Numerical Solution of the Uncertain Characteristic Cauchy Reaction-Diffusion Equation by Variational Iteration Method

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Abstract

In this paper, the uncertain characteristic Cauchy reaction-diffusion equation is solved by the variational iteration method. The uncertain characteristic Cauchy problem is converted to a system of characteristic Cauchy problems by parametric representation and LU-fuzzy representation of fuzzy numbers. Also, using the variational iteration method and 2 representations of fuzzy numbers, 2 representations of approximate fuzzy solution are found and compared. Finally, the method is explained by 2 illustrative examples.

Keywords: Uncertain characteristic Cauchy problem, LU-fuzzy representation, parametric representation, variational iteration method

Introduction

The basic arithmetic structure was developed for fuzzy numbers as a collection of α-cuts, $0 \le \alpha \le 1$, and the arithmetic operational on fuzzy numbers is usually approached either by the use of the extension principle in domain of the membership function or by the interval arithmetic in domain of the α-cuts [1-3].

In 1986, Goetschel and Voxman presented a repetition of fuzzy numbers that is applied in fuzzy integral equations [4], or fuzzy differential and integral equations [5-7] and is called the Parametric Representation (PR) of fuzzy numbers [8]. Recently, Stefanini *et al*. introduced another representation of fuzzy numbers called LU-Fuzzy Representation (LUFR) of fuzzy numbers [9-11]. In here, 2 representations of fuzzy numbers are applied in solving Uncertain Characteristic Cauchy Problem (UCCP) by Variational Iteration Method (VIM). The VIM has been used to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions [12-15]. The convergence analysis of the VIM has been studied in [16-18].

The paper is organized as follows, respectively: the basic notations of fuzzy numbers are presented, the PR of UCCP and the LUFR of UCCP are discussed, the described technique is explained using a system of partial differential, the VIM is applied on two test problems of UCCP to show the efficiency of the method and finally is denoted a brief conclusion.

Preliminaries

Basic fuzzy calculus of PR

Definition 1 (see [8]) A PR of continuous fuzzy number w is any pair (w^-, w^+) of functions $w^\pm : [0,1] \to \mathbb{R}$ satisfying the following requirements;

(i) w^- is a nondecreasing function on $[0,1]$,

(ii) w^{+} is a nonincreasing function on $[0,1]$,

(iii) w^{-} and w^{+} are bounded and left continuous on $(0,1]$, and right continuous at $\alpha = 0$, and

(iv) $w^{-} \leq w^{+}, 0 \leq \alpha \leq 1$.

The notation;

$$[w]^{\alpha} = [w_{\alpha}^{-}, w_{\alpha}^{+}], \tag{1}$$

denotes explicitly the α -cuts of w. This paper will also refer to w^{-} and w^{+} as the left (lower) and the right (upper) branches on w, respectively.

$F = \{w \mid w = (w^{-}, w^{+})\}$ defines the PR of all fuzzy numbers. For arbitrary $w = (w^{-}, w^{+})$ and $z = (z^{-}, z^{+})$ of F the arithmetic operations are defined as follows;

Addition: $w + z = (w^{-} + z^{-}, w^{+} + z^{+})$,
Scalar multiplication;

$$\begin{cases} \varepsilon w = (\varepsilon w^{-}, \varepsilon w^{+}) & if\ \varepsilon > 0, \\ \varepsilon w = (\varepsilon w^{+}, \varepsilon w^{-}) & if\ \varepsilon < 0, \end{cases} \tag{2}$$

Subtraction;

$$w - z = w + (-z) = (w^{-} - z^{+}, w^{+} - z^{-}). \tag{3}$$

Definition 2 (see [8]) For arbitrary fuzzy numbers w and z, the quantity;

$$D(w,z) = \sup_{0 \leq \alpha \leq 1} d_H([w]^{\alpha}, [z]^{\alpha}), \tag{4}$$

is the distance between w and z, where

$$d_H([w]^{\alpha}, [z]^{\alpha}) = \max\{| w_{\alpha}^{-} - z_{\alpha}^{-} |, | w_{\alpha}^{+} - z_{\alpha}^{+} |\}, \tag{5}$$

is the Hausdorff distance.

