Numerical Solution for Fractional Partial Differential Equations Using Crank-Nicolson Method with Shifted Grünwald Estimate

Abdollah BORHANIFAR[*] and Sohrab VALIZADEH

Department of Mathematics, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran

([*]**Corresponding author; e-mail: borhani@uma.ac.ir**)

Abstract

In this paper, the novel hybrid finite difference type Crank-Nicolson scheme with the aid of shifted Grünwald estimate is proposed to solve fractional partial differential equations. Consistency of the proposed method is confirmed using fractional Taylor's expansion. Error analysis and properties of the scheme are proved. It is proved that the truncation error for this scheme is of the order of the fractional. Stability and convergence of the proposed method is proved. The exact solution is obtained via two-steps Adomian decomposition method. Companions are made between this proposed scheme and the closed analytical form solution. Numerical results are given.

Keywords: Crank-Nicolson method, fractional partial differential equations, fractional Taylor's series, Riemann-Liouville fractional derivative, shifted Grünwald Estimate, two-step Adomian decomposition method

Introduction

Various powerful methods have been presented to find the approximation solutions of nonlinear partial differential equations and fractional PDE, for example, Exp-function, G'/G-function, DTM method, tanh function method, sinh function method and so on [1-8]. In recent years, fractional calculus is used for many problems in physics, mathematics and engineering in this direction we refer for example to the papers by Ibrahim and Momani [9], Letinivov [10]. Oeser and Freitag [11] in 2009 used fractional differential equations for the behavior of rheological materials that exhibit special load history characteristics. In the last decade fractional partial differential equations have many applications in science and technology by Fitt, *et al.* in [12]. Meerschaert and Tadjeran [13] in 2006 gave numerical solutions for specific fractional PDEs using finite difference types of explicit and implicit Euler methods. In this paper, we examine fractional finite difference methods of Crank-Nicolson type to solve the general fractional partial differential equations (FPDEs) of the form

$$\frac{\partial U(x,t)}{\partial t} = c(x)\frac{\partial^\alpha U(x,t)}{\partial x^\alpha} + s(x,t) \qquad (1)$$

on a finite domain

$$\Omega = \{(x,t)|L \le x \le R, 0 \le t \le T\}.$$

Here, we consider the case $1 \le \alpha \le 2$, where the parameter α is the fractional order of the spatial derivative. The function $s(x,t)$ is a source or sink term. The functions $c(x) \ge 0$ may be interpreted as transport related coefficients. We also assume an initial value $U(x,0) = f(x)$ for $L \le x \le R$ and zero Dirichlet boundary conditions. For $\alpha = 1$ and $\alpha = 2$ Eq. (1) reduces to the following classical hyperbolic and parabolic PDEs respectively

$$\frac{\partial U(x,t)}{\partial t} = c(x)\frac{\partial U(x,t)}{\partial x} + s(x,t) \qquad (2)$$

$$\frac{\partial U(x,t)}{\partial t} = c(x)\frac{\partial^2 U(x,t)}{\partial x^2} + s(x,t). \tag{3}$$

Definitions

Riemann-Liouville Fractional Derivatives

Let function $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$, has continuous derivatives of order n.

$$D_x^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x - t)^{n-\alpha-1} f(t)dt \tag{4}$$

which is a Riemann-Liouville fractional derivative of order α, where n is an integer such that $n - 1 \leq \alpha \leq n$ and Γ is the gamma function (see [14-16]). For the function $f(x) = x^m$, α derivative of x^m is as follows

$$D_x^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{m-\alpha}. \tag{5}$$

Lax-Richtmyer's equivalence theorem

Given a properly posed linear initial-value problem and a linear finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence [17].

Taylor's expansion of fractional order

Assume that the continuous function $f : \mathbb{R} \to \mathbb{R}, x \to f(x)$, has fractional derivative of order $k\alpha$ for any positive integer k and any α, $1 \leq \alpha \leq 2$. Then the following fractional Taylor's series holds

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)}f^{(k\alpha)}(x), \tag{6}$$

where $f^{(k\alpha)}(x)$ is the derivative of order $k\alpha$ of $f(x)$. Formally, one has

$$f(x+h) = E_\alpha\big(h^\alpha D_x^\alpha\big)f(x), \tag{7}$$

where $E_\alpha(x)$ denotes the Mittag-Leffler function defined by the expression

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}. \tag{8}$$

we draw attention to the point that $\Gamma(1+k\alpha) := (k\alpha)!$ [8,18].

Grünwald estimate

Standard Grünwald estimate

when $1 \leq \acute{a} \leq 2$, we define the standard Grünwald formula

$$\frac{d^{\acute{a}}f}{dx^{\acute{a}}} = \lim_{M\to\infty} \frac{1}{h^{\acute{a}}}\sum_{k=0}^{M} g_k.f(x - (k+1)h), \tag{9}$$

that defines the following standard Grünwald estimate to the fractional derivative

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{h^\alpha}\sum_{k=0}^{M} g_k.f(x - (k+1)h) + O(h^\alpha). \tag{10}$$

Shifted Grünwald estimate

When $1 \leq \acute{a} \leq 2$, we define the shifted Grünwald formula

$$\frac{d^\alpha f}{dx^\alpha} = \lim_{M\to\infty} \frac{1}{h^\alpha}\sum_{k=0}^{M} g_k.f(x - (k-1)h), \tag{11}$$

that defines the following shifted Grünwald estimate to the fractional derivative

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{h^\alpha}\sum_{k=0}^{M} g_k.f(x - (k-1)h) + O(h^\alpha), \tag{12}$$

where M are positive integers, $h = \frac{R-L}{M}$, Γ is the gamma function, and the normalized Grünwald weights are defined by

$$g_k = (-1)^k \frac{\tilde{A}(\acute{a}+1)}{\tilde{A}(k+1)\,\tilde{A}(\acute{a}-k+1)} \tag{13}$$

for $k = 0,1,2,...$,

or

$$g_0 = 1 \text{ and } g_k = (-1)^k\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} \tag{14}$$

for $k = 1,2,3,...$

Note that these normalized weights only depend on the order α and the index k (see [16,19-20]).

