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Fractional Homotopy Analysis Transforms Method for Solving a Fractional Heat-Like Physical Model

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Abstract

The main aim of the present work is to present a new and simple algorithm for time fractional heat like physical models by using the new fractional homotopy analysis transform method (FHATM). The proposed method is an innovative adjustment in the Laplace transform algorithm (LTA) for fractional partial differential equations and makes the calculation much simpler. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive. A good agreement between the obtained solution and some well-known results has been obtained.

Keywords: Heat equation, Fractional derivatives, Analytical solution, Mittag-Leffler function, Laplace transform method, fractional homotopy analysis transform method

Introduction

Fractional differential equations have drawn the interest of many researchers [1-4] due to their important applications in science and engineering, such as modeling of anomalous diffusive and sub-diffusive systems, description of fractional random walk and unification of diffusion and wave propagation phenomena. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Our concern in this work is to consider the numerical solution of the time-fractional heat equations. Heat-like models can describe many physical problems in different fields of science and engineering. These models play important roles in applied science, so finding their analytical solutions has fundamental significance in various field of science and engineering. In this paper, we consider the time fractional heat-like equation with variable coefficients described by the following 3 dimensional IBVP;

$$D_t^{\alpha} u(x,y,z,t) = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz}, \qquad t>0, \tag{1}$$

where $0<x<a,\ 0<y<b,\ 0<z<c,$ subject to the Neumann boundary conditions;

$$u_x(0,y,z,t) = f_1(y,z,t), \quad u_x(a,y,z,t) = f_2(y,z,t), \tag{2}$$

$$u_y(x,0,z,t) = g_1(x,z,t), \quad u_y(x,b,z,t) = g_2(x,z,t), \tag{3}$$

$$u_z(x,y,0,t)=h_1(x,y,t),\quad u_z(x,y,c,t)=h_2(x,y,t), \tag{4}$$

with initial conditions;

$$u(x,y,z,0)=\phi(x,y,z), \tag{5}$$

where α is a parameter describing the order of the time fractional derivatives. The function $u(x,y,z,t)$ is assumed to be a causal function of time and space. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of $\alpha=1$ the fractional equation reduces to the classical heat equation. The different types of the proposed problem have been solved in [5-8].

In this paper, the fractional homotopy analysis transform method (FHATM) basically illustrates how the Laplace transform can be used to the approximate and analytical solutions of the heat-like fractional models by manipulating the homotopy analysis method. The proposed method involves coupling of the homotopy analysis method and Laplace transform. The main advantage of this method is its capability of combining 2 powerful methods for obtaining rapid convergent series for fractional partial differential equations. The homotopy analysis method (HAM) was first proposed and applied by Liao [9-12] based on homotopy, a fundamental concept in topology and differential geometry. HAM is based on construction of a homotopy which continuously deforms an initial guess approximation to the exact solution of a given problem. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary parameter is used to control the region of convergence of the solution series. HAM provides greater flexibility in choosing initial approximations and auxiliary linear operators and hence a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub-problems, as shown by Liao and Tan [13]. HAM has been successfully applied by many researchers for solving linear and non-linear partial differential equations [14-23]. In recent years, many authors have studied the solutions of linear and nonlinear differential and integral equations by using various methods combined with the Laplace transform. Among these are Laplace decomposition methods [24,25], and the homotopy perturbation transform method [26,27]. Recently, Khan *et al*. [28] has applied it to obtain the solutions of the Blasius flow equation on a semi-infinite domain by coupling of the homotopy analysis and Laplace transform methods. Also, this method is used for solving various nonlinear equations [29], linear and nonlinear partial differential equations [30] and the time-space fractional gas dynamics equation [31].

The main aim of this article is to present approximate analytical solutions of heat-like physical models with time fractional derivative α $(0<\alpha\le 1)$ in the form of a rapidly convergent series with easily computable components by using FHATM.

Basic definitions of fractional calculus and Laplace transform

Fractional calculus unifies and generalizes the notions of integer-order differentiation and n-fold integration [2-4]. We give some basic definitions and properties of fractional calculus theory which shall be used in this paper.

Definition 1 A real function $f(t), t>0$ is said to be in the space $C_\mu,\ \mu\in R$ if there exists a real number $p>\mu,$ such that $f(t)=t^p f_1(t)$ where $f_1(t)\in C(0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)}\in C_\mu,\ n\in \mathrm{N}.$