Basic fuzzy calculus of LUFR

Definition 3 (see [11]) Let $[w]^{\alpha} = [w_{\alpha}^{-}, w_{\alpha}^{+}]$, and the lower and upper branches w_{α}^{-} and w_{α}^{+} of w are differentiable. A LUFR of a fuzzy number can be written as;

$$\begin{aligned} &w = (\alpha, w^{-}, \delta w^{-}, w^{+}, \delta w^{+}), or \\ &w = (w^{-}, \delta w^{-}, w^{+}, \delta w^{+}), \end{aligned} \tag{6}$$

the slopes of which corresponding to w^{-} and w^{+} are denoted by δw^{-} and δw^{+}, respectively, and

$\delta w^- \geq 0, \quad \delta w^+ \leq 0, \quad for\, all \quad \alpha \in [0,1].$

$F_{LU} = \{w \mid w = (w^-, \delta w^-, w^+, \delta w^+)\}$ defines the LUFR of all fuzzy numbers. Assume $w, z \in F_{LU}$, therefore the arithmetic operations are defined as follows;

Addition;

$$w + z = (w^- + z^-, \delta w^- + \delta z^-, w^+ + z^+, \delta w^+ + \delta z^+), \tag{7}$$

Scalar multiplication;

$$\begin{cases} \varepsilon w = (\varepsilon w^-, \varepsilon \delta w^-, \varepsilon w^+, \varepsilon \delta w^+) & if\ \varepsilon > 0, \\ \varepsilon w = (\varepsilon w^+, \varepsilon \delta w^+, \varepsilon w^-, \varepsilon \delta w^-) & if\ \varepsilon < 0, \end{cases} \tag{8}$$

Subtraction;

$$w - z = w + (-z) = (w^- - z^+, \delta w^- - \delta z^+, w^+ - z^-, \delta w^+ - \delta z^-). \tag{9}$$

Definition 4 (see [11]) For arbitrary LUFR of fuzzy numbers $w = (w^-, \delta w^-, w^+, \delta w^+)$ and $z = (z^-, \delta z^-, z^+, \delta z^+)$, the quantity;

$$\begin{aligned} &D_E(w,z) = \frac{\|w - z\|_N}{4N}, where \\ &\|w - z\|^2{}_N = \sum_{i=0}^{N} [(w^-_{\alpha_i} - z^-_{\alpha_i})^2 + (w^+_{\alpha_i} - z^+_{\alpha_i})^2 + (\delta w^-_{\alpha_i} - \delta z^-_{\alpha_i})^2 + (\delta w^+_{\alpha_i} - \delta z^+_{\alpha_i})^2], \\ &\quad and \quad 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_i < \cdots < \alpha_N = 1, \end{aligned} \tag{10}$$

is the Euclidean-like distance on F_{LU} between w and z.

Presentation of UCCP

Reaction-diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [19-21]. Consider the characteristic Cauchy reaction-diffusion equation;

$$u_y(x, y) = \eta u_{xx}(x, y) + p(x, y)u(x, y), \tag{11}$$

subject to;

$$u(x,0) = f(x), \tag{12}$$

where $u(x, y)$ is the concentration, p is the reaction parameter and known continuous function, $\eta > 0$ is the diffusion coefficient and $(x, y) \in \Omega = (x_1, x_2) \times [0, y_1)$ so that $x_1, x_2 \in \mathbb{R}, y_1 \in \mathbb{R}^+$.

If suppose $f(x)$ be the fuzzy extension of real-valued differentiable function, i.e., $f(x)$ be a fuzzy function, then the Eqs. (11) - (12) are called an UCCP. In an UCCP, it is assumed $u(x,y)$ be continuous and H-differentiable, therefore, $u_y(x,y)$ and $u_{xx}(x,y)$ are partial derivatives, see [22]. Now, the UCCP is converted to equations in the crisp case the use of the PR and the LUFR.

PR of UCCP

It is assumed that $f:[x_1,x_2]\to F$ is a fuzzy function where the PR of $f(x)$ is as follows;

$$f(x)=(f^-(x),f^+(x)), \tag{13}$$

therefore in Eq. (11) $u:\Omega\to F$ as a fuzzy function is obtained;

$$u(x,y)=(u^-(x,y),u^+(x,y)). \tag{14}$$

Without losing generality, suppose the sign of $p(x,y)$ does not change and $p(x,y)<0$; then, $u(x,y)$ can be obtained by solving the following system of equations in the crisp case;

$$\begin{aligned}
&u_y^- = \eta u_{xx}^- + p(x,y)u^+,\\
&u_y^+ = \eta u_{xx}^+ + p(x,y)u^-,\\
&u^\pm(x,0) = f^\pm(x).
\end{aligned} \tag{15}$$