Approximating the fractional partial differential equation

We examine Crank-Nicolson finite difference methods of order α in space for the solution of the following time-dependent fractional partial differential equations

$$U_t - c(x)U_{\alpha x} - s(x,t) = 0. \tag{15}$$

Along with the initial value and Dirichlet boundary conditions

$$U(x, t_0) = f(x), \qquad L \leq x \leq R. \tag{16}$$

And also boundary conditions if we are working on a bounded domain, e.g., the zero Dirichlet conditions

$$U(L, t) = 0 \quad \text{for } 0 \leq t \leq T$$
$$U(R, t) = 0 \quad \text{for } 0 \leq t \leq T. \tag{17}$$

In practice we generally apply a set of finite difference equations on a discrete grid with grid points (x_i, t_j) where $i = 1, 2, \ldots, K, \ j = 1, 2, \ldots, n,$

$x_i = L + ih, \ t_j = jk$. Here $h = \Delta x$ is the mesh spacing on the x-axis and $k = \Delta t$ is the time step. Let $u_{i,j} = u(x_i, t_j)$ represent the numerical approximation at grid point (x_i, t_j). Since the fluid flow is an evolution equation that can be solved forward in time, we set up our difference equations in a form where we can march forward in time, determining the values $U_{i,j+1}$ for all i from the values $U_{i,j}$ at the previous time level, or perhaps using also values at earlier time levels with a multistep formula. One natural discretization of Eq. (15) based on shifted Grünwald estimates Eq. (12) would be

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \frac{c_i}{2h^\alpha} \sum_{z=0}^{1} \sum_{p=0}^{i+1} g_k \cdot u_{i-p+1,j+z} - \frac{s_{i,j} + s_{i,j+1}}{2} = 0, \tag{18}$$

or for $r = \frac{\Delta t}{2h^\alpha}$ we have

$$u_{i,j+1} = u_{i,j} + rc_i \sum_{z=0}^{1} \sum_{p=0}^{i+1} g_k \cdot u_{i-p+1,j+z} + k \frac{s_{i,j} + s_{i,j+1}}{2}, \tag{19}$$

i.e.,

$$rg_0 c_i u_{i+1,j+1} + (1 - rg_1 c_i) u_{i,j+1} - rc_i \sum_{k=2}^{i+1} g_k \cdot u_{i-k+1,j+1}$$

$$= rg_0 c_i u_{i+1,j} + (1 + rg_1 c_i) u_{i,j} + rc_i \sum_{k=2}^{i+1} g_k \cdot u_{i-k+1,j} + k \frac{s_{i,j} + s_{i,j+1}}{2}. \tag{20}$$

The computational molecule corresponding to Eq. (19) is shown in **Figure 1**. Denote $u_{i,j+1}$ by $u_{i,j}$ for $i = 1, 2, \ldots, K$
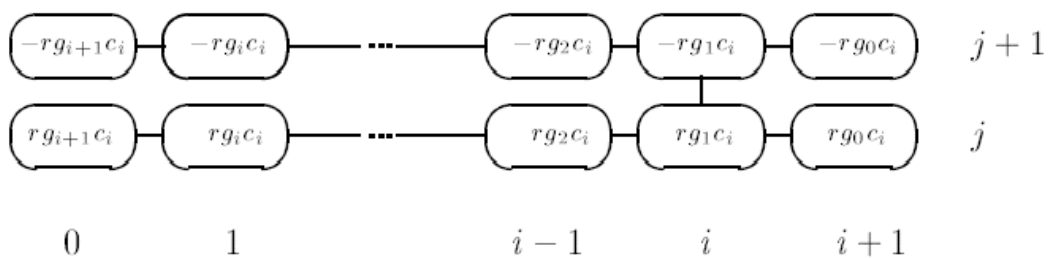

Figure 1 The molecule of fractional Crank-Nicolson method in rows j and $j + 1$.

The local truncation error and consistency

Let $F_{i,j}(u) = 0$ represent fractional difference equation approximating the fractional partial differential equation at the (i, j)th mesh point, with exact solution u. If u is replaced by U at the mesh points of the difference equation, where U is the exact solution of the fractional partial difference equation, the value of $F_{i,j}(U)$ is called the local truncation error $T_{i,j}$ at the (i, j) mesh point. $F_{i,j}(U)$ measures the amount by which the exact solution values of the fractional partial differential equation at the mesh points of the difference equation do not satisfy the difference equation at the point $(ih, j\Delta t)$.

Using standard and fractional Taylor's expansions, it is easy to express $T_{i,j}$ in terms of powers of h, Δt, partial derivatives and fractional partial derivatives of U at $(ih, j\Delta t)$.

Although U and its derivatives are generally unknown, the analysis is useful because it provides a method for comparing the local accuracies of different difference schemes approximating the fractional partial differential equation.