Definition 2 The left sided Riemann-Liouville fractional integral operator of order $\mu\ge 0,$ of a function $f\in C_\alpha, \alpha\ge -1$ is defined as [32,33];

$$I^{\mu} f(t)=\begin{cases}\dfrac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1} f(\tau)d\tau, \ \mu>0, \ t>0, \\ f(t), \qquad\qquad\qquad \mu=0\end{cases} \tag{6}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 3 The left sided Caputo fractional derivative of f, $f\in C_{-1}^{m}$, $m\in \mathrm{N}\cup\{0\}$ is defined as [4,31];

$$D^{\mu} f(t)=\frac{d^{\mu} f(t)}{dt^{\mu}}=\begin{cases} I^{m-\mu}\left[\dfrac{d^m f(t)}{dt^m}\right], & m-1<\mu<m, \quad m\in N, \\ \dfrac{d^m f(t)}{dt^m}, & \mu=m.\end{cases} \tag{7}$$

Note that [4,31];

(i) $I_t^{\mu} f(x,t)=\dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^t \frac{f(x,s)}{(t-s)^{1-\mu}}ds, \qquad \mu>0, \ t>0,$

(ii) $D^{\mu} f(x,t)=I_t^{m-\mu}\dfrac{\partial^m f(x,t)}{\partial t^m}, \qquad m-1<\mu\le m.$

Definition 4 If the Laplace transform of the function $f(t)$ is $F(s)$, then the Laplace transform of the Riemann-Liouville fractional integral $I^{\alpha} f(t)$ is defined as [3];

$$L[I^{\alpha} f(t)]=s^{-\alpha}F(s). \tag{8}$$

Definition 5 The Laplace transform of the Caputo fractional derivative is defined as [4];

$$L[D_t^{n\alpha}u(r,t)]=s^{n\alpha}L[u(r,t)]-\sum_{k=0}^{n-1}s^{(n\alpha-k-1)}u^{(k)}(r,0), \qquad n-1<n\alpha\le n, \quad n\in \mathrm{N}. \tag{9}$$

Definition 6 The Mittag-Leffler function $E_{\alpha}(z)$ with $\alpha>0$ is defined by following series representation, valid in the whole complex plane [32];

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{\alpha}}{\Gamma(n\alpha+1)}, \ \alpha>0, \ \ z\in C. \tag{10}$$

FHATM

To illustrate the basic idea of the HATM, we consider the following fractional partial differential equation;

$$D_t^\alpha u(r,t)+R[r]u(r,t)+N[r]u(r,t)=g(r,t), \quad t>0,\ r\in \mathrm{R}^3,\ \ n-1<\alpha\le n, \quad n\in \mathrm{N}, \tag{11}$$

where $D_t^\alpha=\dfrac{\partial^\alpha}{\partial t^\alpha}$, $R[r]$ is a linear operator in $r\in \mathrm{R}^3$, $N[r]$ is the general nonlinear operator in $r\in R^3$ and $g(r,t)$ is a continuous function. For simplicity we ignore all initial and boundary conditions, which can be treated in a similar way. Now the methodology consists of applying the Laplace transform first on both sides of Eq. (11), we get;

$$L[D_t^\alpha u(r,t)]+L[R[r]u(r,t)+N[r]u(r,t)]=L[g(r,t)]. \tag{12}$$

Now, using the differentiation property of the Laplace transform, we have;

$$L[u(r,t)]-\frac{1}{s^\alpha}\sum_{k=0}^{n-1}s^{(\alpha-k-1)}u^{(k)}(r,0)+\frac{1}{s^\alpha}L\big(R[r]u(r,t)+N[r]u(r,t)-g(r,t)\big)=0. \tag{13}$$

We define the nonlinear operator;

$$N[\phi(r,t;q)]=L[\phi(r,t;q)]-\frac{1}{s^\alpha}\sum_{k=0}^{n-1}s^{(\alpha-k-1)}u^{(k)}(r,0)+\frac{1}{s^\alpha}L\big(R[r]\phi(r,t;q)+N[r]\phi(r,t;q)-g(r,t)\big), \tag{14}$$

where $q\in[0,1]$ to be an embedding parameter and $\phi(r,t;q)$ is the real function of r,t and q. By means of generalizing the traditional homotopy methods, the great mathematician Liao [9-12] constructed the zero order deformation equation as follows;

$$(1-q)L[\phi(r,t;q)-u_0(r,t)]=\hbar qH(r,t)N[\phi(r,t;q)], \tag{15}$$

where $\hbar$ is a nonzero auxiliary parameter, $H(r,t)\neq 0$ is an auxiliary function, $u_0(r,t)$ is an initial guess of $u(r,t)$ and $\phi(r,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliaries in HATM. Obviously, when $q=0$ and $q=1$, it holds;

$$\phi(r,t;0)=u_0(r,t), \qquad \phi(r,t;1)=u(r,t). \tag{16}$$

Thus, as q increases from 0 to 1 the solution varies from the initial guess $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;q)$ in Taylor's series with respect to q, we have;

$$\phi(r,t;q)=u_0(r,t)+\sum_{m=1}^{\infty}q^m u_m(r,t), \tag{17}$$

where

$$u_m(r,t)=\frac{1}{m!}\frac{\partial^m\phi(r,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{18}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (17) converges at $q = 1$, we have;

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t), \tag{19}$$

which must be one of the solutions of the original nonlinear equations.

