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Numerical Analysis of the One-Demential Wave Equation Subject to a Boundary Integral Specification

Babak SOLTANALIZADEH[1,*], Hamidreza ESMALIFALAK[2], Rasoul HEKMATI[3], Zahra SARMAST[3] and Sepideh SHABANI[4]

[1]*Department of Mathematics, Prairie View A&M University, Prairie View, TX, USA*
[2]*Department of Economics and Finance, University of Tasmania, Hobart, Australia*
[3]*Department of Mathematics, University of Houston, Houston, TX, USA*
[4]*Department of Management and Economics, Islamic Azad University, Tabriz, Iran*

([*]Corresponding author's e-mail: babak.soltanalizadeh@gmail.com)

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Abstract

In this paper a numerical technique is developed for the one-dimensional wave equation that combines classical and integral boundary conditions. A new matrix formulation technique with arbitrary polynomial bases is proposed for the analytical solution of this kind of partial differential equation. Not only have the exact solutions been achieved by the known forms of the series solutions, but also, for the finite terms of series, the corresponding numerical approximations have been computed. We give a simple and efficient algorithm based on an iterative process for numerical solution of the method. Some numerical examples are included to demonstrate the validity and applicability of the technique.

Keywords: Wave equation, nonlocal boundary conditions, expansion methods, matrix formulation

Introduction

In 1981, Ortiz and Samara [1] proposed an operational technique for the numerical solution of nonlinear ordinary differential equations, with some supplementary conditions based on the Tau method [2]. The stability and convergence of the Tau approximations have been proved in [3]. During recent years, many authors have used this method for solving various types of equations. For example, in [4], this method has been used for linear ordinary differential eigenvalue problems, and, in [5,6], it has been used for partial differential equations and integral and integro-differential equations [7,8]. Abbasbandi, in [9], used this method for the system of nonlinear Volterra integro-differential equations.

In 1963, nonlocal boundary equation were presented by Cannon [10] and Batten [11] independently. Then, parabolic initial-boundary problems with nonlocal integral conditions for parabolic equations were investigated by Kamynin [12] and Ionkin [13]. There are many physical phenomena that have been formulated in nonlocal form of PDEs. The development of numerical techniques for the solution of hyperbolic non-local boundary value problems has been an important research topic in many branches of science and engineering. Hyperbolic initial-boundary value problems in one dimension, which involve non-local boundary conditions, are studied by several authors. Beilin [14] investigated the non-local analogue to classical mixed problems, which involve initial, boundary integral conditions. Bouziani [15] studied the existence, uniqueness, and continuousness of a hyperbolic partial differential equation. The wave equation is an important second-order linear partial differential equation of waves, such as sound waves, light waves, and water waves. It arises in fields such as acoustics, electromagnetics, and fluid dynamics.

In this paper, a new matrix formulation is presented for the problem of obtaining numerical/analytical approximations to $u(x,t)$ which satisfies the wave equation;

$$u_{tt} - \alpha u_{xx} = f(x,t), \quad 0 < x < L, 0 < t \le T \tag{1}$$

with the initial condition;

$$u(x,0) = r(x), \quad 0 \le x \le L, \tag{2}$$

$$u_t(x,0) = s(x), \quad 0 \le x \le L, \tag{3}$$

and boundary condition;

$$u(0,t) = p(t), \quad 0 < t \le T, \tag{4}$$

$$\int_0^L k(x)u(x,t)dx = m(t), \quad 0 < t \le T, \tag{5}$$

where the functions $f(x,t)$, $r(x), s(x), k(x), p(t)$, and $q(t)$ and the constants α, L, and T are known. The function $f(x,t)$ is often called the source function because, in practice, it describes the effects of the sources of waves on the medium carrying them. Physical examples of source functions include the force driving a wave on a string, or the charge or current density in the Lorenz gauge of electromagnetism.

Beilin proved the existence and uniqueness of a classical solution of (1) - (5) and found its Fourier representation [14]. In [16], Dehghan *et al*. presented several finite differential schemes for the numerical solution of (1) - (5). These 3 level techniques are based on 2 second-order schemes (one explicit and one weighted), and a fourth-order technique (a weighted explicit). Also, in [3], the shifted Legendre Tau method technique was developed for the solution of the studied model. In [17], variational iteration method was used for solving the studied model. The authors of [18] presented a numerical technique based on finite difference and spectral methods. Another method was presented in [19]. A new matrix formulation technique with arbitrary polynomial bases was proposed for the numerical/analytical solution of the heat equation with nonlocal boundary condition [20]. The authors of [21] presented a matrix formulation method based on shifted standard and shifted Chebyshev bases for solving a wave equation with a boundary integral condition. Similar problems can be found in [22-39].