LUFR of UCCP

It is assumed that $f:[x_1,x_2]\to F_{LU}$ is a fuzzy function, hence the LUFR of $f(x)$ is as follows;

$$f(x)=(f^-(x),\delta f^-(x),f^+(x),\delta f^+(x)), \tag{16}$$

therefore in Eq. (11) $u:\Omega\to F_{LU}$ as a fuzzy function is obtained;

$$u(x,y)=(u^-(x,y),\delta u^-(x,y),u^+(x,y),\delta u^+(x,y)). \tag{17}$$

Without losing generality, suppose the sign of $p(x,y)$ does not change and $p(x,y)<0$; then, $u(x,y)$ can be obtained by solving the following system of equations in the crisp case;

$$\begin{aligned}
&u_y^- = \eta u_{xx}^- + p(x,y)u^+,\\
&u_y^+ = \eta u_{xx}^+ + p(x,y)u^-,\\
&u^\pm(x,0) = f^\pm(x),
\end{aligned} \tag{18}$$

and to determine the corresponding slopes, the following system of equations is added to (18);

$$\begin{aligned} &\delta u_y^- = \eta \delta u_{xx}^- + p(x,y)\delta u^+, \\ &\delta u_y^+ = \eta \delta u_{xx}^+ + p(x,y)\delta u^-, \\ &\delta u^{\pm}(x,0) = \delta f^{\pm}(x). \end{aligned} \tag{19}$$

The variational iteration method

Consider the system of partial differential equations in an operator form;

$$\begin{aligned} &L_y u = \eta L_{xx} u + p(x,y) I v, \\ &L_y v = \eta L_{xx} v + p(x,y) I u, \end{aligned} \tag{20}$$

subject to;

$$\begin{aligned} &u(x,0) = g_1(x), \\ &v(x,0) = g_2(x), \end{aligned} \tag{21}$$

where I is the identity operator, g_1 and g_2 are real-valued differentiable functions, $L_y = \dfrac{\partial}{\partial y}$ and $L_{xx} = \dfrac{\partial^2}{\partial x^2}$ define the partial differential operators.

To solve Eq. (20) by means of VIM, a correctional functional is constructed which reads;

$$\begin{aligned} &u_{k+1}(x,y) = u_k(x,y) + \int_0^y \lambda_1 (L_s u_k - \eta L_{xx} \tilde{u}_k - p\tilde{v}_k) ds, \\ &v_{k+1}(x,y) = v_k(x,y) + \int_0^y \lambda_2 (L_s v_k - \eta L_{xx} \tilde{v}_k - p\tilde{u}_k) ds, \end{aligned} \tag{22}$$

where λ_1 and λ_2 are general Lagrange multipliers [23], the subscript k denotes the k th order iteration, and it is assumed that the restricted variation of $L_{xx}\tilde{u}_k$, $\tilde{u}_k$, $L_{xx}\tilde{v}_k$ and $\tilde{u}_k$ are zero [24]. Its stationary conditions can be obtained as follows;

$$\begin{aligned} &\lambda_1'(y) = 0, \\ &1 + \lambda_1(y) = 0, \\ &\lambda_2'(y) = 0, \\ &1 + \lambda_2(y) = 0. \end{aligned} \tag{23}$$

The Lagrange multipliers, therefore, can be identified as $\lambda_1 = \lambda_2 = -1$, and the following variational iteration formula can be obtained by;

$$\begin{aligned} &u_{k+1}(x,y) = u_k(x,y) - \int_0^y (L_s u_k - \eta L_{xx} u_k - p v_k) ds, \\ &v_{k+1}(x,y) = v_k(x,y) - \int_0^y (L_s v_k - \eta L_{xx} v_k - p u_k) ds, \end{aligned} \tag{24}$$

starting with initial approximations $u_0(x,y) = g_1(x)$ and $v_0(x,y) = g_2(x)$ by Eq. (24), an approximate solution of (20) is obtained.

Results and discussion

Two numerical examples are considered to illustrate the method.

Example 1 Consider Eq. (11) with $\eta = 1$ and $p(x,y) = -1$ as follows;

$$u_y(x,y) = u_{xx}(x,y) - u(x,y), \quad (x,y) \in (-3,3) \times [0,2). \tag{25}$$

(i) The fuzzy initial condition by the PR is as follows;

$$\begin{aligned} u^-(x,0) &= (x^2 + 8x + 20)a, \\ u^+(x,0) &= (x^2 + 8x + 20)b, \end{aligned} \tag{26}$$

where

$$\begin{aligned} a &= \frac{13}{15}(\alpha^2 + \alpha) + \frac{2}{15}(4 - \alpha^3 - \alpha), \\ b &= \frac{2}{15}(\alpha^2 + \alpha) + \frac{13}{15}(4 - \alpha^3 - \alpha). \end{aligned}$$