Define the vectors;

$$\vec{u}_n = \{u_0(r,t), u_1(r,t), u_2(r,t), \dots, u_n(r,t)\}. \tag{20}$$

Differentiating equation (15) m time with respect to embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we obtain the $m\,th$ order deformation equation;

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar\, q\, H(r,t)\, R_m(\vec{u}_{m-1}, r, t). \tag{21}$$

Operating the inverse Laplace transform on both sides, we get;

$$u_m(r,t) = \chi_m u_{m-1}(r,t) + \hbar\, q\, L^{-1}\left[H(r,t) R_m(\vec{u}_{m-1}, r, t)\right], \tag{22}$$

where

$$R_m(\vec{u}_{m-1}, r, t) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}\phi(r,t;q)}{\partial q^{m-1}}\right|_{q=0}, \tag{23}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

In this way, it is easy to obtain $u_m(r,t)$ for $m \ge 1$, at $M\,th$ order, we have;

$$u(r,t) = \sum_{m=0}^{M} u_m(r,t), \tag{24}$$

when $M \to \infty$, we get an accurate approximation of the original equation (11).

Illustrative examples

In this section, 3 examples of time fractional heat-like physical models are solved to demonstrate the performance and the efficiency of the HAM with coupling of the Laplace transform method.

Example 1 We first consider the one dimensional initial boundary value problems [7,8] as;

$$D_t^{\alpha} u(x,t) = \frac{1}{2} x^2 u_{xx}(x,t), \quad 0 < x < 1,\ t > 0, \quad 0 < \alpha \le 1, \tag{25}$$

subject to the boundary conditions;

$$u(0,t)=0, \quad u(1,t)=e^{t}, \tag{26}$$

and the initial condition;

$$u(x,0)=x^{2}. \tag{27}$$

Applying the Laplace transform on both sides in Eq. (25) and after using the differentiation property of Laplace transform, we get;

$$s^{\alpha}L[u(x,t)]-s^{\alpha-1}u(x,0)-\frac{1}{2}L[x^{2}u_{xx}]=0. \tag{28}$$

On simplifying;

$$L[u(x,t)]-s^{-1}x^{2}-\frac{1}{2}s^{-\alpha}L[x^{2}u_{xx}]=0. \tag{29}$$

We choose the linear operator as;

$$£[\phi(x,t;q)]=L[\phi(x,t;q)], \tag{30}$$

with property $£[c]=0,$ where c is a constant. We now define a nonlinear operator as;

$$N[\phi(x,t;q)]=L[\phi(x,t;q)]-s^{-1}x^{2}-\frac{s^{-\alpha}}{2}L[x^{2}\phi_{xx}(x,t;q)]. \tag{31}$$

Using the above definition, with the assumption $H(x,t)=1,$ we can construct the zero-th order deformation equation;

$$(1-q)£[\phi(x,t;q)-u_{0}(x,t)]=\hbar qN[\phi(x,t;q)]. \tag{32}$$

Obviously, when $q=0$ and $q=1,$

$$\phi(x,t;0)=u_{0}(x,t), \qquad \phi(x,t;1)=u(x,t). \tag{33}$$

Thus, we obtain the mth order deformation equation;

$$£[u_{m}(x,t)-\chi_{m}u_{m-1}(x,t)]=\hbar\, qR_{m}(\vec{u}_{m-1},x,t). \tag{34}$$

Operating the inverse Laplace transform on both sides in Eq. (34), we get;

$$u_{m}(x,t)=\chi_{m}u_{m-1}(x,t)+\hbar\, q\, L^{-1}\left[R_{m}(\vec{u}_{m-1},x,t)\right], \tag{35}$$

where

$$R_m(\vec{u}_{m-1}, x, t) = L[u_{m-1}(x,t)] - (1-\chi_m)\frac{x^2}{s} - \frac{s^{-\alpha}}{2} L[x^2 (u_{m-1})_{xx}]. \tag{36}$$

Now the solution of mth order deformation equations;

$$u_m(x,t) = (\chi_m + \hbar)u_{m-1} - \hbar(1-\chi_m)x^2 - \frac{\hbar}{2} L^{-1}\left(s^{-\alpha} L[x^2 (u_{m-1})_{xx}]\right), \quad m \geq 1. \tag{37}$$

Using the initial approximation $u_0(x,t) = u(x,0) = x^2$ and from iterative scheme (35), we obtain the various iterates;

$$u_1(x,t) = -\frac{\hbar x^2 t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = -\frac{\hbar(1+\hbar)x^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{\hbar^2 x^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = -\frac{\hbar(1+\hbar)^2 x^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{2\hbar^2(1+\hbar)x^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\hbar^3 x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$u_4(x,t) = -\frac{\hbar(1+\hbar)^3 x^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{3\hbar^2(1+\hbar)^2 x^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{3\hbar^3(1+\hbar)x^2 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\hbar^4 x^2 t^{4\alpha}}{\Gamma(4\alpha+1)}, \dots .$$