Description of Matrix formulation

In Eqs. (1) - (5), the functions $f(x,t)$, $p(x), k(x), g(t)$, and $m(t)$ generally are not polynomials. We assume that these functions are polynomial, or that they can be approximated by polynomials to any degree of accuracy. For this purpose, one may use one or 2 variate Taylor or Chebyshev series or other suitable methods. So, we can write:

$$\begin{cases} f(x,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} f_{ij}x^i t^j = X^T FT,\ r(x) \cong \sum_{i=0}^{n} r_i x^i = X^T R, \\ s(x) \cong \sum_{i=0}^{n} s_i x^i = X^T S,\ k(x) \cong \sum_{i=0}^{n} k_i x^i = X^T K, \\ g(t) \cong \sum_{j=0}^{n} g_j t^j = GT,\ m(t) \cong \sum_{j=0}^{n} m_j t^j = MT, \end{cases} \tag{6}$$

where

$$\begin{cases} X = [1, x, x^2, ..., x^n]^T, T = [1, t, t^2, ..., t^n]^T, \\ S = [s_0, s_1, s_2, ..., s_n]^T, R = [r_0, r_1, r_2, ..., r_n]^T, K = [k_0, k_1, k_2, ..., k_n]^T, \\ G = [g_0, g_1, g_2, ..., g_n]^T,\ M = [m_0, m_1, m_2, ..., m_n]^T, \\ F = [F_0, F_1, F_2, ..., F_n], F_i = [f_{0i}, f_{1i}, f_{2i}, ..., f_{ni}]^T, i = 0,1,2,...,n. \end{cases} \tag{7}$$

Therefore, we consider the approximate solution of the form;

$$U_n(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij}x^i t^j = X^T UT, \tag{8}$$

where $U = [U_0, U_1, U_2, ..., U_n]$, with $U_i = [u_{0i}, u_{1i}, u_{2i}, ..., u_{ni}]^T$. The matrix U is an $(n+1)\times(n+1)$ matrix which contains $(n+1)^2$ unknown coefficients of $U_n(x,t)$. To find these unknowns, we proceed as follows.

We first consider the initial condition;

$$u(x,0) = r(x), \quad 0 \le x \le L. \tag{9}$$

then, with due attention to Eq. (6) and Eq. (8), we obtain;

$$X^T U_0 = X^T R, \tag{10}$$

which implies;

$$U_0 = R. \tag{11}$$

since X is a basis vector. From above equation, we can find the first column of U.

Now, consider;

$$u(0,t) = p(t),\ 0 < t \le T. \tag{12}$$

Substituting Eqs. (6) and (8) in Eq. (10), we get;

$$X_1^T UT = PT. \tag{13}$$

where $X_1 = [1, 0, 0, ..., 0]$. Since T is a basis vector, we have;

$$X_1^T U = P. \tag{14}$$

Now, we consider the nonlocal boundary condition;

$$\int_0^L k(x)u(x,t)dx = m(t). \tag{15}$$

We substitute from Eqs. (6) and (8) and obtain;

$$\int_0^L (\sum_{h=0}^{n} k_h x^h)(\sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij} x^i t^j)dx = \sum_{j=0}^{n} m_j t^j, \tag{16}$$

hence;

$$\begin{aligned}\sum_{j=0}^{n} m_j t^j &= (\sum_{h=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{n} k_h u_{ij} t^j \int_0^L x^{h+i} dx) \\ &= (\sum_{h=0}^{n}\sum_{i=0}^{n}\sum_{j=0}^{n} k_h u_{ij} t^j v_{h+i+1}), \qquad 0 < t \le T.\end{aligned} \tag{17}$$

or equivalently;

$$DUT = MT. \tag{18}$$

where

$$D = [d_0, d_1, d_2, ..., d_n], \tag{19}$$

with

$$d_i = \sum_{h=0}^{n} k_h v_{h+i+1}, \quad i = 0,1,2,...,n, \tag{20}$$

and

$$v_{h+i+1} = \frac{L^{h+i+1}}{h+i+1}, \quad h,i = 0,1,2,...,n. \tag{21}$$

The equivalent from (18) is;

$$DU = M, \tag{22}$$

since T is a basis vector. Now, we recall the following lemma from [4,8], to write Eq. (1) in the matrix form to determine remainder equations. This lemma is proved by induction.