The exact fuzzy solution $u(x,y) = (u^-(x,y), u^+(x,y))$ in this case is given by;

$$\begin{aligned} u^-(x,y) &= (x^2 + 8x + 20 + 2y)(a\cosh y - b\sinh y), \\ u^+(x,y) &= (x^2 + 8x + 20 + 2y)(b\cosh y - a\sinh y). \end{aligned} \tag{27}$$

If $u(x,y) = (u^-(x,y), u^+(x,y))$, then the system of Eqs. (25) by (15) is;

$$\begin{aligned} u_y^- &= u_{xx}^- - u^+, \\ u_y^+ &= u_{xx}^+ - u^-, \end{aligned} \tag{28}$$

with the initial conditions (26). According to the VIM for the Eq. (28), u_k^- and u_k^+ are calculated for $k = 1,2,\ldots,30$ and u_{30}^- and u_{30}^+ are considered as approximations of exact solutions (27). The numerical results by this approximation are summarized in **Table 1** and the error function by D distance is plotted in **Figure 1**.

(ii) The fuzzy initial condition by the LUFR is exact and gives;

$$u(x,0) = (u^-(x,0), \delta u^-(x,0), u^+(x,0), \delta u^+(x,0)),$$

with

$$\begin{aligned} u^-(x,0) &= (x^2+8x+20)a, \\ u^+(x,0) &= (x^2+8x+20)b, \end{aligned} \tag{29}$$

$$\begin{aligned} \delta u^-(x,0) &= (x^2+8x+20)c, \\ \delta u^+(x,0) &= (x^2+8x+20)d, \end{aligned} \tag{30}$$

that

$$\begin{aligned} c &= \frac{13}{15}(2\alpha+1)+\frac{2}{15}(-3\alpha^2-1), \\ d &= \frac{2}{15}(2\alpha+1)+\frac{13}{15}(-3\alpha^2-1). \end{aligned}$$

If

$$u(x,y) = (u^-(x,y), \delta u^-(x,y), u^+(x,y), \delta u^+(x,y)), \tag{31}$$

so the exact fuzzy solution in this case is as follows;

$$\begin{aligned} u^-(x,y) &= (x^2+8x+20+2y)(a\cosh y - b\sinh y), \\ u^+(x,y) &= (x^2+8x+20+2y)(b\cosh y - a\sinh y), \\ \delta u^-(x,y) &= (x^2+8x+20+2y)(c\cosh y - d\sinh y), \\ \delta u^+(x,y) &= (x^2+8x+20+2y)(d\cosh y - c\sinh y). \end{aligned} \tag{32}$$

In result, the 2 systems of Eqs. (18) and (19) are;

$$\begin{aligned} u_y^- &= u_{xx}^- - u^+, \\ u_y^+ &= u_{xx}^+ - u^-, \end{aligned} \tag{33}$$

with the initial conditions (29), and

$$\begin{aligned} \delta u_y^- &= \delta u_{xx}^- - \delta u^+, \\ \delta u_y^+ &= \delta u_{xx}^+ - \delta u^-, \end{aligned} \tag{34}$$

with the initial conditions (30). The VIM is applied to the Eqs. (33) and (34), thus, u_k^-, u_k^+, δu_k^- and δu_k^+ are obtained for $k=1,2,\ldots,30$ and u_{30}^-, u_{30}^+, δu_{30}^- and δu_{30}^+ are considered as approximations of exact solutions (32). **Figure 2** by Euclidean-like distance (D_E) presents the error function and **Table 2** shows the numerical results.

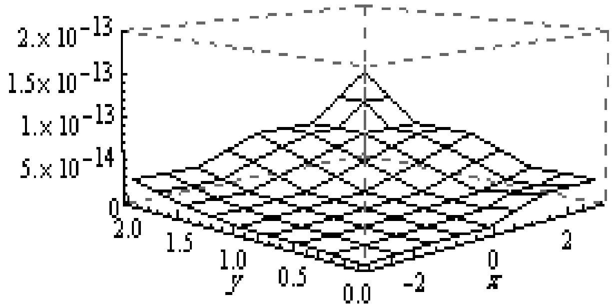

Figure 1 Plot of the error function by D distance in Example 1.