Theorem 1. The local truncation error of the fractional Crank-Nicolson difference approximation with $1 \le \alpha \le 2$ for fractional equation

$$\frac{\partial U}{\partial t} - c \frac{\partial^\alpha U}{\partial x^\alpha} - s = 0 \tag{21}$$

at the point $(ih, j\Delta t)$ is $T_{i,j} = O(\Delta t^2) + O(h^\alpha) + O(h^\alpha \Delta t)$.

Proof. Since u is the exact solution of the Crank-Nicolson from Eq. (18) we can write

$$F_{i,j}(u) = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} - \frac{c_i}{2h^\alpha} \sum_{z=0}^{1} \sum_{k=0}^{i+1} g_k. u_{i-k+1,j+z} - \frac{s_{i,j} + s_{i,j+1}}{2} = 0 \tag{22}$$

$$T_{i,j} = F_{i,j}(U) = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} - \frac{c_i}{2h^\alpha} \sum_{z=0}^{1} \sum_{k=0}^{i+1} g_k. U_{i-k+1,j+z} - \frac{s_{i,j} + s_{i,j+1}}{2}. \tag{23}$$

By standard Taylor's expansion

$$U_{i,j+1} = U(x_i, t_{j+1}) = U_{i,j} + \Delta t \left(\frac{\partial U}{\partial t}\right)_{i,j} + \frac{1}{2}\Delta t^2 \left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} + \frac{1}{6}\Delta t^3 \left(\frac{\partial^3 U}{\partial t^3}\right)_{i,j} + \cdots \tag{24}$$

The fractional Taylor's expansion from Eq. (6) gives

$$U_{i-k+1,j} = U(x_{i-k+1}, t_j) = U_{i,j} + \frac{[(1-k)h]^\alpha}{\alpha!}\left(\frac{\partial^\alpha U}{\partial x^\alpha}\right)_{i,j} + \frac{[(1-k)h]^{2\alpha}}{(2\alpha)!}\left(\frac{\partial^{2\alpha} U}{\partial x^{2\alpha}}\right)_{i,j} + \cdots \tag{25}$$

$$U_{i-k+1,j+1} = U(x_{i-k+1}, t_{j+1}) = U_{i,j} + \Delta t \left(\frac{\partial U}{\partial t}\right)_{i,j} + \frac{1}{2}\Delta t^2 \left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} + \cdots$$

$$+ \frac{[(1-k)h]^\alpha}{\alpha!}\left[\left(\frac{\partial^\alpha U}{\partial x^\alpha}\right)_{i,j} + \Delta t \left(\frac{\partial^{\alpha+1} U}{\partial x^\alpha \partial t}\right)_{i,j} + \frac{1}{2}\Delta t^2 \left(\frac{\partial^{\alpha+2} U}{\partial x^\alpha \partial t^2}\right)_{i,j} + \cdots\right]$$

$$+ \frac{[(1-k)h]^{2\alpha}}{(2\alpha)!}\left[\left(\frac{\partial^{2\alpha} U}{\partial x^{2\alpha}}\right)_{i,j} + \Delta t \left(\frac{\partial^{2\alpha+1} U}{\partial x^{2\alpha} \partial t}\right)_{i,j} + \frac{1}{2}\Delta t^2 \left(\frac{\partial^{2\alpha+2} U}{\partial x^{2\alpha} \partial t^2}\right)_{i,j} + \cdots\right] + \cdots \tag{26}$$

We consider the Taylor's expansion for $s_{i,j+1}$, thus

$$s_{i,j+1} = s(x_i, t_{j+1}) = s_{i,j} + \Delta t \left(\frac{\partial s}{\partial t}\right)_{i,j} + \frac{1}{2}\Delta t^2 \left(\frac{\partial^2 s}{\partial t^2}\right)_{i,j} + \frac{1}{6}\Delta t^3 \left(\frac{\partial^3 s}{\partial t^3}\right)_{i,j} + \cdots \tag{27}$$

Substituting Eq. (24), Eq. (25), Eq. (26) and Eq. (27) into Eq. (23) gives

$$T_{i,j} = F_{i,j}(U) = \left(\frac{\partial U}{\partial t}\right)_{i,j} + \frac{1}{2}\Delta t \left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} + \frac{1}{6}\Delta t^2 \left(\frac{\partial^3 U}{\partial t^3}\right)_{i,j} + \cdots$$

$$-\frac{c_i}{2h^\alpha}\sum_{k=0}^{i+1}g_k\cdot\left[U_{i,j}+\frac{[(1-k)h]^\alpha}{\alpha!}\left(\frac{\partial^\alpha U}{\partial x^\alpha}\right)_{i,j}+\frac{[(1-k)h]^{2\alpha}}{(2\alpha)!}\left(\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}}\right)_{i,j}+\cdots+U_{i,j}+\Delta t\left(\frac{\partial U}{\partial t}\right)_{i,j}\right.$$

$$+\frac{1}{2}\Delta t^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j}+\cdots$$

$$+\frac{[(1-k)h]^\alpha}{\alpha!}\left(\left(\frac{\partial^\alpha U}{\partial x^\alpha}\right)_{i,j}+\Delta t\left(\frac{\partial^{\alpha+1}U}{\partial x^\alpha\,\partial t}\right)_{i,j}+\frac{1}{2}\Delta t^2\left(\frac{\partial^{\alpha+2}U}{\partial x^\alpha\,\partial t^2}\right)_{i,j}+\cdots\right)$$

$$+\frac{[(1-k)h]^{2\alpha}}{(2\alpha)!}\left.\left(\left(\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}}\right)_{i,j}+\Delta t\left(\frac{\partial^{2\alpha+1}U}{\partial x^{2\alpha}\,\partial t}\right)_{i,j}+\frac{1}{2}\Delta t^2\left(\frac{\partial^{2\alpha+2}U}{\partial x^{2\alpha}\,\partial t^2}\right)_{i,j}+\cdots\right)+\cdots\right]$$

$$-\left(s_{i,j}+\frac{\Delta t}{2}\left(\frac{\partial s}{\partial t}\right)_{i,j}+\frac{1}{4}\Delta t^2\left(\frac{\partial^2 s}{\partial t^2}\right)_{i,j}+\cdots\right).\tag{28}$$