Lemma 1 *The effect of r repeated differentiation on coefficients vector* $\underline{a} = [a_0, a_1, a_2, ..., a_n]$ *of a polynomial* $y_n(x) = \underline{a}X$ *is the same as that of post-multiplication of* $\underline{a}$ *by the matrix* η^r*;*

$$\frac{d^r}{dx^r} y_n(x) = \underline{a}\eta^r X, \tag{23}$$

where η is the $(n+1)\times(n+1)$ operational matrix of derivation ;

$$\eta = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3 & 0 & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & n & 0 \end{bmatrix}_{(n+\_1)(n+1)}$$

Corollary 1 *By using lemma 1, we have;*

$$u_{xx} = X^T(\eta^T)^2 UT, \quad u_t = X^T U\eta T, \quad u_{tt} = X^T U\eta^2 T. \tag{24}$$

Now, consider;

$$u_t(x,0) = s(x), \quad 0 \le x \le L, \tag{25}$$

Applying Eqs. (6), (8) and (24), we obtain;

$$U\eta T(0) = S, \quad 0 \le x \le L, \tag{26}$$

where $T(0) = [1,0,0,...,0]^T$. Finally, consider;

$$u_{tt} - \alpha u_{xx} = f(x,t), \quad 0 < x < L, 0 < t \le T. \tag{27}$$

Therefore, substituting from Eqs. (6), (8) and (24) in Eq. (24) leads to;

$$X^T U \eta^2 T - \alpha X^T (\eta^T)^2 U T = X^T F T,$$

or

$$U\eta^2 - \alpha(\eta^T)^2 U = F. \tag{28}$$

since X and T are basis vectors.

Determining final linear equations

In this section, we arrange the linear equations obtained in the previous section to have a system of $(n+1)(n+1)$ equations for the $(n+1)(n+1)$ unknowns. We find $(n+1)$ equations from Eq. (8). Note that, since the Eq. (1) and boundary conditions (4) and (5) are not defined for $x \in [0,1]$ and $t \in [0,T]$, we choose n equations from Eq. (14), n equations from Eq. (17), n equations from Eq. (18), and $n(n-2)$ equations from Eq. (28).

By Eq. (9), we have;

$$U(i,0) = r_i, \ i = 0,1,2,...,n, \tag{29}$$

and from Eq. (26), we have;

$$U(0,j) = p_j, \ j = 1,2,...,n, \tag{30}$$

and from Eq. (14), we get;

$$U(i,1) = s_i, \ i = 1,2,...,n, \tag{31}$$

from Eq. (22), we get;

$$\sum_{i=0}^{n} d_i U(i,j) = m_j, \ j = 1,2,...,n. \tag{32}$$

Finally, for $j = 1,2,...,n$ By Eq. (20), we have;

$$\begin{cases} U(i,j) = \dfrac{1}{(j-1)(j)}(f(i,j-2) + \alpha(i+2)(i+1)U(i+2,j-2)), \\ \ i = 1,...,n-2, \\ U(n-1,j) = \dfrac{1}{(j-1)(j)}(f(n-1,j)). \end{cases} \tag{33}$$

The Eqs. (29) - (33) generate a system of $(n+1)(n+1)$ equations. By solving this system of equations, the unknown coefficients of U can be calculated. For solving this system, we introduce a simple interesting process. We know that the first column of U by Eq. (29) and the first row of U by Eq. (30) and the second column by Eq. (31) were obtained. In every step, we obtain one column of U by an interesting process. For example, for finding the second column of U, we know that the value of $U(0,1)$ was calculated. Thus, from Eq. (33), the values of $U(2,1),U(3,1),...,U(n-1,1)$ can be found. Now, if we utilize Eq. (32), then the value of $U(n,1)$ will be found. Here, the 3rd column of U has been obtained. By a similar process, we obtain the remainder columns of U. Therefore, the corresponding algorithm can be introduced to under case:

Algorithm

step 1: Choose $n \in N$ as the degree of approximate solution.
step 2: Determine the vectors K, P, G, D and M and the matrix F.
step 3: Set $X=[1,x,x^2,x^3,...,x^n]^T, T=[1,t,t^2,t^3,...,t^n]^T$.
step 4: For $i=0,1,2,...,n,$ set $U(i,0)=r_i$.
step 5: For $j=1,2,...,n,$ set $U(0,j)=p_j$.
step 6: For $i=1,2,...,n,$ set $U(i,1)=s_i$.
step 7: For $j=2,3,...,n$

$$U(n-1,j)=\frac{1}{(j-1)(j)}f(n-1,j-1).$$

for i=1,2,3,...,n-2 {

$$U(i,j)=\frac{1}{(j-1)(j)}(f(i,j-1)+\alpha(i+2)(i+1)U(i+2,j-1)).$$

}

$$U(n,j)=\frac{1}{d_n}(m_j-\sum_{i=0}^{n-1}d_iU(i,j)).$$

step 8: Set $U_n(x,t)=X^TUT$.
step 9: End

Numerical examples

In this section, we apply the process presented in this paper and solve 5 examples. These examples are chosen such that their exact solutions are known. The numerical computations have been done by the software Matlab. Let $e_n(x,t)=u_n(x,t)-u(x,t)$; we calculate the following norms of the error for different values of n.