Proceeding in this manner, the rest of the components $u_n(x,t)$ for $n \geq 5$ can be completely obtained and the series solution is thus entirely determined. Hence, the solution of Eq. (25) is given as;

$$u(x,t) = u_0(x,t) + \sum_{m=0}^{\infty} u_m(x,t). \tag{38}$$

If we select $\hbar = -1$ and $\alpha = 1$, then;

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right) = x^2 e^t. \tag{39}$$

Clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty} u_m(x,t)$ converges to the solution $u(x,t) = x^2 e^t$, which is an exact solution of the given Eq. (25) for $\alpha = 1$. The above result is in complete agreement with [5-8].

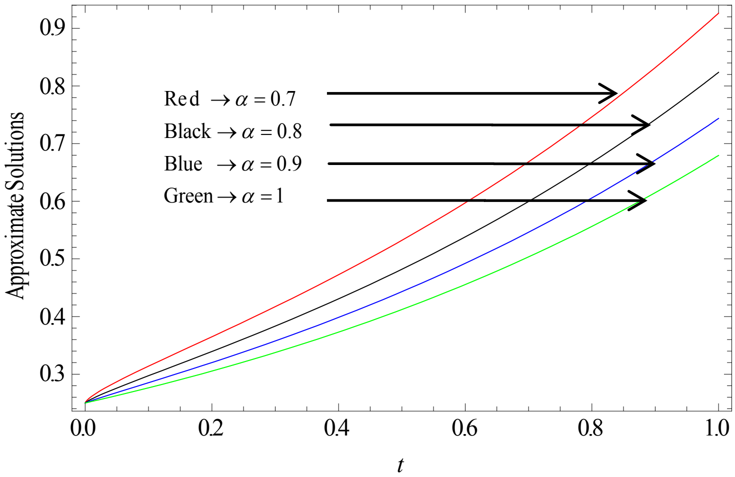

Figure 1 Plot of the solutions obtained by the proposed method at different values of $\alpha = 0.7,\ 0.8,\ 0.9$ and 1 for Example 1.

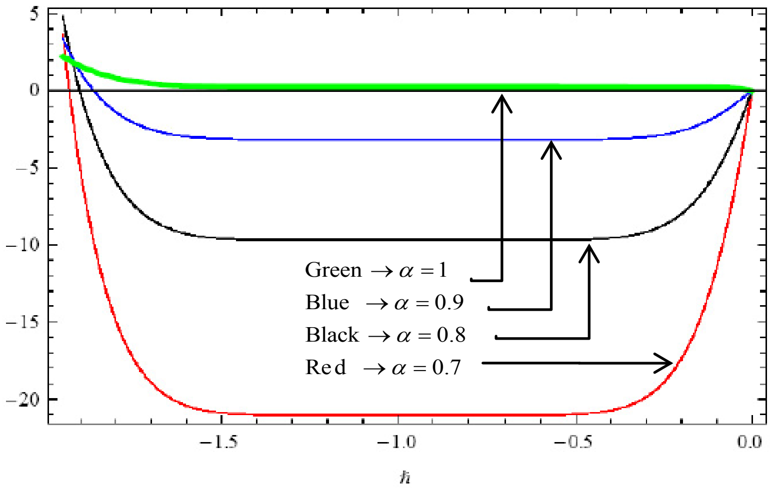

Figure 2 Plot of the $\hbar$ curves for different values of $\alpha = 0.7, 0.8, 0.9$ and 1 for Example 1.

Figure 1 shows the behavior of the approximate solution $u(x,t)$ for different fractional Brownian motion $\alpha = 0.7,\ 0.8,\ 0.9$ and for standard motion i.e., at $\alpha = 1$ for example 1. It is seen from **Figure 1**, that the solution obtained by FHATM increases very rapidly with the increases in t at the value of $x = 0.5$ and $\hbar = -1$.

As pointed out by Liao [10], the convergence and rate of approximation for the HAM solution strongly depends on the value of auxiliary parameter $\hbar$. Even if the initial approximation $u_0(x,t)$, the auxiliary linear operator L, and the auxiliary function $H(x,t)$ are given, we still have great freedom to choose the value of the auxiliary parameter $\hbar$. So, the auxiliary parameter $\hbar$ provides us with an additional way to conveniently adjust and control the convergence region and rate of solution series. By means of the so-called $\hbar$-curves it is easy to find out the so-called valid regions of $\hbar$ to gain a convergent solution series. When the valid region of $\hbar$ is a horizontal line segment then the solution is converged.

