FUZZY AND $l$-FUZZY SUBSET IN LOCALLY CONVEX TOPOLOGY

Abstract. In this paper, the concepts of sectional fuzzy continuous mappings, and $l$-fuzzy compact sets, were introduced in locally convex topology generated by fuzzy $n$-norms. Schauder-type and other fixed point theorems were established in locally convex topology generated by fuzzy $n$-norms.

1. INTRODUCTION AND PRELIMINARIES

In [9], S. Gähler introduced $n$-norms on a linear space. A detailed theory of $n$-normed linear space can be found in [10, 12, 14, 15, 16]. In [10], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$-norm from the $n$-norm in such a way that the convergence and completeness in the $n$-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1, 4, 6, 7, 8, 11, 13, 14]. In [17], A. Narayanan and S. Vijayabalaji have extend $n$-normed linear space to fuzzy $n$-normed linear space. The main objective of this paper is to introduce concepts of sectional fuzzy continuous mappings, and $l$-fuzzy compact sets and in the same time to perform the Schauder-type [22] and other fixed point theorems in locally convex topology generated by fuzzy $n$-norms. In section 1, we quote some basic definitions, In section 2, we introduce concepts of sectional fuzzy continuous mappings, and $l$-fuzzy compact sets as well as we present our new results.

Let $n$ be a positive integer, and let $X$ be a real vector space of dimension at least $n$. We recall the definitions of an $n$-seminorm and a fuzzy $n$-norm from [22].

**Definition 1.1.** A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from $X^n$ to $[0, \infty)$ is called an $n$-seminorm on $X$ if it has the following four properties:

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ if $x_1, x_2, \ldots, x_n$ are linearly dependent;
(S2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;
(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, \ldots, x_{n-1}, x_n\|$ for any real $c$;
(S4) $\|x_1, \ldots, x_{n-1}, y + z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An $n$-seminorm is called a $n$-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

**Definition 1.2.** A fuzzy subset $N$ of $X^n \times \mathbb{R}$ is called a fuzzy $n$-norm on $X$ if and only if:

(F1) For all $t \leq 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$;
(F2) For all \( t > 0 \), \( N(x_1, x_2, \ldots, x_n, t) = 1 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent;
(F3) \( N(x_1, x_2, \ldots, x_n, t) \) is invariant under any permutation of \( x_1, x_2, \ldots, x_n \);
(F4) For all \( t > 0 \) and \( c \in \mathbb{R} \), \( c \neq 0 \),
\[
N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});
\]
(F5) For all \( s, t \in \mathbb{R} \),
\[
N(x_1, \ldots, x_{n-1}, y+z, s+t) \geq \min \{ N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1} z, t) \}.
\]
(F6) \( N(x_1, x_2, \ldots, x_n, t) \) is a non-decreasing function of \( t \in \mathbb{R} \) and
\[
\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1.
\]

**Definition 1.3.** [3] Let \((X, N)\) be a fuzzy normed space, a subset \( A \) of \( X \) is said to be \( l \)-fuzzy closed if for any sequence \( \{x_n\} \) and for each \( \alpha \in (0, 1) \), and \( x \in A \)
\[
(1.1) \quad \lim_{k \to \infty} N(x_n - x, t) \geq \alpha
\]
for all \( t > 0 \).

**Definition 1.4.** [5] Let \((X, N)\) is a fuzzy \( n \)-normed space, that is, \( X \) is real vector space and \( N \) is fuzzy \( n \)-norm on \( X \). We form the family of \( n \)-seminorms \( \| \bullet, \bullet, \ldots, \bullet \|_\alpha, \) \( \alpha \in (0, 1) \), this family generates a family \( \mathcal{F} \) of seminorms
\[
\| x_1, \ldots, x_{n-1}, \bullet \|_\alpha, \quad \text{where } x_1, \ldots, x_{n-1} \in X \text{ and } \alpha \in (0, 1).
\]
The family \( \mathcal{F} \) generates a locally convex topology on \( X \); a basis of neighborhoods at the origin is given by
\[
\{ x \in X : p_i(x) \leq \epsilon_i \text{ for } i = 1, 2, \ldots, n \},
\]
where \( p_i \in \mathcal{F} \) and \( \epsilon_i > 0 \) for \( i = 1, 2, \ldots, n \). We call this the locally convex topology generated by the fuzzy \( n \)-norm \( N \).

**Definition 1.5.** [3] Let \((X, N)\) be a fuzzy normed space, a mapping \( T : (X, N_1) \to (Y, N_2) \) is said to be fuzzy continuous at \( x_0 \in X \), if for a given \( \epsilon > 0 \) and \( \alpha \in (0, 1) \) there exist \( \delta = \delta(\alpha, \epsilon) > 0 \) and \( \beta = \beta(\alpha, \epsilon) \in (0, 1) \) such that for each \( \epsilon > 0 \), there exists \( \delta > 0 \) and
\[
N_1(x - x_0, \delta) > \beta \Rightarrow N_2(T(x) - T(x_0), \epsilon) > \alpha
\]
for all \( x \in X \).
If \( T \) is fuzzy continuous at each point of \( X \), then \( T \) is said to be sectional fuzzy continuous on \( X \).
Definition 1.6. [3] Let $(X, N)$ be a fuzzy normed space, a mapping $T : (X, N_1) \rightarrow (Y, N_2)$ is said to be sectional fuzzy continuous at $x_0 \in X$, if there exists $\alpha_0 \in (0, 1)$ such that for each $\epsilon > 0$, there exists $\delta > 0$ and
\[
N_1(x - x_0, \delta) \geq \alpha_0 \Rightarrow N_2(T(x) - T(x_0), \epsilon) \geq \alpha_0
\]
for all $x \in X$.