Since

$$\sum_{k=0}^{\infty}g_k=0\tag{29}$$

(see [13]), one can prove that

$$\sum_{k=0}^{\infty}g_k(1-k)^\alpha=\alpha!,\tag{30}$$

Because $\sum_{k=0}^{\infty}g_k(1-k)^\alpha$ is an α derivative of $(1+x)^\alpha$.

Thus new from on Eq. (28), Eq. (29) and Eq. (30) we can write

$$T_{i,j}=\left(\frac{\partial U}{\partial t}-c\frac{\partial^\alpha U}{\partial x^\alpha}-s\right)_{i,j}+\frac{1}{2}\Delta t\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j}+\cdots-c_i\frac{(1-k)^{2\alpha}}{(2\alpha)!}h^\alpha\left(\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}}\right)_{i,j}+\cdots$$

$$-c_i\frac{(1-k)^\alpha}{2(\alpha!)}\Delta t\left(\frac{\partial^{\alpha+1}U}{\partial x^\alpha\,\partial t}\right)_{i,j}+\cdots-c_i\frac{(1-k)^{2\alpha}}{2((2\alpha)!)}h^\alpha\Delta t\left(\frac{\partial^{2\alpha+1}U}{\partial x^{2\alpha}\,\partial t}\right)_{i,j}+\cdots$$

$$-c_i\frac{(1-k)^{2\alpha}}{2((2\alpha)!)}h^\alpha\frac{\Delta t}{2}\left(\frac{\partial^{2\alpha+2}U}{\partial x^{2\alpha}\,\partial t^\alpha}\right)_{i,j}+\cdots$$

$$-\frac{\Delta t}{2}\left(\frac{\partial s}{\partial t}\right)_{i,j},\tag{31}$$

but U is the exact solution of the differential Eq. (21) so $\left(\frac{\partial U}{\partial t}-c\frac{\partial^\alpha U}{\partial x^\alpha}-s\right)_{i,j}=0$ and

$$\left(\frac{\partial^2 U}{\partial t^2}-c\frac{\partial^{\alpha+1}U}{\partial x^\alpha\partial t}-\frac{\partial s}{\partial t}\right)_{i,j}=0.$$

Therefore the principal part of the local truncation error is

$$\left[\left(\frac{1}{6}\frac{\partial^3 U}{\partial t^3}-\frac{1}{4}\frac{\partial^2 s}{\partial t^2}\right)\Delta t^2-ch^\alpha\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}}-c\frac{1}{2}h^\alpha\Delta t\frac{\partial^{2\alpha+1}U}{\partial x^{2\alpha}\partial t}\right]_{i,j}.$$

Hence $T_{i,j}=O(\Delta t^2)+O(h^\alpha)+O(h^\alpha\Delta t)$ i.e., for any $r=\frac{\Delta t}{2h^\alpha}$, $T_{i,j}$ is $O(\Delta t^2)$ or $O(h^\alpha)$ or $O(h^\alpha\Delta t)$, as one would expect. This is in the contrast with Eq. (12) in [13]

This shows that the difference equation is consistent and the local truncation error vanishes as $h\to0$ and $\Delta t\to0$.

Note that for $s(x,t)=0$, when $\alpha=1$ (hyperbolic), from Eq. (14) $g_0=1$, $g_1=-1$ and $g_2=g_3=\cdots=0$ and we have the standard Crank-Nicolson finite difference approximation

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left(\frac{u_{i+1,j+1} - u_{i,j+1}}{h}\right) + \frac{1}{2}\left(\frac{u_{i+1,j} - u_{i,j}}{h}\right),$$

while for $\alpha = 2$ (parabolic), from Eq. (14) $g_0 = 1$, $g_1 = -2$, $g_2 = 1$ and $g_3 = g_4 = \cdots = 0$ the resulting Crank-Nicolson method is

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right) + \frac{1}{2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right).$$

Stability of the fractional Crank-Nicolson method

From the system of equations defined by Eq. (20), together with the Dirichlet boundary conditions Eq. (17), define a linear system

$$\mathbb{U}_{j+1} = (I - A)^{-1}(I + A)\mathbb{U}_j + \Delta t(I - A)^{-1}\mathbb{S}_j, \tag{32}$$

where $r = \frac{\Delta t}{2h^\alpha}$, $\mathbb{U}_j = \left[u_{0,j}, u_{1,j}, u_{2,j}, \ldots, u_{K,j}\right]^T$ and

$$\mathbb{S}_j = \left[0, \frac{s_{1,j}+s_{1,j+1}}{2} +, \frac{s_{2,j}+s_{2,j+1}}{2}, \ldots, \frac{s_{K-1,j}+s_{K-1,j+1}}{2}, 0\right]^T.$$