Figure 2 shows the $\hbar$-curves obtained from the 20th-order FHATM approximation solution of time fractional heat Eq. (25) at $x = 0.5$. In our study, it is obvious from **Figure 2** that the acceptable range of auxiliary parameter $\hbar$ is $-1.95 \le \hbar \le 0$. We still have freedom to choose the auxiliary parameter according to the $\hbar$ curve. Thus, the arbitrary point of this interval i.e., $\hbar = 0$, is an appropriate selection for $\hbar$ in which numerical solution converges. From **Figure 2**, the valid regions of convergence correspond to the line segments nearly parallel to the horizontal axis.

Example 2 In this example, we consider the following 2 dimensional heat-like models [5-8] as;

$$D_t^\alpha u(x,y,t) = \frac{1}{2}(y^2 u_{xx} + x^2 u_{yy}), \quad 0 < x, y < 1,\ t > 0,\ 0 < \alpha \le 1, \tag{40}$$

subject to the Neumann boundary conditions;

$$u_x(0,y,t) = 0, \quad u_x(1,y,t) = 2\sinh t, \quad u_y(x,0,t) = 0, \quad u_y(x,1,t) = 2\cosh t, \tag{41}$$

and the initial condition;

$$u(x,y,0) = y^2. \tag{42}$$

Operating the Laplace transform on both sides in Eq. (40) and after using the differentiation property of Laplace transform, we get;

$$L[u(x,y,t)] - s^{-1}y^2 - \frac{1}{2}s^{-\alpha}L[y^2 u_{xx} + x^2 u_{yy}] = 0. \tag{43}$$

We choose the linear operator as;

$$£[\phi(x,y,t;q)] = L[\phi(x,y,t;q)], \tag{44}$$

with property $£[c] = 0$, where c is a constant. We now define a nonlinear operator as;

$$N[\phi(x,y,t;q)] = L[\phi(x,y,t;q)] - s^{-1}y^2 - \frac{s^{-\alpha}}{2}L[y^2\phi_{xx}(x,y,t;q) + x^2\phi_{yy}(x,y,t;q)]. \tag{45}$$

Using the above definition, with the assumption $H(x,y,t) = 1$, we can construct the zero-th order deformation equation;

$$(1 - \mathrm{q})£[\phi(x,y,t;q) - u_0(x,y,t)] = q\hbar\, N[\phi(x,y,t;q)]. \tag{46}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,y,t;0)=u_0(x,y,t), \qquad \phi(x,y,t;1)=u(x,y,t). \tag{47}$$

Thus, we obtain the mth order deformation equation;

$$£[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)]=\hbar\, q R_m(\vec{u}_{m-1},x,y,t), \tag{48}$$

Operating the inverse Laplace transform on both sides in Eq. (48), we get;

$$u_m(x,y,t)=\chi_m u_{m-1}(x,y,t)+\hbar\, q\, L^{-1}\left[R_m(\vec{u}_{m-1},x,y,t)\right], \tag{49}$$

where

$$R_m(\vec{u}_{m-1},x,y,t)=L[u_{m-1}(x,y,t)]-(1-\chi_m)\frac{y^2}{s}-\frac{s^{-\alpha}}{2}L[y^2(u_{m-1})_{xx}+x^2(u_{m-1})_{yy}]. \tag{50}$$

Now the solution of mth order deformation equations;

$$u_m(x,y,t)=(\chi_m+\hbar)u_{m-1}-\hbar(1-\chi_m)y^2-\frac{\hbar}{2}L^{-1}\left(s^{-\alpha}L[y^2(u_{m-1})_{xx}+x^2(u_{m-1})_{yy}]\right),\ m\geq 1. \tag{51}$$

Using the initial approximation $u_0(x,y,t)=u(x,y,0)=y^2$ and from the iterative scheme (51), we obtain the various iterates;

$$u_1(x,y,t)=-\frac{\hbar x^2 t^{\alpha}}{\Gamma(\alpha+1)},$$
$$u_2(x,y,t)=-\frac{\hbar(1+\hbar)x^2 t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\hbar^2 y^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3(x,y,t)=-\frac{\hbar(1+\hbar)^2 x^2 t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2\hbar^2(1+\hbar)y^2 t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\hbar^3 x^2 t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$u_4(x,y,t)=-\frac{\hbar(1+\hbar)^3 x^2 t^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\hbar^2(1+\hbar)^2 y^2 t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{3\hbar^3(1+\hbar)x^2 t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\hbar^4 y^2 t^{4\alpha}}{\Gamma(4\alpha+1)},\dots .$$