$$E_\infty=\|e_n(x,t)\|_\infty=\max\{|e_n(x,t)|,\ 0\le x\le L,\ 0\le t\le T\}. \tag{34}$$

Example 1 *([34])Consider the heat equation presented in Eqs. (1) - (5) with;*

$$f(x,t) = (2x^5 + 2x^3 - 2x^2) - (20x^3 + 6x - 2)(t^2 - t),$$

$$r(x) = 0, \quad s(x) = x^2 - x^3 - x^5, \quad k(x) = 1,$$

$$g(t) = 0, \quad m(t) = -\frac{t(t-1)}{12}$$

$$\alpha = 1, \quad L = 1, \quad T = 1.$$

The authors in [34] solved this example by 2 forms of MOL, and obtained the results of **Table 1**.

Table 1 The norm $\|u_{exact} - u_{MOL}\|_{x_i,\infty}$in [34].

x	MOL 1	MOL 2
0.1	$1.4589452381\times10^{-4}$	$2.27205158021\times10^{-8}$
0.2	$2.7172977818\times10^{-4}$	$1.59301172808\times10^{-8}$
0.3	$3.7488135133\times10^{-4}$	$2.98653940467\times10^{-8}$
0.4	$4.3322433196\times10^{-4}$	$4.39071972114\times10^{-8}$
0.5	$4.1659346493\times10^{-4}$	5.996250296×10^{-8}
0.6	$2.9994415140\times10^{-4}$	$7.65543484160\times10^{-8}$
0.7	5.996250296×10^{-5}	$9.44377208656\times10^{-8}$
0.8	$3.3337581084\times10^{-4}$	$1.134464433505\times10^{-7}$
0.9	$8.9980978039\times10^{-4}$	$1.326242955457\times10^{-7}$
1.0	$1.66653416461\times10^{-3}$	$2.4346786631213\times10^{-6}$

Selecting $n = 2$ and using Eqs. (26) and (27), we get;

$$U(0,0) = 2, \; U(1,0) = 0, \; U(2,0) = 4,$$

$$U(0,1) = 1, \; U(0,2) = 8.$$

Now, from Eq. (29) and then Eq. (28), we obtain;

$$U(1,1) = 0, \; U(2,1) = 4,$$

and also

$$U(1,2) = 0, \; U(2,2) = 32,$$

then

$$U=\begin{bmatrix}2 & 1 & 8\\ 0 & 0 & 0\\ 8 & 4 & 32\end{bmatrix}$$

and by using Eq. (7), we have;

$$U(x,t)=2+t+8t^2+8x^2+4x^2t+32x^2t^2,$$

which it is the exact solution of the problem.

Example 2 *Consider the Eqs. (1)-(5) with;*

$$f(x,t)=(x^2-2)e^t,\quad r(x)=s(x)=x^2,\quad k(x)=1,$$

$$g(t)=0,\quad m(t)=\frac{e^t}{3},\quad \alpha=L=1,\quad T=0.5.$$

Applying Eqs. (26) - (27), we obtain;

$$U(0,j)=0,\quad j=1,2,...,n,$$

$$U(0,0)=0,\ U(1,0)=0,\ U(2,0)=1,\ U(3,0)=...=U(n,0)=0,$$

and then

$$U(1,1)=0,\ U(2,1)=1,\ U(3,1)=...=U(n-2,1)=0,$$

$$U(n-1,1)=1(f(n-1,1))=0,$$

$$U(n,1)=(n+1)(\frac{1}{3}-\frac{1}{3})=0.$$

By continuing this process, we get;

$$u(x,t)=x^2(1+t+\frac{t^2}{2!}+...+\frac{t^n}{n!}),$$

which is the Taylor expansion of the $u(x,t)=x^2e^t$, which is the exact solution of this example. The numerical results of this example are reported in **Tables 2** and **3** and a plot of corresponding error function is shown in **Figure 1**.

