n \ge 2$. Our results generalize results in [12,14-16,19].

Materials and methods

Throughout this paper, the main method in proving our results is modular arithmetic. We also need the following results from [12,20,21] which are necessary to the proof of our main theorem. We will start this section by stating Catalan's conjecture [22] that was conjectured in 1844 and proven in 2004 by Mihailescu [20].

Lemma 1. [20] (3,2,2,3) is a unique solution (a,b,x,y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with min $\{a,b,x,y\} > 1$.

Lemma 2. [12] (1,3) is a unique solution (x, z) for the Diophantine equation $8^x + 1 = z^2$ where x and z are non-negative integers.

Lemma 3. [21] The Diophantine equation $p^x + 1 = z^2$, where p is an odd prime number, has exactly 1 non-negative integer solution (x, z, p) = (1, 2, 3).

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Results and discussion

Before we prove the results in this paper, we first recall that there have been numerous studies conducted to describe all non-negative solutions of the equation $8^x + p^y = z^2$ for many prime numbers p. In some specific cases, for $p \in \{7, 13, 19, 59, 61, 67\}$, they obtained that (1, 0, 3) is the only non-negative solution of the equation. In this paper, one of our objectives is to describe all non-negative solutions of the entitled equation when p is an arbitrary odd prime number which is not congruent to 1 modulo 8. Therefore, our results generalize those specific cases that have been previously done. We now present the following 2 lemmas, which play an important role in proving the main results.

Lemma 4. Let p be an odd prime number and (x, y, z) be a non-negative integer solution of the Diophantine equation $8^{x} + p^{y} = z^{2}$. If p is not congruent to 1 modulo 8, then x = 0 or y is an even number.

Proof. Assume that the conditions hold. Let us suppose for the sake of contradiction that $x \ge 1$ and y is an odd number. Then y = 2r + 1 for some integer $r \ge 0$. Thus $z^2 = 8^x + p^y = 8^x + p^{2r+1}$. Since $x \ge 1$ and p is odd, we have $8^x + p^y$ is odd which implies that z is odd. We write z = 2q + 1 for some integer $q \ge 0$. Then $z^2 = 4q^2 + 4q + 1$ and thus

$$z^2 - 1 = 4q^2 + 4q = 8(\frac{q^2 + q}{2}),$$

where $\frac{q^2 + q}{2}$ is always an integer, no matter what q is even or odd. This implies $z^2 \equiv 1 \pmod{8}$. Since p is odd, we have $p^2 \equiv 1 \pmod{8}$ and thus $(p^2)^r \equiv 1 \pmod{8}$. Now, suppose that t is the remainder after dividing p by 8, then $p \equiv t \pmod{8}$. It follows that

$$z^{2} = 8^{x} + p^{2r+1} = 8^{x} + p(p^{2})^{r} \equiv t \pmod{8}.$$

By the transitive property of congruence, we have $p \equiv 1 \pmod{8}$, this leads to a contradiction. Therefore, x = 0 or y is an even number.

To make a self-contained paper, we recall that Mersenne numbers are those numbers of the form $M_k = 2^k - 1$ for some positive integer k. It is known that if M_p is prime, then p is a prime number (see [23] p. 221), in this case we call M_p a Mersenne prime number. But the converse is not true, for example $M_{11} = 2^{11} - 1 = 2,047$ is not prime since $2,047 = 23 \cdot 89$.

The proof of the next result follows some ideas from [12] Theorem 3.1.

Lemma 5. Let p be an odd prime number and (x, y, z) be a non-negative integer solution of the Diophantine equation $8^{x} + p^{y} = z^{2}$. If $x \neq 0$ and y is an even positive number, then y = 2 and p is a Mersenne prime number of the form $p = 2^{3x-2} - 1$.

