Fuzzy and l-Fuzzy Subset in a Locally Convex Topology

Sayed Khalil ELAGAN\(^1,2\) and Rabha Waell IBRAHIM\(^3,*\)

\(^1\)Department of Mathematics and Statistics, Faculty of Science, Taif University, El-Haweiah, Kingdom of Saudi Arabia
\(^2\)Department of Mathematics and Statistics, Faculty of Science, Menofya University, Shebin Elkom, Egypt
\(^3\)Faculty of Computer Science and Information Technology, University of Malaya, Kuala Lumpur, Malaysia

(*Corresponding author’s e-mail: rabhaibrahim@yahoo.com, sayed_khalil2000@yahoo.com)

Received: 8 November 2013, Revised: 13 February 2014, Accepted: 18 March 2014

Abstract

In this paper, the concepts of sectional fuzzy continuous mappings, and l-fuzzy compact sets, are introduced in locally convex topology generated by fuzzy n-norms. Schauder-type and other fixed point theorems are established in locally convex topology generated by fuzzy n-norms.

Keywords: Locally convex topology, Schauder fixed point theorem

Introduction and preliminaries

In Gähler [1] introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [2-8]. Gunawan and Mashadi [2] gave a simple way to derive an \((n-1)\)-norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived \((n-1)\)-norm. Narayanan and Vijayabalaji extended n-normed linear space to fuzzy n-normed linear space. The main objective of this paper is to introduce concepts of sectional fuzzy continuous mappings and l-fuzzy compact sets, and in the same time, to perform the Schauder-type [9] and other fixed point theorems in locally convex topology generated by fuzzy n-norms. In section 1, we quote some basic definitions, and in section 2, we introduce concepts of sectional fuzzy continuous mappings and l-fuzzy compact sets, as well as presenting our new results. Let \(n\) be a positive integer, and let \(X\) be a real vector space of dimension of at least \(n\). We recall the definitions of an n-seminorm and a fuzzy n-norm from [10,11].

**Definition 1** A function \( (x_1, x_2, \ldots, x_n) \mapsto \| x_1, \ldots, x_n \| \) from \( X^n \) to \([0, \infty)\) is called an n-seminorm on \(X\) if it has the following four properties:

\[(S_1) \quad \| x_1, x_2, \ldots, x_n \| = 0 \quad \text{if} \quad x_1, x_2, \ldots, x_n \quad \text{are linearly dependent};\]

\[(S_2) \quad \| x_1, x_2, \ldots, x_n \| \quad \text{is invariant under any permutation of} \quad x_1, x_2, \ldots, x_n;\]

\[(S_3) \quad \| x_1, \ldots, x_{n-1}, cx_n \| = |c| \| x_1, \ldots, x_{n-1}, x_n \| \quad \text{for any real} \quad c;\]

\[(S_4) \quad \| x_1, \ldots, x_{n-1}, y + z \| \leq \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \|.\]
An $n$-seminorm is called a $n$-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

**Definition 2** A fuzzy subset $N$ of $X^n \times \mathbb{R}$ is called a fuzzy $n$-norm on $X$ if and only if:

(F1) For all $t \leq 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$;

(F2) For all $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(F3) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(F4) For all $t > 0$ and $c \in \mathbb{R}$, $c \neq 0$,

\[ N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|}); \]

(F5) For all $s, t \in \mathbb{R}$,

\[ N(x_1, x_2, \ldots, x_{n-1}, y + z, s + t) \geq \min \{ N(x_1, x_2, \ldots, x_{n-1}, y, s), N(x_1, x_2, \ldots, x_{n-1}, z, t) \}; \]

(F6) $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1$.

**Definition 3** [2] Let $(X, N)$ be a fuzzy normed space; a subset $A$ of $X$ is said to be $l$-fuzzy closed if for any sequence $\{x_n\}$ and for each $\alpha \in (0, 1)$, and $x \in A$;

\[
\lim_{n \to \infty} N(x_n - x, t) \geq \alpha
\]

for all $t > 0$.