Table 2 Absolute errors of the presented method for n = 15.

x	$t = 0.1$	$t = 0.3$	$t = 0.5$
0	0	0	0
0.1	3.469×10^{-18}	1.735×10^{-18}	3.469×10^{-18}
0.2	1.388×10^{-17}	6.939×10^{-18}	1.388×10^{-17}
0.3	4.163×10^{-17}	1.388×10^{-17}	5.551×10^{-17}
0.4	5.551×10^{-17}	2.776×10^{-17}	5.551×10^{-17}
0.5	1.110×10^{-16}	5.551×10^{-17}	1.110×10^{-16}
0.6	1.665×10^{-16}	5.551×10^{-17}	2.220×10^{-16}
0.7	2.220×10^{-16}	1.110×10^{-16}	2.220×10^{-16}
0.8	2.220×10^{-16}	1.110×10^{-16}	2.220×10^{-16}
0.9	3.331×10^{-16}	2.220×10^{-16}	4.441×10^{-16}
1.0	4.441×10^{-16}	2.220×10^{-16}	4.441×10^{-16}

Table 3 Maximum errors for $x = 0 : 0.05 : 1.0,\ \ t = 0 : 0.025 : 0.5$.

	$n=5$	$n=10$	$n=13$	$n=15$
E_∞	2.335×10^{-5}	1.276×10^{-11}	1.110×10^{-15}	6.661×10^{-16}

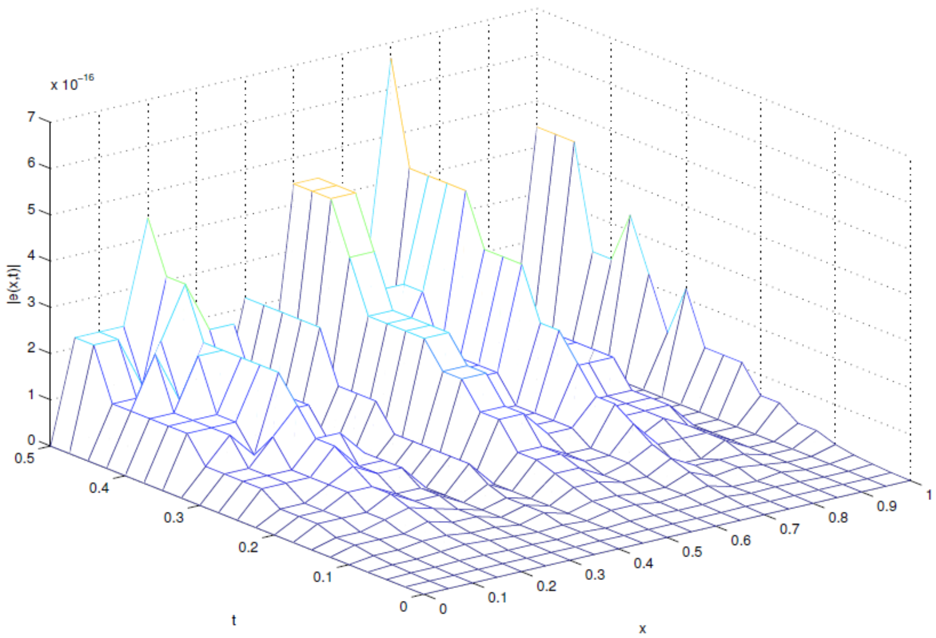

Figure 1 Plot of error function from Example 2. for n = 15.

Example 3 *Consider the Eqs. (1)-(5) with;*

$$f(x,t)=e^{(x+t)},\quad r(x)=e^{x},\quad s(x)=e^{x},\quad k(x)=1,$$

$$p(t)=e^{t},\quad m(t)=(e-1)e^{t},\quad \alpha=2,\quad L=T=1,$$

using Eqs. (26) and (27), we get;

$$U(i,0)=\frac{1}{i!},\ i=0,1,2,...,n,$$

$$U(0,j)=\frac{1}{j!},\ j=1,2,...,n,$$

$$U(i,1)=\frac{1}{i!},\ i=1,2,...,n,$$

and from Eq. (29), we have;

$$U(i,2)=\frac{1}{2!i!},\ i=1,2,...,n-2,$$

$$U(n-1,2)=\frac{1}{2!(n-1)!},\ i=1,2,...,n-2,$$

Then, by using Eq. (28) and the above numerical results, we get;

$$U(n,2)=e-1-1-\frac{1}{2!2!}-\frac{1}{2!3!}-\frac{1}{2!4!}-...-\frac{1}{2!(n-1)!};\frac{1}{2!n!}.$$

By using this recursive scheme, the remainder values of $U(i,j)$ are obtained. Now, by using Eq. (7), we get;

$$u(x,t);1+t+\frac{1}{2!}t^2+...+\frac{1}{n!}t^n$$
$$+x+xt+\frac{1}{2!}xt^2+...+\frac{1}{n!}xt^n++\frac{1}{2!}x^2+\frac{1}{2!}x^2t+\frac{1}{2!2!}x^2t^2+...+\frac{1}{n!n!}x^nt^n+...,$$

which is the Taylor expansion of the $u(x,t)=e^{x+t}$, which is the exact solution of this example. The numerical results of the absolute errors with $n=5,10,15,20$ obtained by using our described method are given in **Tables 4** and **5** and a plot of corresponding error function is shown in **Figure 2**.