If $T$ is sectional fuzzy continuous at each point of $X$, then $T$ is said to be sectional fuzzy continuous on $X$.

Definition 1.7. [3] Let $(X, N)$ be a fuzzy normed space, a subset $A$ of $X$ is said to be $l$-fuzzy compact if for any sequence $\{x_n\}$ and for each $\alpha \in (0, 1)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in A$ (both depending on $\{x_n\}$ and $\alpha$) such that
\[
\lim_{k \to \infty} N(x_{n_k} - x, t) \geq \alpha
\]
for all $t > 0$.

2. Schauder fixed point theorem

In this section, we establish Schauder fixed point theorems in the locally convex topology generated by fuzzy $n$-normed spaces.

Definition 2.1. A subset $A$ of $X$ is said to be $l$-fuzzy closed in the locally convex topology generated by $N$ if for any sequence $\{x_n\}$ and for each $\alpha \in (0, 1)$, and $x \in A$
\[
\lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_n - x, t) \geq \alpha
\]
for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

Definition 2.2. A mapping $T : (X, N_1) \rightarrow (Y, N_2)$ is said to be fuzzy continuous at $x_0 \in X$ in the locally convex topology generated by $N$, if for a given $\epsilon > 0$ and $\alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ and $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that for each $\epsilon > 0$, there exists $\delta > 0$ and
\[
N_1(a_1, \ldots, a_{n-1}, x - x_0, \delta) > \beta \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(x_0), \epsilon) > \alpha
\]
for all $a_1, \ldots, a_{n-1}, x, x_0 \in X$.

If $T$ is fuzzy continuous at each point of $X$, then $T$ is said to be sectional fuzzy continuous on $X$.

Definition 2.3. A mapping $T : (X, N_1) \rightarrow (Y, N_2)$ is said to be sectional fuzzy continuous at $x_0 \in X$, in the locally convex topology generated by $N$ if there exists $\alpha_0 \in (0, 1)$ such that for each $\epsilon > 0$, there exists $\delta > 0$ and
\[
N_1(a_1, \ldots, a_{n-1}, x - x_0, \delta) \geq \alpha_0 \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(x_0), \epsilon) \geq \alpha_0
\]
for all $a_1, \ldots, a_{n-1}, x, x_0 \in X$.

If $T$ is sectional fuzzy continuous at each point of $X$, then $T$ is said to be sectional fuzzy continuous on $X$. 
Definition 2.4. A subset $A$ of $X$ is said to be $l$-fuzzy compact in the locally convex topology generated by $N$ if for any sequence $\{x_n\}$ and for each $\alpha \in (0,1)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in A$ (both depending on $\{x_n\}$ and $\alpha$) such that
\[(2.2) \lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) \geq \alpha\]
for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

Lemma 2.1. A subset $A$ of $X$ is $l$-fuzzy compact in the locally convex topology generated by $N$ iff $A$ is compact w.r.t. $|||\alpha |||$ for each $\alpha \in (0,1)$.

Proof. First suppose that $A$ is $l$-fuzzy compact. Take $\alpha_0 \in (0,1)$. Let $\{x_n\}$ be a sequence in $A$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x$ in $A$ (both depend on $\alpha_0$) such that
\[(2.3) \lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) \geq \alpha_0\]
for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$. This implies that for a given $\epsilon > 0$ with $\alpha_0 - \epsilon > 0$ and for a given $t > 0$, there exists a positive integer $K(\epsilon, t)$ such that
\[N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) > \alpha_0 - \epsilon\]
for all $n \geq K(\epsilon, t)$.

Which implies to
\[||a_1, a_2, \ldots, a_n||_{\alpha_0-\epsilon} \leq t \text{ for all } n \geq K(\epsilon, t).\]

This implies to $A$ is compact. Since $\alpha_0 \in (0,1)$ and $\epsilon > 0$ are arbitrary, it follows that $A$ is compact w.r.t. $|||\alpha |||$ for each $\alpha \in (0,1)$. Conversely, suppose that $A$ is compact w.r.t. $|||\alpha |||$ for each $\alpha \in (0,1)$. Let $\{x_n\}$ be a sequence in $A$. Take $\alpha_0 \in (0,1)$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x$ in $A$ (both depend on $\alpha_0$) such that
\[(2.4) \lim_{k \to \infty} ||a_1, a_2, \ldots, a_{n-1}, x_{n_k} - x||_{\alpha_0} = 0.\]
for all $a_1, a_2, \ldots, a_{n-1} \in X$. This implies to for a given $\epsilon > 0$, there exists a positive integer $K(\epsilon)$ such that
\[||a_1, a_2, \ldots, a_{n-1}, x_{n_k} - x||_{\alpha_0} < \epsilon\]
for all $k \geq K(\epsilon)$.