Note that this matrix A is a non-sparse matrix. To illustrate this matrix pattern, we list from Eq. (20) the corresponding first three and last two equations for $i = 1, 2, 3, K-2$ and $K-1$.

$$-rg_0c_1u_{2,j+1} + (1 - rg_1c_1)u_{1,j+1} - rg_2c_1u_{0,j+1} =$$

$$rg_0c_1u_{2,j} + (1 + rg_1c_1)u_{1,j} + rg_2c_1u_{0,j} + \Delta t\frac{s_{1,j}+s_{1,j+1}}{2}$$

$$-rg_0c_2u_{3,j+1} + (1 - rg_1c_2)u_{2,j+1} - rg_2c_2u_{1,j+1} - rg_3c_2u_{0,j+1}$$

$$= rg_0c_2u_{3,j} + (1 + rg_1c_2)u_{2,j} + rg_2c_2u_{1,j} + rg_3c_2u_{0,j} + \Delta t\frac{s_{2,j}+s_{2,j+1}}{2}$$

$$-rg_0c_3u_{4,j+1} + (1 - rg_1c_3)u_{3,j+1} - rg_2c_3u_{2,j+1} - rg_3c_3u_{1,j+1} - rg_4c_3u_{0,j+1}$$

$$= rg_0c_3u_{4,j} + (1 + rg_1c_3)u_{3,j} + rg_2c_3u_{2,j} + rg_3c_3u_{1,j} + rg_4c_3u_{0,j} + \Delta t\frac{s_{3,j}+s_{3,j+1}}{2}$$

$$-rg_0c_{K-2}u_{K-1,j+1} + (1 - rg_1c_{K-2})u_{K-2,j+1} - rc_{K-2}\sum_{k=2}^{K-1}g_k \cdot u_{K-k-1,j+1} =$$

$$rg_0c_{K-2}u_{K-1,j} + (1 + rg_1c_{K-2})u_{K-2,j} + rc_{K-2}\sum_{k=2}^{K-1}g_k \cdot u_{K-k-1,j} + \Delta t\frac{s_{K-2,j}+s_{K-2,j+1}}{2}$$

$$-rg_0c_{K-1}u_{K,,j+1} + (1 - rg_1c_{K-1})u_{K-1,j+1} - rc_{K-1}\sum_{k=2}^{K}g_k \cdot u_{K-k,j+1} =$$

$$rg_0c_{K-1}u_{K,,j} + (1 + rg_1c_{K-1})u_{K-1,j} + rc_{K-1}\sum_{k=2}^{K}g_k \cdot u_{K-k,j} + \Delta t\frac{s_{K-1,j}+s_{K-1,j+1}}{2}.$$

Thus the matrix entries $A_{i,j}$ for $i, j = 1, 2, \ldots K - 1$ can be defined by

$$A_{i,j} = \begin{cases} 0, & \text{for } j \geq i + 2 \\ rg_1c_i, & \text{for } j = i \\ rg_{i-j+1}c_i, & \text{otherwise}, \end{cases}$$

while $A_{0,j} = A_{K,j} = A_{j,0} = A_{j,K} = 0$ for $j = 0, 1, \ldots K$.

Note that from Eq. (14) $g_1 = -\alpha$, for $1 \leq \alpha \leq 2$. For $i \neq 1$ we have $g_i \geq 0$ (the strict inequality holds for non-integer values of α).

We also have $-g_1 \geq \sum_{k=0,k\neq 1}^{k=K} g_k$, which follows from Eq. (29).

According to the Greschgorin theorem the eigenvalues of the matrix A in (5.2) lie in the union of the K circles centered at $A_{i,i}$ with radius

$$r_i = \sum_{k=0,k\neq 1}^{K} |A_{i,k}|.$$

Here we have,

$$A_{i,i} = rg_1 c_i = -r\alpha c_i$$

and

$$r_i = \sum_{k=0,k\neq 1}^{K} |A_{i,k}| = \sum_{k=0,k\neq 1}^{i+1} |A_{i,k}| = rc_i \sum_{k=0,k\neq 1}^{i+1} g_k \leq r\alpha c_i.$$

Therefore, the eigenvalues of matrix A are in the left half of the complex plane.

The translate $T(z) = \frac{1+z}{1-z}$ brings the left half of the complex plane into the circle with radius 1 (see [21]) and so if μ is the eigenvalues of the matrix A, when $\frac{1+\mu}{1-\mu}$ is the eigenvalues of the matrix $(1-A)^{-1}(1+A)$ (see [22]). Thus we result that the eigenvalues of the matrix $(1-A)^{-1}(1+A)$ are in the circle with radius 1. Hence any error in $\mathbb{U}_j$ is not magnified, and therefore the method is unconditionally stable. Now from Lax-Richtmyer's equivalence theorem since from theorem 1 this method is consistence of order $O(\Delta t^2) + O(h^\alpha) + O(h^\alpha \Delta t)$ therefore we can express that the proposed method in section (20) is convergent too.

Numerical results

Example problem

Consider the following fractional partial differential equation defined on

$$\Omega = \{(x,t)|0 \leq x \leq 1, 0 \leq t \leq 1\}$$
$$\frac{\partial u(x,t)}{\partial t} = c(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + s(x,t), \qquad (33)$$

with the coefficient function $c(x) = \Gamma(0.2)x^{1.8}$, the forcing function

$$s(x,t) = -(2x - 11x^2)e^{-t},$$

initial value $u(x,0) = x(1-x)$, and zero Dirichlet boundary conditions.