Proceeding in this manner, the rest of the components $u_n(x,y,t)$ for $n\geq 5$ can be completely obtained and the series solution is thus entirely determined. Hence, the solution of the given problem is given as;

$$u(x,y,t)=u_0(x,y,t)+\sum_{m=0}^{\infty}u_m(x,y,t). \tag{52}$$

Setting $\hbar=-1$ and $\alpha=1$ in the above expressions are exactly the same as those given by the homotopy perturbation method by Özis and Agırseven [7]. However, mostly, the results given by the Adomian decomposition method and homotopy perturbation method converge to the corresponding numerical solutions in a rather small region. But, different from those 2 methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$. For $\hbar = -1$, we have the following solution;

$$u(x,y,t) = x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right) + y^2\left(1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \ldots.\right),$$

$$= x^2\sum_{k=0}^{\infty}\frac{t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} + y^2\sum_{k=0}^{\infty}\frac{t^{2k\alpha}}{\Gamma(2k\alpha+1)} = x^2\sinh(t^{\alpha},\alpha) + y^2\cosh(t^{\alpha},\alpha), \tag{53}$$

where the functions $\sinh(z,\alpha)$ and $\cosh(z,\alpha)$ are defined as follows;

$$\sinh(z,\alpha) = \frac{E_{\alpha}(z) - E_{\alpha}(-z)}{2} \text{ and } \cosh(z,\alpha) = \frac{E_{\alpha}(z) + E_{\alpha}(-z)}{2}. \tag{54}$$

As $\alpha = 1$, this series has the closed form $u(x,y,t) = x^2\sinh(t) + y^2\cosh(t)$, which is the exact solution of the classical heat equation. The above result is in complete agreement with [7].

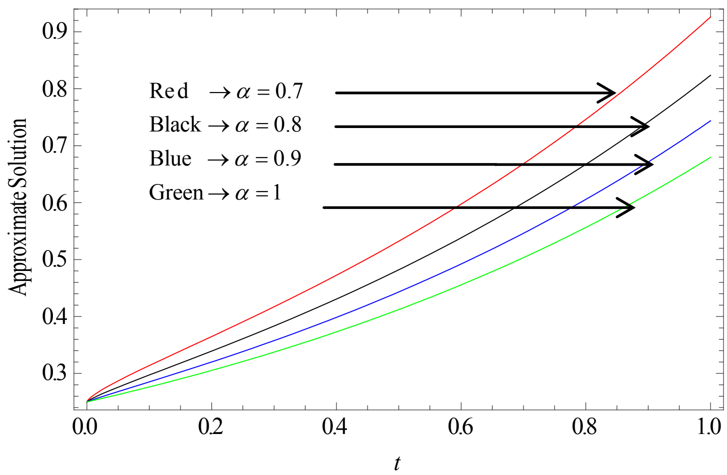

Figure 3 Plot of solutions obtained by the proposed method (FHATM) at different value of $\alpha = 0.7,\ 0.8,\ 0.9$ and 1 for Example 2.

Figure 3 shows the behavior of the approximate solution $u(x,y,t)$ for different fractional Brownian motions $\alpha = 0.7,\ 0.8,\ 0.9$ and for standard motion i.e., at $\alpha = 1$ for example 2. It is seen from **Figure 3** that the solution obtained by FHATM increases very rapidly with the increases in t at the value of $x = 0.5,\ y = 0.5$ and $\hbar = -1$. **Figure 4** shows the $\hbar$-curves obtained from the 20$^{\text{th}}$-order FHATM approximation solution of 2 dimensional time fractional heat equation at $x = 0.5$ and $y = 0.5$. In our study, it is obvious from **Figure 4** that the acceptable range of the auxiliary parameter $\hbar$ is

$-2.02 \le \hbar \le 0.$ We still have freedom to choose the auxiliary parameter according to the $\hbar$ curves. Thus, the arbitrary point of this interval i.e., $\hbar = 0,$ is an appropriate selection for $\hbar$ in which numerical solution converges. From **Figure 4**, the valid regions of convergence correspond to the line segments nearly parallel to the horizontal axis.

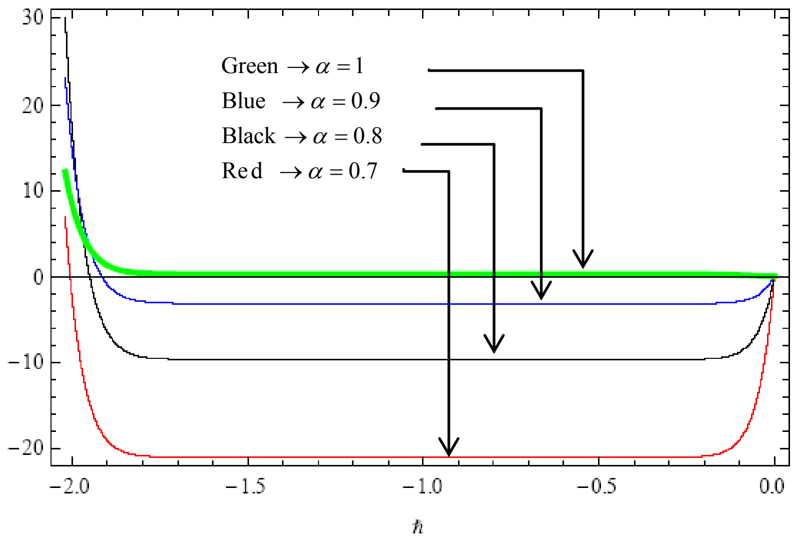

Figure 4 Plot of the $\hbar$ curves for different values of $\alpha = 0.7,\ 0.8,\ 0.9$ and 1 for Example 2.