Form this we conclude that
\[N(a_1, \ldots, a_{n-1}, x, t) > \alpha_0\]
for all $a_1, a_2, \ldots, a_{n-1} \in X$. Since $\epsilon$ is arbitrary, so
\[\lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_{n_k} - x, t) > \alpha_0\]
for all $t > 0$.

Since $\alpha_0 \in (0,1)$ is arbitrary, it follows that $A$ is $l$-fuzzy compact. \qed

Lemma 2.2. A mapping $T : (X, N_1) \to (Y, N_2)$ is sectional fuzzy continuous in the locally convex topology generated by $N$ iff $T : (X, |||\alpha |||^1) \to (Y, |||\alpha |||^2)$ is continuous for some $\alpha \in (0,1)$. 
Proof. First we suppose that, \( T : (X, N_1) \to (Y, N_2) \) is sectional fuzzy continuous. Thus there exists \( \alpha_0 \in (0, 1) \) such that for each \( \epsilon > 0 \), there exists \( \delta > 0 \) and
\[
N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \geq \alpha_0 \Rightarrow N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \epsilon) \geq \alpha_0
\]
for all \( a_1, \ldots, a_{n-1}, x, y \in X \).

Choose \( \eta_0 \) such that \( \delta_1 = \delta - \eta_0 > 0 \). Let \( \|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0}^1 \leq \delta_1 \). Then \( \|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0}^1 \leq \delta \), this leads to \( N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \geq \alpha_0 \), since \( T : (X, N_1) \to (Y, N_2) \) is sectional fuzzy continuous, this implies to \( N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \epsilon) \geq \alpha_0 \) for all \( a_1, \ldots, a_{n-1}, x, y \in X \), and hence \( \|a_1, a_2, \ldots, a_{n-1}, T(x) - T(y)\|_{\alpha_0}^2 \leq \epsilon \). Thus \( T : (X, N_1) \to (Y, N_2) \) is continuous w.r.t. \( |||_\alpha^1 \) and \( |||_\alpha^2 \). Conversely, suppose that \( T : (X, N_1) \to (Y, N_2) \) is continuous w.r.t. \( |||_\alpha^1 \) and \( |||_\alpha^2 \). Thus
\[
\|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0}^1 \leq \delta \Rightarrow \|a_1, a_2, \ldots, a_{n-1}, T(x) - T(y)\|_{\alpha_0}^2 \leq \frac{\epsilon}{2}
\]
for all \( a_1, \ldots, a_{n-1}, x, y \in X \). Let \( N_1(a_1, \ldots, a_{n-1}, x - y, \delta) \geq \alpha_0 \), so \( \|a_1, a_2, \ldots, a_{n-1}, x - y\|_{\alpha_0}^1 \leq \delta \), which implies to \( \|a_1, a_2, \ldots, a_{n-1}, T(x) - T(y)\|_{\alpha_0}^2 \leq \epsilon \). Therefore
\[
N_2(a_1, \ldots, a_{n-1}, T(x) - T(y), \epsilon) \geq \alpha_0 \text{ for all } a_1, \ldots, a_{n-1}, x, y \in X.
\]

Thus the mapping \( T : (X, N_1) \to (Y, N_2) \) is sectional fuzzy continuous. \( \square \)

Theorem 2.1. (Schauder fixed point theorem). Let \( K \) be a nonempty convex, \( l \)-fuzzy compact subset in the locally convex topology generated by \( N \) and \( T : K \to K \) is sectional fuzzy continuous. Then \( T \) has a fixed point.

Proof. For every \( \alpha \in (0, 1) \), \( \|\bullet, \bullet, \ldots, \bullet\|_\alpha \) is an \( n \)-seminorm on \( X \). As \( K \) is an \( l \)-fuzzy compact of \( X \), thus \( K \) is a compact subset of \( (X, \|\|_\alpha) \) for each \( \alpha \in (0, 1) \) ( by Lemma 3.1), since \( T : K \to K \) be sectional fuzzy continuous, it follows by Lemma 3.2 \( T : K \to K \) is continuous w.r.t. \( \|\|_\alpha \) for some \( \alpha_0 \in (0, 1) \). Therefore, we get \( K \) is a nonempty convex and compact subset of a normed linear space \( (X, \|\|_{\alpha_0}) \) and \( T : K \to K \) is a continuous mapping. By Schauder fixed point theorem [22] it follows that \( T \) has a fixed point. \( \square \)

3. Conclusion

We investigated the concepts of sectional fuzzy continuous mappings, and \( l \)-fuzzy compact sets, in locally convex topology generated by fuzzy \( n \)-normed spaces as an extension of the fuzzy normed space. In this new frame we established the Schauder-type and other fixed point theorems as well as some results in locally convex topology generated by fuzzy \( n \)-normed spaces which are useful tools in the development of the fuzzy set theory.

References