**Definition 4** [5] Let $(X, N)$ is a fuzzy $n$-norm space, that is, $X$ is real vector space, and $N$ is fuzzy $n$-norm on $X$. We form the family of $n$-seminorms $\|\cdot, \cdot, \ldots, \cdot\|_\alpha$, $\alpha \in (0, 1)$, and this family generates a family $\mathcal{F}$ of seminorms $\|x_1, x_2, \ldots, x_n\|_\alpha$, where $x_1, x_2, \ldots, x_n$ and $(0,1)$. The family $\mathcal{F}$ generates a locally convex topology on $X$; a basis of neighborhoods at the origin is given by:

\[ \{x \in X : p_i(x) \leq \varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots, n\}, \]

where $p_i \in \mathcal{F}$ and $\varepsilon_i > 0$ for $i = 1, 2, \ldots, n$. We call this the locally convex topology generated by the fuzzy $n$-norm $N$.

**Definition 5** [2] Let $(X, N)$ be a fuzzy normed space; a mapping $T : (X, N_1) \to (Y, N_2)$ is said to be fuzzy continuous at $x_0 \in X$, if for a given $\varepsilon > 0$ and $\alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \varepsilon) > 0$ and $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$ such that for each $\varepsilon > 0$, there exists $\delta > 0$ and...
Fuzzy and I-Fuzzy Subset in a Locally Convex Topology

Sayed Khalil ELAGAN and Rabha Waell IBRAHIM

http://wjst.wu.ac.th

\[ N_1(x-x_0, \delta) > \beta \Rightarrow N_2(T(x) - T(x_0), \varepsilon) > \alpha \]
for all \( x \in X \).

If \( T \) is fuzzy continuous at each point of \( X \), then \( T \) is said to be sectional fuzzy continuous on \( X \).

**Definition 6** [2] Let \((X, N)\) be a fuzzy normed space; a mapping \( T : (X, N_1) \rightarrow (Y, N_2) \) is said to be sectional fuzzy continuous at \( x_0 \in X \), if there exists \( \alpha_0 \in (0,1) \) such that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) and
\[ N_1(x-x_0, \delta) \geq \alpha_0 \Rightarrow N_2(T(x) - T(x_0), \varepsilon) \geq \alpha_0 \]
for all \( x \in X \).

If \( T \) is sectional fuzzy continuous at each point of \( X \), then \( T \) is said to be sectional fuzzy continuous on \( X \).

**Definition 7** [3] Let \((X, N)\) be a fuzzy normed space; a subset \( A \) of \( X \) is said to be \( l \)-fuzzy compact if for any sequence \( \{x_n\} \) and for each \( \alpha \in (0,1) \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and \( x \in A \) (both depending on \( \{x_n\} \) and \( \alpha \)) such that;
\[ \lim_{k \to \infty} N(x_{n_k} - x, t) \geq \alpha \]
for all \( t > 0 \).

**Schauder fixed point theorem**

In this section, we establish Schauder fixed point theorems in the locally convex topology generated by fuzzy \( n \)-normed spaces.

**Definition 8** A subset \( A \) of \( X \) is said to be \( l \)-fuzzy closed in the locally convex topology generated by \( N \) if for any sequence \( \{x_n\} \) and for each \( \alpha \in (0,1) \), and \( x \in A \);
\[ \lim_{n \to \infty} N(a_1, \ldots, a_{n-1}, x_n - x, t) \geq \alpha \]
for all \( a_1, \ldots, a_{n-1} \in X \) and all \( t > 0 \).

**Definition 9** A mapping \( T : (X, N_1) \rightarrow (Y, N_2) \) is said to be fuzzy continuous at \( x_0 \in X \) in the locally convex topology generated by \( N \), if for a given \( \varepsilon > 0 \) and \( \alpha \in (0,1) \) there exist \( \delta = \delta(\alpha, \varepsilon) > 0 \) and \( \beta = \beta(\alpha, \varepsilon) \in (0,1) \) such that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) and
Fuzzy and $l$-Fuzzy Subset in a Locally Convex Topology  
Sayed Khalil ELAGAN and Rabha Waell IBRAHIM 

\[ N_1(a_1, \ldots, a_{n-1}, x - x_0, \delta) > \beta \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(x_0), \varepsilon) > \alpha \]
for all \( a_1, \ldots, a_{n-1}, x, x_0 \in X \).

(6)

If \( T \) is fuzzy continuous at each point of \( X \), then \( T \) is said to be sectional fuzzy continuous on \( X \).