Proof. Suppose that $x \neq 0$ and y is an even positive number. Then $8^x + p^y$ is odd, consequently z is odd. Now, we write y = 2s for some positive integer s. We have $8^x + p^{2s} = z^2$, which implies; $2^{3x} = z^2 - (p^s)^2 = (z + p^s)(z - p^s).$

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Thus, there exists a non-negative integer u such that $z - p^s = 2^u$ and $z + p^s = 2^{3x-u}$ where 3x > 2u. Then $2^{3x-u} - 2^u = 2p^s$. It follows that;

$$2^{u}(2^{3x-2u}-1) = 2p^{s}.$$
(2)

By dividing both sides of (2) by 2, we have;

$$2^{u-1}(2^{3x-2u}-1) = p^s.$$
(3)

Since $z - p^s$ is even and $z - p^s = 2^u$, we have $u \ge 1$. Now, if u > 1, then the left-hand side of (3) is even while the right-hand side is odd, a contradiction. Hence, we deduce that u = 1. Consequently, (3) implies that;

$$2^{3x-2} - 1 = p^s. (4)$$

If s > 1, then $2^{3x-2} = p^s + 1 > p + 1 \ge 2^2$ which implies 3x - 2 > 2. Hence, min $\{2, p, 3x - 2, s\} > 1$ and thus (4) contradicts with Lemma 1. Therefore s = 1, that is y = 2 and (4) implies $p = 2^{3x-2} - 1$ as required.

Next, we consider the Diophantine equations $8^x + p^y = z^2$ and later in this section we study the equations $(8^n)^x + p^y = z^2$ for $n \ge 2$.

Theorem 1. Let p be an odd prime number for which p is not congruent to 1 modulo 8. Then the following are all non-negative solutions (x, y, z) of the equation $8^x + p^y = z^2$:

(i)
$$(1, 0, 3)$$
 and $(0, 1, 2)$, when $p = 3$;

(*ii*) (1, 0, 3) and $(\frac{k+2}{3}, 2, p+2)$, when p is a Mersenne prime number $2^k - 1$ where $k \equiv 1 \pmod{3}$;

(*iii*) (1, 0, 3), otherwise.

Proof. Let (x, y, z) be a non-negative integer solution of $8^x + p^y = z^2$ and p is not congruent to 1 modulo 8.

(i) Suppose that p = 3, by Lemma 4 we have x = 0 or y is an even number.

Case 1: If x = 0, then the equation $8^x + p^y = z^2$ can be reduced to $1 + 3^y = z^2$ and so $3^y = z^2 - 1 = (z+1)(z-1)$.

Thus, there exist non-negative integers u and v where u > v and u + v = y such that $z + 1 = 3^u$ and $z - 1 = 3^v$. Then $3^u - 3^v = (z+1) - (z-1) = 2$, this is possible only when u = 1 and v = 0. It follows that y = 1 and z = 2. Hence (x, y, z) = (0, 1, 2).

Case 2: If $x \ne 0$, then y is an even number. Suppose that y > 0, by Lemma 5 we have $3 = 2^{3x-2} - 1$, that is 3x-2=2 and so $x = \frac{4}{3}$, contradicts to that x is an integer. Hence y = 0 and the equation $8^x + 3^y = z^2$ can be written as $8^x + 1 = z^2$. Therefore, Lemma 2 implies that (x, y, z) = (1, 0, 3).

(ii) Suppose that $p = 2^k - 1$ where $k \equiv 1 \pmod{3}$. If x = 0, then the equation $8^x + p^y = z^2$ can be written as $1 + p^y = z^2$ and so Lemma 3 implies p = 3, that is k = 2, contradicts to that $k \equiv 1 \pmod{3}$. Thus, $x \neq 0$ and we obtain that y is an even number by Lemma 4.

Case 1: If y = 0, then we have $8^x + 1 = z^2$. Therefore, Lemma 2 implies that (x, y, z) = (1, 0, 3).

Case 2: If y > 0, then Lemma 5 implies that y = 2 and $p = 2^{3x-2} - 1$, which means that k = 3x - 2. Therefore $x = \frac{k+2}{3}$ is an integer (since $k \equiv 1 \pmod{3}$). Now we have;

$$z^{2} = 8^{x} + p^{y} = 8^{\frac{k+2}{3}} + p^{2},$$

which implies;

$$z = \sqrt{8^{\frac{k+2}{3}} + p^2} = \sqrt{2^{k+2} + p^2} = \sqrt{4 \cdot 2^k + p^2} = \sqrt{4(p+1) + p^2} = \sqrt{p^2 + 4p + 4} = p + 2.$$

In this case, we deduce that $(x, y, z) = (\frac{k+2}{3}, 2, p+2)$.