Example 3 Consider the 3-dimensional heat-like equation with variable coefficients [5-8] as;

$$D_t^\alpha u(x,y,z,t) = x^4y^4z^4 + \frac{1}{36}(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}),\ \ 0 < x,y,z < 1,\ t > 0,\ \ 0 < \alpha \le 1, \tag{55}$$

subject to Neumann boundary conditions;

$$u(0,y,z,t) = 0,\quad u(1,y,z,t) = y^4z^4(e^t - 1),\quad u(x,0,z,t) = 0,\quad u(x,1,z,t) = x^4z^4(e^t - 1),$$
$$u(x,y,0,t) = 0,\qquad u(x,y,1,t) = x^4y^4(e^t - 1), \tag{56}$$

and the initial condition;

$$u(x,y,z,0) = 0. \tag{57}$$

Operating the Laplace transform on both sides in Eq. (55) and after using the differentiation property of the Laplace transform, we get;

$$L[u(x,y,z,t)] - \frac{x^4y^4z^4}{s^{\alpha+1}} - \frac{1}{36}s^{-\alpha}L[x^2u_{xx} + y^2u_{yy} + z^2u_{zz}] = 0. \tag{58}$$

We choose the linear operator as;

$$£[\phi(x,y,z,t;q)] = L[\phi(x,y,z,t;q)], \tag{59}$$

with the property $£[c]=0,$ where c is a constant. We now define a nonlinear operator as;

$$N[\phi(x,y,z,t;q)] = L[\phi(x,y,z,t;q)] - \frac{x^4y^4z^4}{s^{\alpha+1}} - \frac{s^{-\alpha}}{36}L[x^2\phi_{xx} + y^2\phi_{yy} + z^2\phi_{zz}]. \tag{60}$$

Thus, we obtain the mth order deformation equation;

$$£[u_m(x,y,z,t) - \chi_m u_{m-1}(x,y,z,t)] = \hbar\, R_m(\vec{u}_{m-1},x,y,z,t), \tag{61}$$

Operating the inverse Laplace transform on both sides in Eq. (61), we get;

$$u_m(x,y,z,t) = \chi_m u_{m-1}(x,y,z,t) + \hbar\, q\, L^{-1}[R_m(\vec{u}_{m-1},x,y,z,t)], \tag{62}$$

where

$$R_m(\vec{u}_{m-1},x,y,z,t) = L[u_{m-1}(x,y,z,t)] - \frac{(1-\chi_m)x^4y^4z^4}{s^{\alpha+1}} - \frac{s^{-\alpha}}{36}L[x^2(u_{m-1})_{xx} + y^2(u_{m-1})_{yy} + z^2(u_{m-1})_{zz}]. \tag{63}$$

Now the solution of the mth order deformation equations ($m \geq 1$);

$$u_m(x,y,z,t) = (\chi_m + \hbar)u_{m-1} - \frac{\hbar(1-\chi_m)x^4y^4z^4t^\alpha}{\Gamma(\alpha+1)} - \frac{\hbar}{36}L^{-1}\left(s^{-\alpha}L[x^2(u_{m-1})_{xx} + y^2(u_{m-1})_{yy} + z^2(u_{m-1})_{zz}]\right), \tag{64}$$

Using the initial approximation $u_0(x,y,z,t) = u(x,y,z,t) = 0$ and from the iterative scheme (64), we obtain the various iterates;

$$u_1(x,y,z,t) = -\frac{x^4y^4z^4\hbar t^\alpha}{\Gamma(\alpha+1)},$$
$$u_2(x,y,z,t) = -\frac{\hbar(1+\hbar)x^4y^4z^4t^\alpha}{\Gamma(\alpha+1)} + \frac{\hbar^2x^4y^4z^4t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3(x,y,z,t) = -\frac{\hbar(1+\hbar)^2x^4y^4z^4t^\alpha}{\Gamma(\alpha+1)} + \frac{2\hbar^2(1+\hbar)x^4y^4z^4t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\hbar^3x^4y^4z^4t^{3\alpha}}{\Gamma(3\alpha+1)},\ldots.$$

Proceeding in this manner, the rest of the components $u_n(x,y,z,t)$ for $n \geq 5$ can be completely obtained and the series solution is thus entirely determined. Consequently, we obtain the series solution as;

$$u(x,y,z,t) = u_0(x,y,z,t) + \sum_{m=0}^{\infty} u_m(x,y,z,t). \tag{65}$$

If we select $\hbar = -1$, then clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty} u_m(x, y, z, t)$ converges to the solution;

$$u(x, y, z, t) = x^4 y^4 z^4 \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \right) = x^4 y^4 z^4 \left(E_\alpha(t^\alpha) - 1 \right). \quad (66)$$

If we choose $\alpha = 1$ then clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty} u_m(x, y, z, t)$ converges to the exact solution $u(x, y, z, t) = x^4 y^4 z^4 (e^t - 1)$. The above result is in complete agreement with Özis and Agırseven [7].