**Definition 10** A mapping \( T : (X, N_1) \to (Y, N_2) \) is said to be sectional fuzzy continuous at \( x_0 \in X \), in the locally convex topology generated by \( N \) if there exists \( \alpha_0 \in (0, 1) \) such that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) and

\[ N_1(a_1, \ldots, a_{n-1}, x - x_0, \delta) \geq \alpha_0 \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(x_0), \varepsilon) \geq \alpha_0 \]
for all \( a_1, \ldots, a_{n-1}, x, x_0 \in X \).

(7)

If \( T \) is sectional fuzzy continuous at each point of \( X \), then \( T \) is said to be sectional fuzzy continuous on \( X \).

**Definition 11** A subset \( A \) of \( X \) is said to be \( l \)-fuzzy compact in the locally convex topology generated by \( N \) if for any sequence \( \{x_n\} \) and for each \( \alpha \in (0, 1) \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and \( x \in A \) (both depending on \( \{x_n\} \) and \( \alpha \)) such that;

\[ \lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) \geq \alpha \]
for all \( a_1, \ldots, a_{n-1} \in X \) and all \( t > 0 \).

(8)

**Lemma 12** A subset \( A \) of \( X \) is \( l \)-fuzzy compact in the locally convex topology generated by \( N \) if and only if \( A \) is compact w.r.t. \( \|\cdot\|_\alpha \) (\( \alpha \)-norm of \( N \)) for each \( \alpha \in (0, 1) \).

**Proof.** First suppose that \( A \) is \( l \)-fuzzy compact. Take \( \alpha_0 \in (0, 1) \). Let \( \{x_n\} \) be a sequence in \( A \). Thus, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and \( x \) in \( A \) (both depend on \( \alpha_0 \)) such that;

\[ \lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) \geq \alpha_0 \]
for all \( a_1, \ldots, a_{n-1} \in X \) and all \( t > 0 \). This implies that for a given \( \varepsilon > 0 \) with \( \alpha_0 - \varepsilon > 0 \) and for a given \( t > 0 \), there exists a positive integer \( K(\varepsilon, t) \) such that;

\[ N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) > \alpha_0 - \varepsilon \]
for all \( n \geq K(\varepsilon, t) \).

(10)

which implies that;
\[ \| a_1, a_2, \ldots, a_n \|_{\alpha_0} - \varepsilon \leq t \text{ for all } n \geq K(\varepsilon, t). \]  

(11)

This implies that \( A \) is compact. Since \( \alpha_0 \in (0,1) \) and \( \varepsilon > 0 \) are arbitrary, it follows that \( A \) is compact w.r.t. \( \| \|_\alpha \) for each \( \alpha \in (0,1) \). Conversely, suppose that \( A \) is compact w.r.t. \( \| \|_\alpha \) for each \( \alpha \in (0,1) \). Let \( \{ x_n \} \) be a sequence in \( A \). Take \( \alpha_0 \in (0,1) \). Then, there exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) and \( x \) in \( A \) (both depend on \( \alpha_0 \)) such that;

\[ \lim_{k \to \infty} \| a_1, a_2, \ldots, a_{n-1}, x_{n_k} - x \|_{\alpha_0} = 0. \]  

(12)

for all \( a_1, a_2, \ldots, a_{n-1} \in X \). This implies that for a given \( \varepsilon > 0 \), there exists a positive integer \( K(\varepsilon) \) such that:

\[ \| a_1, a_2, \ldots, a_{n-1}, x_{n_k} - x \|_{\alpha_0} < \varepsilon \text{ for all } k \geq K(\varepsilon). \]  

(13)

From this we conclude that;

\[ N(a_1, \ldots, a_{n-1}, x_{n_k} - x, \varepsilon) > \alpha_0 \text{ for all } k \geq K(\varepsilon). \]  

(14)

for all \( a_1, a_2, \ldots, a_{n-1} \in X \). Since \( \varepsilon \) is arbitrary, so;

\[ \lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) > \alpha_0 \text{ for all } t > 0. \]  

(15)

Since \( \alpha_0 \in (0,1) \) is arbitrary, it follows that \( A \) is \( l \)-fuzzy compact.