(iii) Suppose that p > 3 and p is not a Mersenne prime number of the form $2^{k} - 1$ for any integer $k \equiv 1 \pmod{3}$. In this case, the proof will follow the pattern in (ii). If x = 0, then Lemma 3 implies p = 3 which contradicts to our assumption. Thus, $x \neq 0$ and Lemma 4 implies that y is an even number. If y > 0, then Lemma 5 implies that $p = 2^{3x-2} - 1$ where $3x - 2 \equiv 1 \pmod{3}$, this contradicts our assumption again. Therefore y = 0 and hence Lemma 2 implies that (x, y, z) = (1, 0, 3).

Example 1. $8,191 = 2^{13} - 1$ is a Mersenne prime number and $13 \equiv 1 \pmod{3}$. By Theorem 1 (ii), the equation $8^x + 8,191^y = z^2$ has 2 non-negative solutions which are (1, 0, 3) and (5, 2, 8,193).

Example 2. 13 is a prime number which is not a Mersenne prime and $31 = 2^5 - 1$ is a Mersenne prime but 5 is not congruent to 1 modulo 3. By Theorem 1 (iii), (1, 0, 3) is the only non-negative solution of the equation $8^x + 13^y = z^2$ and $8^x + 31^y = z^2$.

Now, in the following corollary we will consider all non-negative integer solutions of the equation $(8^n)^x + p^y = z^2$ which is an extension of the Eq. (1).

Corollary 1. Let p be an odd prime number for which p is not congruent to 1 modulo 8 and let n be a positive integer such that $n \ge 2$. Then all non-negative integer solutions (x, y, z) of the equation

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$$(8^n)^x + p^y = z^2$$

are as follows:

(i) If p = 3, then (0, 1, 2) is the only non-negative integer solution of (5).

(ii) If p is a Mersenne prime number $2^k - 1$ where $k \equiv 1 \pmod{3}$, then (5) has a non-negative integer solution if and only if 3n|(k+2). In this case, the solution is unique and equal to

$$(x, y, z) = (\frac{k+2}{3n}, 2, p+2).$$

(iii) If p > 3 and p is not a Mersenne prime number of the form $2^k - 1$ for any intege $k \equiv 1 \pmod{3}$, then (5) has no non-negative integer solution.

Proof. For (i), If $(8^n)^x + 3^y = z^2$ has a non-negative solution, then by Theorem 1, $(nx, y, z) \in \{(1, 0, 3), (0, 1, 2)\}$. If (nx, y, z) = (1, 0, 3), then $x = \frac{1}{n}$ where $n \ge 2$, contradicts to that x is an integer.

Therefore (nx, y, z) = (0, 1, 2) which implies the only solution is (x, y, z) = (0, 1, 2).

The proof of (ii) and (iii) proceed in a similar way to (i) by using Theorem 1 (ii) and Theorem 1 (iii).

Example 3. $p = 2^{61} - 1$ is a Mersenne prime number which was discovered in 1883 by a Russian mathematician Ivan Mikheevich Pervushin [24]. By Corollary 1 (ii), the Diophantine equation $(8^n)^x + (2^{61} - 1)^y = z^2$ has a non-negative integer solution only when 3n | 63, that is n = 3 or n = 7. For example, when n = 3 the unique solution of the equation $(8^3)^x + (2^{61} - 1)^y = z^2$ is

$$(x, y, z) = (\frac{61+2}{9}, 2, (2^{61}-1)+2) = (7, 2, 2^{61}+1),$$

which can be verified as follows:

$$(8^{3})^{7} + (2^{61} - 1)^{2} = 8^{21} + 2^{122} - 2^{62} + 1$$

= $2^{122} + (2^{63} - 2^{62}) + 1$
= $2^{122} + 2^{62} + 1$
= $(2^{61} + 1)^{2}$.

The case n = 7 can be verified in a similar way and we obtain that $(3, 2, 2^{61} + 1)$ is the unique non-negative solution of the equation $(8^7)^x + (2^{61} - 1)^y = z^2$.