It is seen from **Figure 5** that the solution obtained by FHATM increases very rapidly with the increases in t at the value of $x = 0.5$, $y = 0.5$, $z = 0.5$ and $\hbar = -1$. It is obvious from **Figure 6** that the acceptable range of the auxiliary parameter $\hbar$ is $-2.0 \le \hbar \le 0$. Thus, the arbitrary point of this interval i.e. $\hbar = 0$, is an appropriate selection for $\hbar$ in which the numerical solution converges. From **Figure 6**, the valid regions of convergence correspond to the line segments nearly parallel to the horizontal axis.

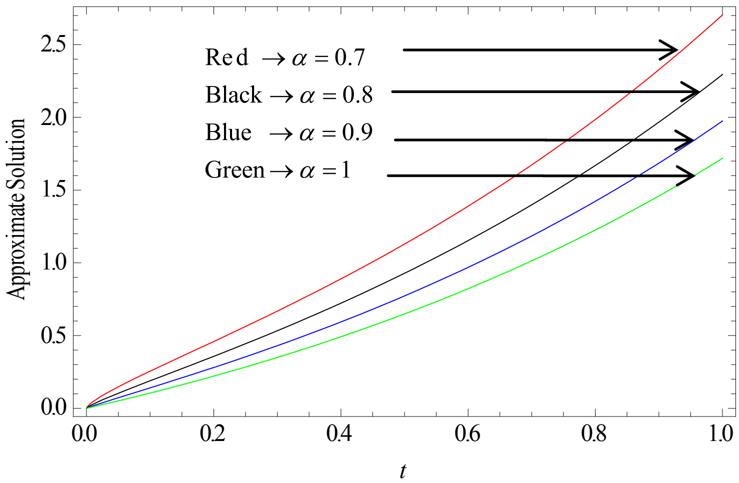

Figure 5 Plot of the solutions obtained by proposed method (FHATM) at different value of $\alpha = 0.7,\ 0.8,\ 0.9$ and 1 for Example 3.

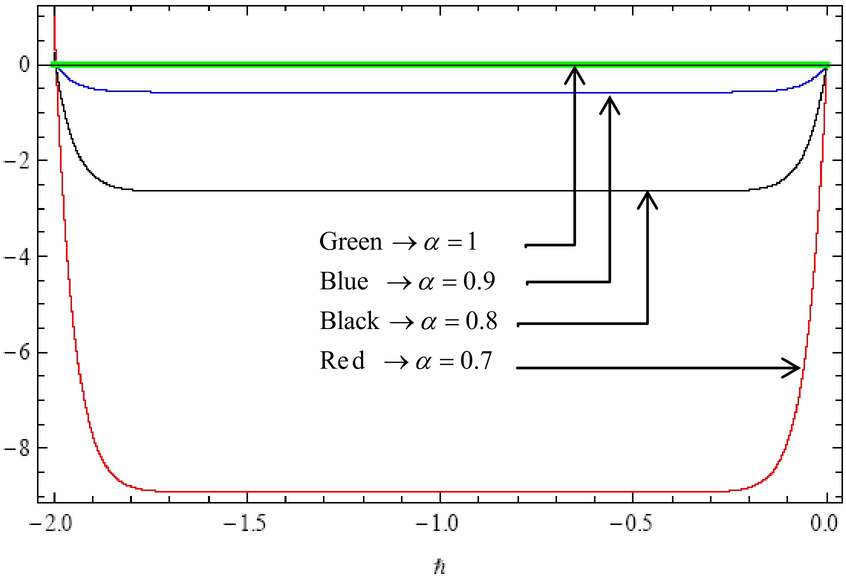

Figure 6 Plot of the $\hbar$ curves for different values of $\alpha = 0.7, 0.8, 0.9$ and 1 for Example 3.

Conclusions

This paper developed an effective modified homotopy analysis method, which coupled the homotopy analysis method and Laplace transform and studied its validity in 3 examples of the time fractional heat equation. An excellent agreement is achieved. The solution is very rapidly convergent by utilizing the homotopy analysis method by modification of the Laplace operator. It may be concluded that the FHATM is very powerful and efficient in finding approximate solutions as well as analytical solutions of many fractional physical models. All numerical results are obtained using Mathematica program 8.

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