Table 4 Absolute errors of the presented method n = 20.

x	$t = 0.1$	$t = 0.3$	$t = 0.5$
0	4.441×10^{-16}	2.220×10^{-16}	4.441×10^{-16}
0.1	6.661×10^{-16}	4.441×10^{-16}	2.220×10^{-16}
0.2	6.661×10^{-16}	2.220×10^{-16}	4.441×10^{-16}
0.3	6.661×10^{-16}	4.441×10^{-16}	8.882×10^{-16}
0.4	8.882×10^{-16}	4.441×10^{-16}	8.882×10^{-16}
0.5	8.882×10^{-16}	8.882×10^{-16}	8.882×10^{-16}
0.6	8.882×10^{-16}	4.441×10^{-16}	8.882×10^{-16}
0.7	1.332×10^{-15}	8.882×10^{-16}	1.332×10^{-15}
0.8	1.332×10^{-15}	8.882×10^{-16}	1.332×10^{-15}
0.9	8.882×10^{-16}	8.882×10^{-16}	8.882×10^{-16}
1.0	1.776×10^{-15}	1.332×10^{-15}	1.776×10^{-15}

Table 5 Maximum errors for $x = 0:0.05:1.0,\ t = 0:0.025:0.5$.

	$n=5$	$n=10$	$n=15$	$n=20$
E_∞	1.767×10^{-3}	2.858×10^{-8}	2.256×10^{-13}	2.220×10^{-15}

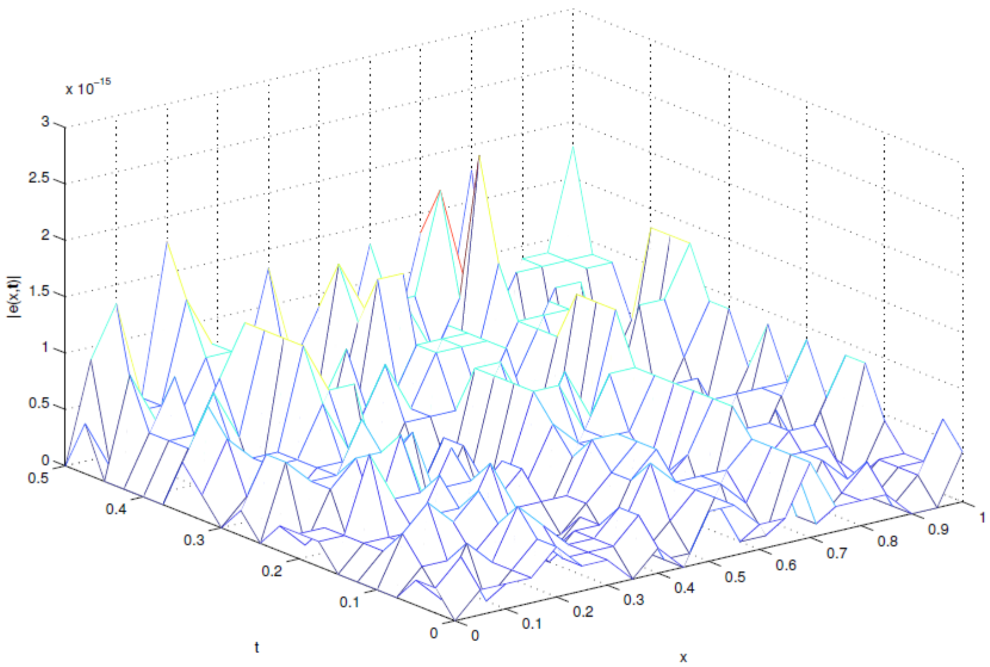

Figure 2 Plot of error function from Example 3. for n = 20.

Example 4 *([35, 36])Consider the Eqs. (1)-(5) with;*

$$f(x,t)=0,\ \ r(x)=0,\ \ s(x)=\pi cos(\pi x),$$
$$p(t)=0,\ \ m(t)=sin(\pi t),\ \ \alpha=1,\ \ L=1,\ \ T=0.5.$$

For this example, we solve the problem by applying the technique described in the preceding section and obtain the closed form of the solution as follows;

$$u_n(x,t);-x(t^2-t^3+\frac{t^4}{2!}-\frac{t^5}{3!}+...)-x^2(t^2-t^3+\frac{t^4}{2!}-\frac{t^5}{3!}+...)$$

which is the Taylor expansion of the $u(x,t)=cos(\pi x)sin(\pi t)$, which is the exact solution of this example. The numerical results of the absolute errors with $n = 10, 20, 25, 26, 30$ obtained by using our described method are given in **Tables 6 - 8**. and a plot of corresponding error function is shown in **Figure 3**.