**Lemma 13** A mapping \( T : (X, N_1) \to (Y, N_2) \) is sectional fuzzy continuous in the locally convex topology generated by \( N \) if and only if \( T : (X, \| \|_\alpha) \to (Y, \| \|_\alpha) \) is continuous for some \( \alpha \in (0,1) \).

**Proof.** First we suppose that, \( T : (X, N_1) \to (Y, N_2) \) is sectional fuzzy continuous. Thus, there exists \( \alpha_0 \in (0,1) \) such that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) and

\[ N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \geq \alpha_0 \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \varepsilon) \geq \alpha_0 \]  

for all \( a_1, \ldots, a_{n-1}, x, y \in X \).

(16)

Choose \( \eta_0 \) such that \( \delta_1 = \delta - \eta_0 > 0 \). Let \( \| a_1, a_2, \ldots, a_{n-1}, x - y \|_{\alpha_0} \leq \delta_1 \). Then
Fuzzy and l-Fuzzy Subset in a Locally Convex Topology  
Sayed Khalil ELAGAN and Rabha Waell IBRAHIM

http://wjst.wu.ac.th

\[ \|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0} \leq \delta. \]  
This leads to \( N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \geq \alpha_0, \) since \( T : (X, N_1) \rightarrow (Y, N_2) \) is sectional fuzzy continuous, this implies that \( N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \varepsilon) \geq \alpha_0 \) for all \( a_1, \ldots, a_{n-1}, x, y \in X, \) and hence \( \|a_1, a_2, \ldots, a_{n-1}, T(x) - T(y)\|_{\alpha_0}^2 \leq \varepsilon. \) Thus \( T : (X, N_1) \rightarrow (Y, N_2) \) is continuous w.r.t. \( \|\|_{\alpha_0} \) and \( \|\|_{\beta_0}^2. \) Conversely, suppose that \( T : (X, N_1) \rightarrow (Y, N_2) \) is continuous w.r.t. \( \|\|_{\alpha_0} \) and \( \|\|_{\beta_0}^2. \) Thus;

\[ \|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0} \leq \delta \Rightarrow \|a_1, a_2, \ldots, a_{n-1}, T(x) - T(y)\|_{\alpha_0}^2 \leq \frac{\varepsilon}{2} \]  
(17)

for all \( a_1, \ldots, a_{n-1}, x, y \in X. \) Let \( N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \geq \alpha_0, \)
so \( \|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0} \leq \delta, \) which implies that \( \|a_1, a_2, \ldots, a_{n-1}, T(x) - T(y)\|_{\alpha_0}^2 \leq \varepsilon. \) Therefore;

\[ N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \varepsilon) \geq \alpha_0 \]  
for all \( a_1, \ldots, a_{n-1}, x, y \in X. \)

Thus, the mapping \( T : (X, N_1) \rightarrow (Y, N_2) \) is sectional fuzzy continuous.

**Theorem 14** (Schauder fixed point theorem). Let \( K \) be a nonempty convex, l-fuzzy compact subset in the locally convex topology generated by \( N \) and \( T : K \rightarrow K \) be sectional fuzzy continuous. Then \( T \) has a fixed point.

Proof. By using theorem. For every \( \alpha \in (0,1), \) \( \|\|_{\alpha_0} \) is an n-seminorm on \( X. \) As \( K \) is an l-fuzzy compact of \( X, \) thus \( K \) is a compact subset of \( (X, \|\|_{\alpha_0}) \) for each \( \alpha \in (0,1) \) (by Lemma 1), since \( T : K \rightarrow K \) be sectional fuzzy continuous, it follows by Lemma 2 \( T : K \rightarrow K \) is continuous w.r.t. \( \|\|_{\alpha_0} \) for some \( \alpha_0 \in (0,1). \) Therefore, we get \( K \) is a nonempty convex and compact subset of a normed linear space \( (X, \|\|_{\alpha_0}) \) and \( T : K \rightarrow K \) is a continous mapping. By Schauder fixed point theorem [18], it follows that \( T \) has a fixed point.

**Conclusions**

We investigated the concepts of sectional fuzzy continuous mappings and l-fuzzy compact sets in locally convex topology generated by fuzzy n-normed spaces as an extension of the fuzzy normed space. In this new frame, we established the Schauder-type and other fixed point theorems, as well as some results in locally convex topology generated by fuzzy n-normed spaces, which are useful tools in the development of the fuzzy set theory.
References