It is obvious that Theorem 1 (iii) is a generalization of [12,14-16,19]. Moreover, Theorem 1 gives some partial results extending form [9]. For example, by [9] Theorem 2.2 we have that the equation $2^x + 8,191^y = z^2$ has a solution (3, 0, 3), but other solutions of the equation have not yet known. Here, from Example 1, we have $8^5 + 8,191^2 = 8,193^2$ which can be written as $2^{15} + 8,191^2 = 8,193^2$. Hence we obtain that (15, 2, 8,193) is also a solution of the equation $2^x + 8,191^y = z^2$.

(5)

When p is a prime number for which $p \equiv 1 \pmod{8}$, if (x, y, z) is a non-negative integer solution of $8^x + p^y = z^2$ we obtain that $x \neq 0$ and $z \neq 0$. To see this, suppose that x = 0. It follows that $1 + p^y = z^2$, then by Lemma 3 we have that p = 3, which contradicts to $p \equiv 1 \pmod{8}$. Similarly, if z = 0, then $0 = z^2 = 8^x + p^y \equiv 1 \pmod{8}$, which is a contradiction again. Note that in case y = 0, we have that $8^x + 1 = z^2$ which implies $2^{3x} = 8^x = z^2 - 1 = (z+1)(z-1)$. Thus, $2^m - 2^n = (z+1) - (z-1) = 2$ for some non-negative integers m > n and m + n = 3x. This implies m = 2 and n = 1 therefore x = 1 and z = 3. In this case we can deduce that (x, y, z) = (1, 0, 3) is a solution of the equation $8^x + p^y = z^2$ and it is the only non-negative solution having zero in its components. By combining this consideration and a results from [17]. Theorem 1 that: If $p \equiv 1 \pmod{8}$ and $p \neq 17$, then the equation $8^x + p^y = z^2$ has at most 2 positive integer solutions, we obtain the following proposition.

Proposition 1. Let p be a prime number for which $p \equiv 1 \pmod{8}$ and $p \neq 17$. Then the nonlinear Diophantine equation $8^x + p^y = z^2$ has at most 3 non-negative integer solutions, one is (1, 0, 3), and the other two (if exist) are both positive solutions.

We observe that, when p is a prime number such that $\sqrt{p+8}$ is an integer, by putting $z = \sqrt{p+8}$ we have that $z^2 = p+8$. This means that, $(1, 1, \sqrt{p+8})$ is a non-negative solution of the equation $8^x + p^y = z^2$ and by Lemma 4 this case does not occur when p is not congruent to 1 modulo 8. This leads us to the following trivial proposition.

Proposition 2. Let *p* be a prime number for which $p \equiv 1 \pmod{8}$. If $\sqrt{p+8}$ is an integer, then $(1, 1, \sqrt{p+8})$ is a solution of the equation $8^x + p^y = z^2$.

In [13] and [18], all non-negative integer solutions of $8^x + p^y = z^2$ were described when p = 17 and p = 113 where both prime numbers are congruent to 1 modulo 8. We end this paper with the following open question: What is the set of all non-negative integer solutions for the equation $8^x + p^y = z^2$ where p is a prime number such that $p \equiv 1 \pmod{8}$ and $p \notin \{17, 113\}$?

Conclusions

In this paper, we have solved the Diophantine equation $8^x + p^y = z^2$ when p is an odd prime number and incongruent with 1 modulo 8. We have shown that when p = 3 the entitled equation has 2 non-negative integer solutions in (x, y, z) i.e. (1, 0, 3) and (0, 1, 2). When p is a Mersenne prime number $2^k - 1$ where $k \equiv 1 \pmod{3}$, the equation has 2 non-negative integer solutions (1, 0, 3) and $(\frac{k+2}{3}, 2, p+2)$. In other cases, the equation has a unique non-negative integer solution (1, 0, 3). We also applied the previous results to describe all non-negative integer solutions of the equation $(8^n)^x + p^y = z^2$ for any positive integer $n \ge 2$. We found that, when p = 3, the equation has a unique solution (0, 1, 2). When p is a Mersenne prime number $2^k - 1$ where $k \equiv 1 \pmod{3}$, the equation has a solution if and only if 3n|(k+2). In this case, the solution is unique and equal to $(\frac{k+2}{3n}, 2, p+2)$. In other cases, the equation.

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