Exact solution for example problem

Applying the one fold integration inverse operator $L_t^{-1} = \int_0^t (.)\, dt$ to (6.2) and using the specified initial condition yields

$$u(x,t) = u(x,0) + L_t^{-1}\left(c(x)D_x^{1.8}\left(\sum_{i=0}^{\infty} u_i\right)\right) + L_t^{-1}(s(x,t)).$$

Now from Appendix $f(x,t)$ may be written as

$$f(x,t) = u(x,0) + L_t^{-1}(s(x,t))$$
$$= x(1-x)$$
$$+ (e^{-t} - 1)(2x - 11x^2).$$

Thus for $f_1(x,t) = x(1-x)$, $f_2(x,t) = x(1-x)e^{-t}$ and $f_3(x,t) = -(2x - 11x^2) + x(1-10x)e^{-t}$.

It is clear that f_1 and f_3 do not satisfy Eq. (33), initial and boundary conditions. Now we choose $\Psi_1 = f_2$ and $\Psi_2 = f_1 + f_3$. Then from the three steps algorithm in Appendix we will obtain
$$u_0 = x(1-x)e^{-t}$$
$$u_1 = (x - 10x^2)e^{-t} - (x - 10x^2) + c(x)D_x^{1.8}u_0$$
$$= (x - 10x^2)e^{-t} - (x - 10x^2)$$
$$- (e^{-t} - 1)(x - 10x^2) = 0$$
$$u_n = 0; \qquad \forall n \geq 2.$$
Therefore, the solution is

$$u(x,t) = x(1-x)e^{-t}. \qquad (34)$$

The fractional Crank-Nicolson absolute-error is identified by $|U_{i,j} - u_{i,j}|$ in which $U_{i,j}$ and $u_{i,j}$ are exact and numerical Crank-Nicolson solutions respectively. The values of absolute-error for the example problem for different values of h and Δt are shown in **Table 1**. Approximate and exact solutions for h = 0.05 and $\Delta t = 0.025$ are shown in **Figure 2** for T = 1 and $\frac{\Delta t}{2h^\alpha} = 2.7464$. **Figure 3** shows the exact and approximate solutions for h = 0.01, $\Delta t = 0.01$ and $\frac{\Delta t}{2h^\alpha} = 19.9054$. Approximate and exact solutions for h = 0.01 and $\Delta t = 0.01$ are shown in **Figure 4** for T = 20 and $\frac{\Delta t}{2h^\alpha} = 19.9054$. **Figure 5** shows the exact and approximate solutions for h = 0.02, $\Delta t = 0.01$ and $\frac{\Delta t}{2h^\alpha} = 5.7163$.

Table 1 Maximum error behavior versus grid size reduction for the example problem.

h	Δt	Final time	Absolute-error of fractional Crank-Nicolson scheme
0.200	0.1000	1	0.0042080
0.100	0.0500	1	0.0039510
0.050	0.0250	1	0.0033609
0.025	0.0125	1	0.0026941
0.010	0.0100	1	0.0019036
0.010	0.0100	10	3.4213e-007
0.020	0.0100	10	4.0654e-007
0.040	0.0100	10	4.6978e-007
0.010	0.0100	20	1.5533e-011
0.020	0.0100	20	1.8457e-011
0.040	0.0100	20	2.1328e-011