Table 6 Absolute errors of Tau and Finite difference methods for t = 0.5.

x	The Optimal explicit [36]	minimum errors of [35]
0.1	3.3×10^{-5}	4.9×10^{-9}
0.2	3.0×10^{-5}	3.2×10^{-9}
0.3	3.2×10^{-5}	6.7×10^{-9}
0.4	3.1×10^{-5}	1.1×10^{-8}
0.5	3.3×10^{-5}	0
0.6	3.4×10^{-5}	1.1×10^{-8}
0.7	3.1×10^{-5}	6.7×10^{-9}
0.8	3.2×10^{-5}	3.2×10^{-9}
0.9	3.4×10^{-5}	4.9×10^{-9}

Table 7 Absolute errors of the presented method for n = 30.

x	$t=0.1$	$t=0.3$	$t=0.5$
0	5.551×10^{-17}	0	2.220×10^{-16}
0.1	5.551×10^{-17}	0	1.110×10^{-16}
0.2	0	1.110×10^{-16}	1.110×10^{-16}
0.3	2.776×10^{-17}	0	1.110×10^{-16}
0.4	1.388×10^{-17}	1.388×10^{-16}	2.220×10^{-16}
0.5	5.567×10^{-17}	2.525×10^{-17}	5.459×10^{-16}
0.6	2.776×10^{-17}	1.110×10^{-16}	7.772×10^{-16}
0.7	8.327×10^{-17}	5.551×10^{-17}	5.551×10^{-16}
0.8	5.551×10^{-17}	3.331×10^{-16}	1.110×10^{-15}
0.9	5.551×10^{-17}	3.331×10^{-16}	5.107×10^{-15}
1.0	5.551×10^{-17}	2.220×10^{-16}	4.663×10^{-15}

Table 8 Maximum errors for $x = 0:0.05:1.0,\ t = 0:0.025:0.5$.

	$n=10$	$n=20$	$n=25$	$n=26$
E_∞	2.449×10^{-3}	7.840×10^{-11}	1.324×10^{-13}	6.217×10^{-15}

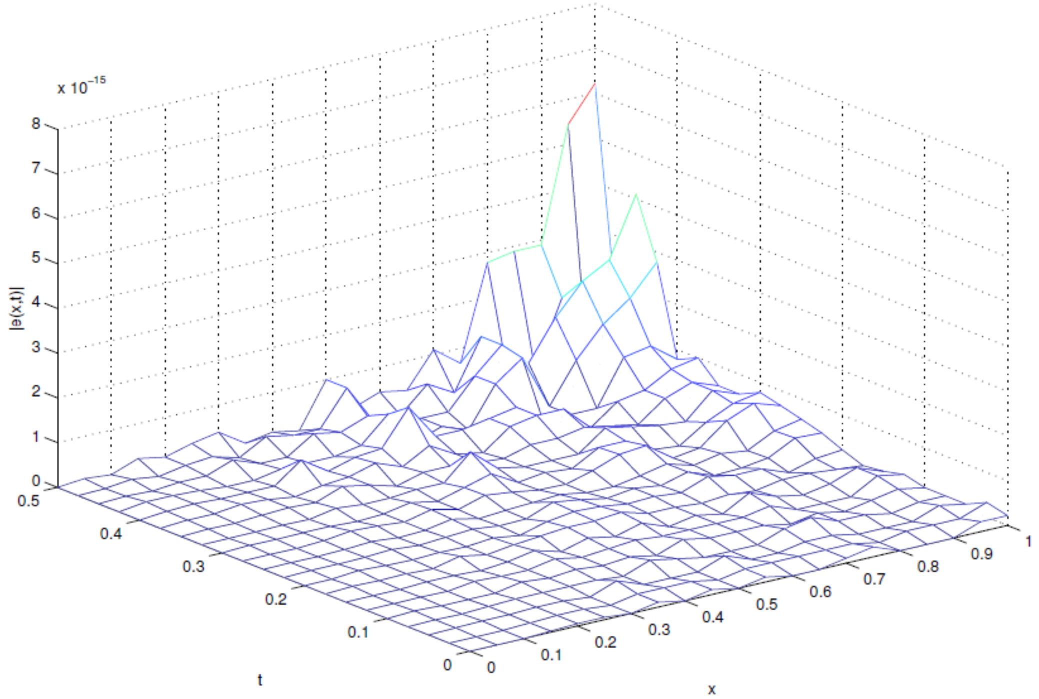

Figure 3 Plot of error function from Example 4. for n = 26.