Figure 2 Comparison between exact and numerical fractional Crank-Nicolson solutions for the example problem with Δt = 0.025 and h = 0.05 at time T = 1.0.
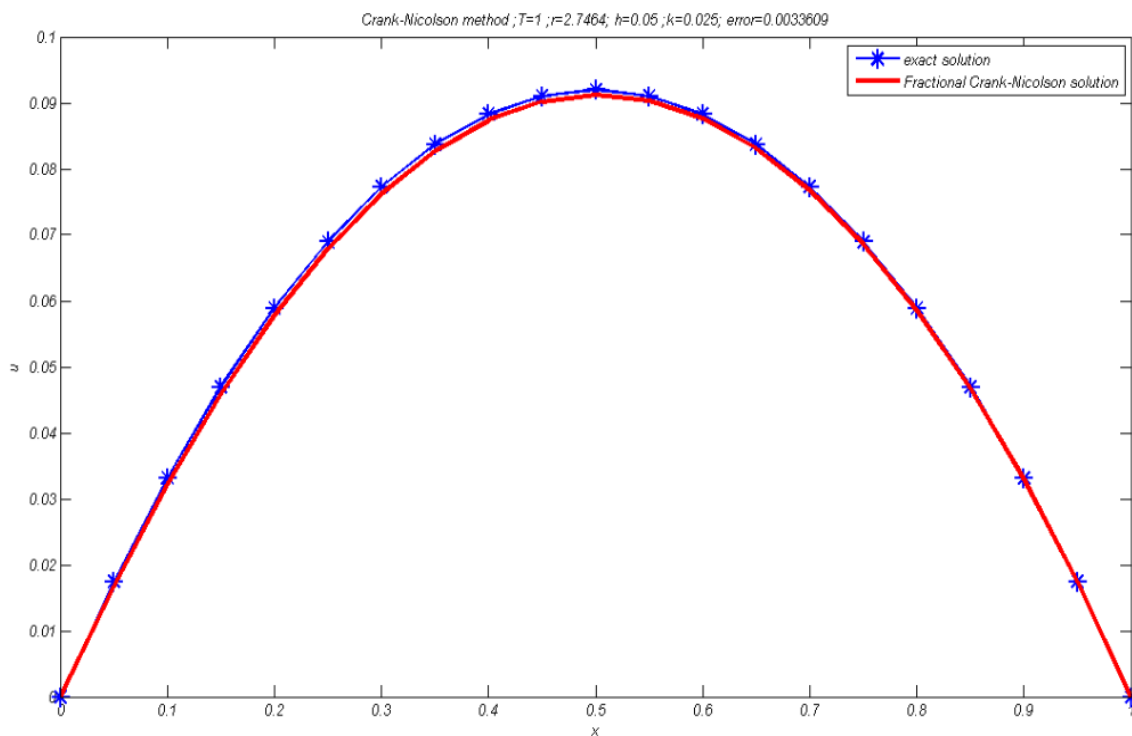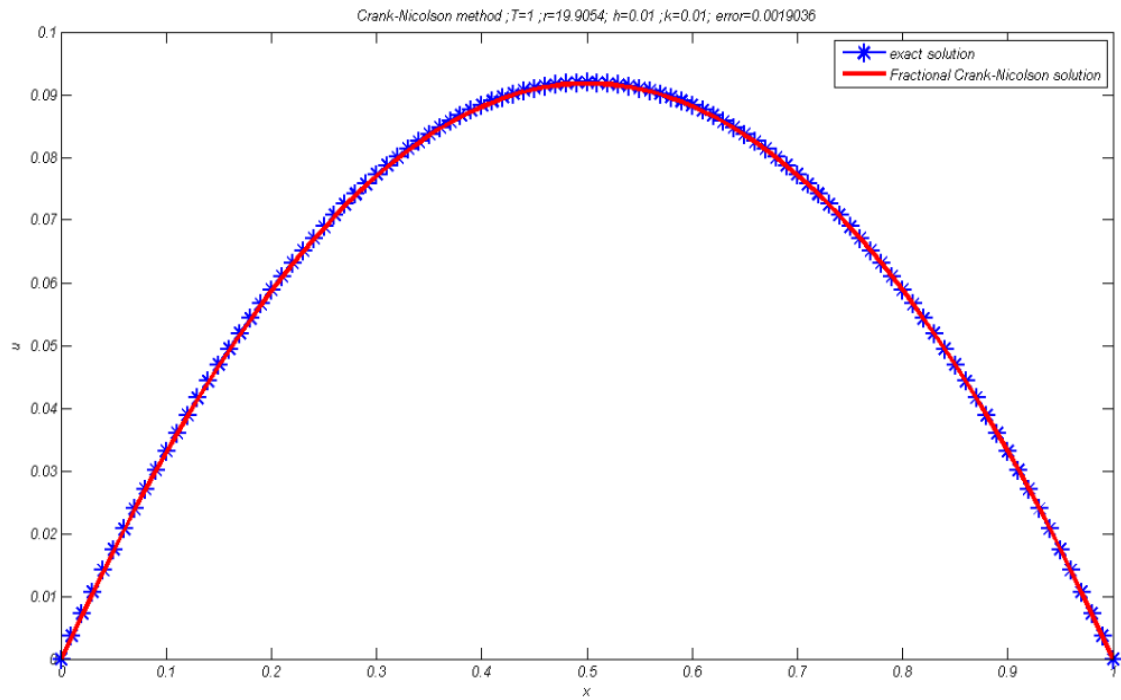
Figure 3 Comparison between exact and numerical fractional Crank-Nicolson solutions for the example problem with $\Delta t = 0.01$ and $h = 0.01$ at time $T = 1.0$.
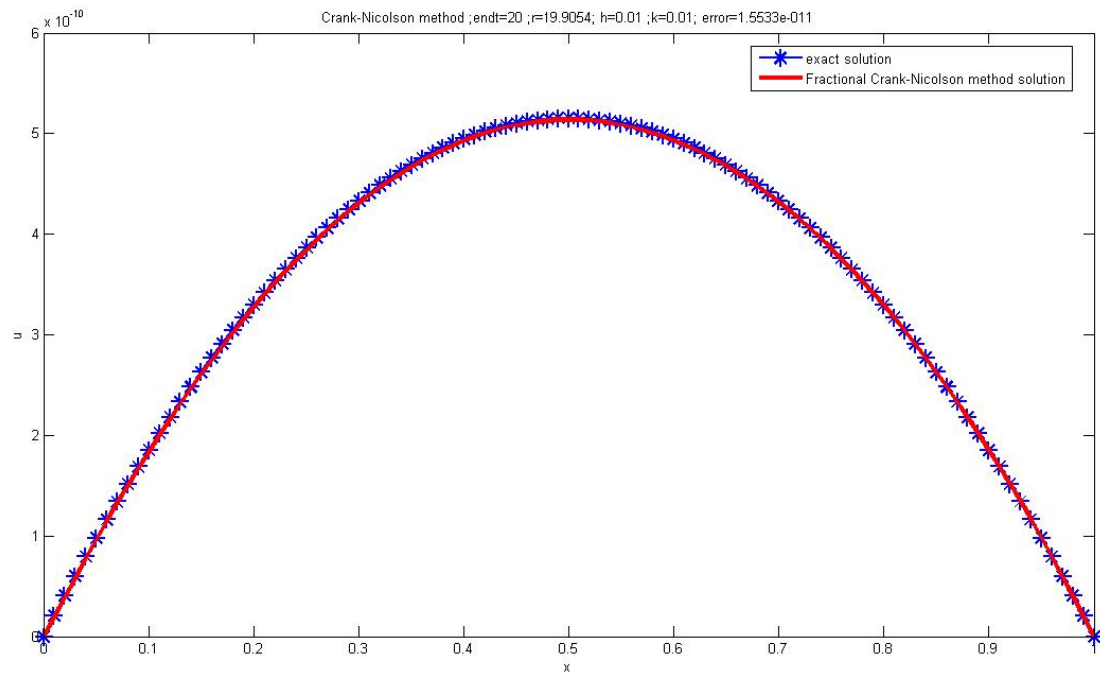

Figure 4 Comparison between exact and numerical fractional Crank-Nicolson solutions for the example problem with $\Delta t = 0.01$ and $h = 0.01$ at time $T = 20$.

Figure 5 Comparison between exact and numerical fractional Crank-Nicolson solutions for the example problem with $\Delta t = 0.01$ and $h = 0.02$ at time $T = 20$.
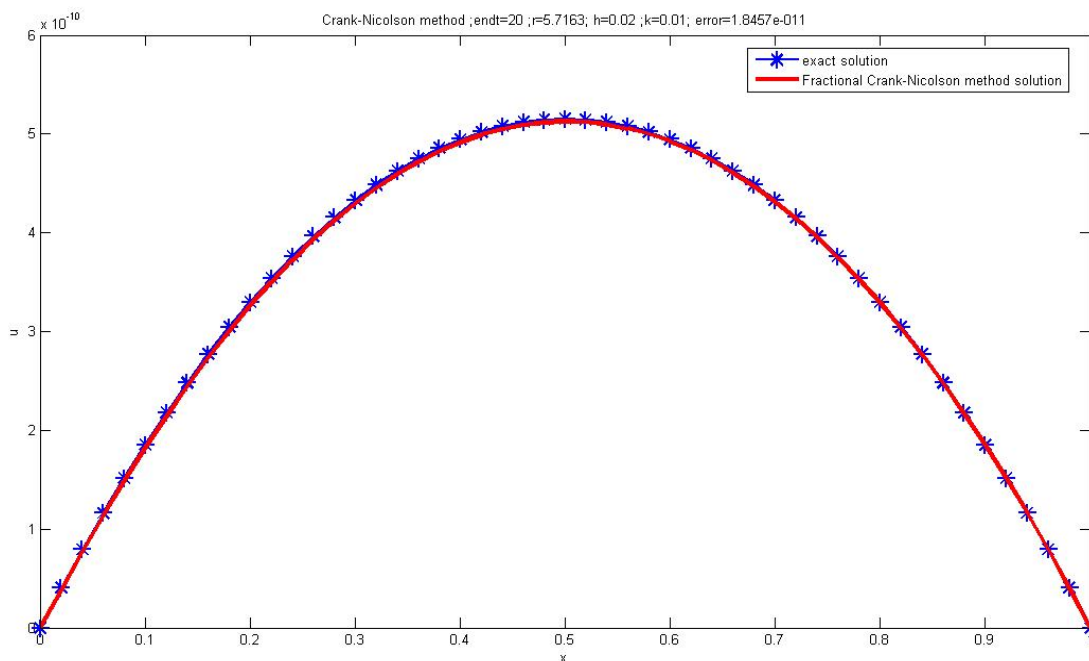
Conclusions

Crank-Nicolson finite difference method using shifted Grünwald estimate is encountered to find the approximate solution for fractional partial differential equations. Two-step Adomian decomposition method is used to obtain the exact solution. It is proved that the truncation error is in the form $O(\Delta t^2) + O(h^\alpha) + O(h^\alpha \Delta t)$. The proposed method is unconditionally stable and the approximate solution based on fractional Crank-Nicolson finite difference method converges to the exact solution successfully.

Acknowledgements

Appendix Two-step Adomian decomposition algorithm

We adopt a two-step Adomian decomposition method to find the closed analytical form solution for the problem $(15) - (17)$. In the light of this method (see [23-25]) we assume that

$$u = \sum_{n=0}^{\infty} u_n. \tag{35}$$

to be the solution of (1).

Now, (1) using (4) and (5) can be rewritten as

$$L_t u(x, t) = c(x) D_x^\alpha u(x, t) + s(x, t), \tag{36}$$

where $D_x^\alpha(.)$ is the Riemann-Liouville derivative of order α, $L_t = \frac{\partial}{\partial t}$ is an invertible linear operator. Thus $L_t^{-1} = \int_0^t (.) \, dt$ is the one fold integration inverse operator.

Let us assume that $f(x, t)$ has the following form of Luo [24]:

$$f(x, t) = u(x, 0) + L_t^{-1}\big(s(x, t)\big), \tag{37}$$

where f has written by sum of three functions

$$f(x, t) = f_1(x, t) + f_2(x, t) + f_3(x, t). \tag{38}$$

Now we choose $\Psi_1 = f_k, k = 1,2,3$ and $\Psi_2 = f - f_k$ where f_k is a function that satisfies initial and boundary conditions.

The two-step Adomian decomposition recursive algorithm is as follows from the papers [26-29]:

step 0: $u_0 = \Psi_1$
step 1: $u_1 = \Psi_2 + L_t^{-1}(c(x)D_x^\alpha u_0)$
step 2: $u_{n+1} = L_t^{-1}(c(x)D_x^\alpha u_n)$, $n \geq 1$.

The practical solution will be the n-term approximation φ_n

$$\varphi_n = \sum_{i=0}^{n-1} u_i \quad , \quad n \geq 1 \tag{39}$$

with

$$\lim_{n \to \infty} \varphi_n = u(x,t). \tag{40}$$

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