Example 5 *([35,36])Consider the Eqs. (1)-(5) with;*

$$f(x,t) = 0,\quad r(x) = cos(\pi x),\quad s(x) = 0,$$

$$p(t) = cos(\pi t),\quad m(t) = 0$$

$$\alpha = 1,\quad L = 1,\quad T = 0.25.$$

For this example, we solve the problem by applying the technique described in the preceding section and obtain the closed form of the solution as follows;

$$u_n(x,t)=(1-\frac{1}{2}\pi^2t^2+\frac{1}{4!}\pi^4t^4-\frac{1}{6!}\pi^6t^6+\ldots)$$
$$-\frac{1}{2}\pi^2x^2(1-\frac{1}{2}\pi^2t^2+\frac{1}{4!}\pi^4t^4-\frac{1}{6!}\pi^6t^6+\ldots)$$
$$+\frac{1}{24}\pi^4x^4(1-\frac{1}{2}\pi^2t^2+\frac{1}{4!}\pi^4t^4-\frac{1}{6!}\pi^6t^6+\ldots),$$

which is the Taylor expansion of the $u(x,t)=\frac{1}{2}(cos(\pi(x+t)))+(cos(\pi(x-t))),$ which is the exact solution of this example. The numerical results of the absolute errors with $n = 10, 20, 25. 26$ obtained by using our described method are given in **Table 9**, and a plot of corresponding error function is shown in **Figure 4**.

Table 9 Maximum errors for $x = 0:0.05:1.0,\ t = 0:0.025:0.5$.

	$n=10$	$n=20$	$n=25$	$n=26$
E_∞	1.829×10^{-3}	7.565×10^{-11}	6.417×10^{-14}	2.887×10^{-15}

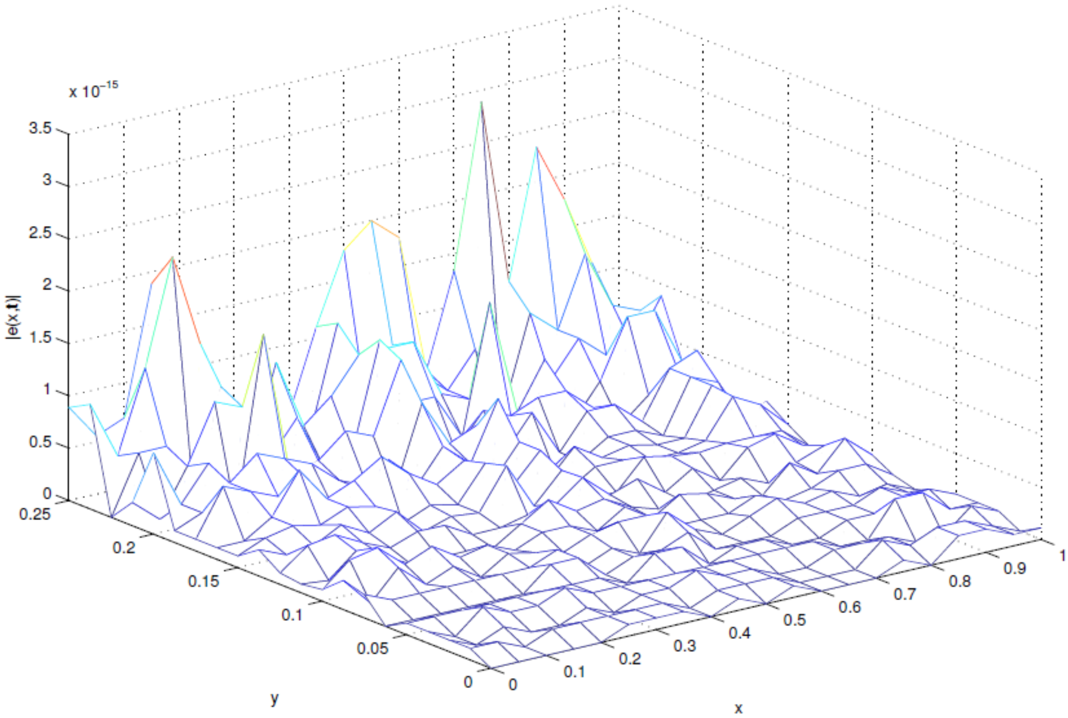

Figure 4 Plot of error function from Example 5. for n = 26.

Conclusions

In this paper, we focus on the Wave equation with nonlocal boundary condition and present an operational matrix formulation to solve this equation. By using this method, numerical/analytical results are obtained by a simple iterative process. This method reduces the computational difficulties of the other methods and all the calculations can be made with a simple iterative process. Also, we can increase the accuracy of the series solution by increasing the number of terms in the series solution. Consequently, it is seen that this method can be an alternative way for the solution of partial differential equations that have no analytic solutions.

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