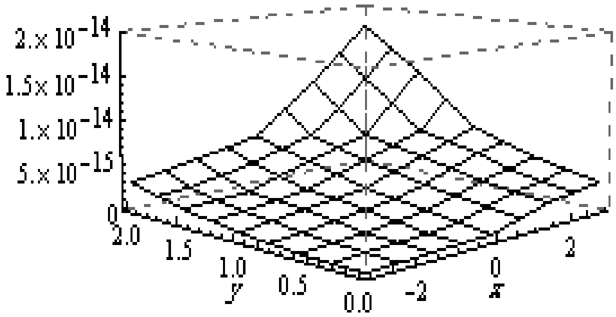

Figure 2 Plot of the error function by D_E distance in Example 1.

Table 1 The error and approximate values with the PR in Example 1.

x	***y***	**Approximate values by VIM (**$k = 30, \alpha = 0.5$**)**	**Error (*D*)**
-3	0	(5.5000,15.125)	0
-2.5	0.25	(2.5002,19.1845)	3.55271×10^{-15}
0	0.5	(-7.0544,59.5951)	8.88178×10^{-15}
2.5	1	(-89.629,162.848)	5.68434×10^{-14}
3	1.5	(-215.792,267.335)	5.68434×10^{-14}

Table 2 The error and approximate values with the LUFR in Example 1.

x	***y***	**Approximate values by VIM (**$k = 30, \alpha = 0.5$**)**	**Error (**D_E**)**
-3	0	(5.5000,7.5000,15.125,-6.2500)	0
-2.5	0.25	(2.50021,12.5745,19.1845,-11.2602)	3.2186×10^{-16}
0	0.5	(-7.05442,49.199,59.5951,-46.0147)	1.66328×10^{-15}
2.5	1	(-89.629,182.56,162.848,-178.122)	6.98725×10^{-15}
3	1.5	(-215.792,346.652,267.335,-343.528)	1.0693×10^{-14}

Example 2 Consider Eq. (11) with $\eta = 1$ and $p(x, y) = -16y$,

$$u_y(x, y) = u_{xx}(x, y) - 16yu(x, y), (x, y) \in (-1,1) \times [0,1). \tag{35}$$

(i) Eq. (35) with the fuzzy initial conditions in the PR form is considered as the following;

$$\begin{aligned} u^-(x,0) &= e^{x-6}(2\alpha - 2), \\ u^+(x,0) &= e^{x-6}(2 - 2\alpha). \end{aligned} \tag{36}$$

The exact fuzzy solution $u(x, y) = (u^-(x, y), u^+(x, y))$ is written as;

$$\begin{aligned} u^-(x, y) &= e^{x+8y^2+y-6}(2\alpha - 2), \\ u^+(x, y) &= e^{x+8y^2+y-6}(2 - 2\alpha). \end{aligned} \tag{37}$$

The system of Eqs. (35) is given as;

$$\begin{aligned} u_y^- &= u_{xx}^- - 16yu^+, \\ u_y^+ &= u_{xx}^+ - 16yu^-, \end{aligned} \tag{38}$$

with the initial conditions (36). The system of Eqs. (38) with initial conditions (36) is solved by VIM. u_k^- and u_k^+ are calculated for $k = 1,2,\ldots,30$ and u_{30}^- and u_{30}^+ are considered as approximations of exact solutions (37). The numerical results and the error function are denoted in **Table 3** and **Figure 3**, respectively.

(ii) Eq. (35) is considered in the LUFR form, so the fuzzy initial condition is written as;

$$\begin{aligned} u^-(x,0) &= e^{x-6}(2\alpha - 2), \\ u^+(x,0) &= e^{x-6}(2 - 2\alpha), \end{aligned} \tag{39}$$

$$\begin{aligned} \delta u^-(x,0) &= 2e^{x-6}, \\ \delta u^+(x,0) &= -2e^{x-6}. \end{aligned} \tag{40}$$

The exact fuzzy solution in this case is;

$$u(x, y) = (u^-(x, y), \delta u^-(x, y), u^+(x, y), \delta u^+(x, y)), \tag{41}$$

that

$$\begin{aligned} u^-(x, y) &= e^{x+8y^2+y-6}(2\alpha - 2), \\ u^+(x, y) &= e^{x+8y^2+y-6}(2 - 2\alpha), \\ \delta u^-(x, y) &= 2e^{x+8y^2+y-6}, \\ \delta u^+(x, y) &= -2e^{x+8y^2+y-6}. \end{aligned} \tag{42}$$

Hence the 2 systems of Eqs. (35) are written as;

$$\begin{aligned} u_y^- &= u_{xx}^- - 16yu^+, \\ u_y^+ &= u_{xx}^+ - 16yu^-, \end{aligned} \tag{43}$$

with the initial conditions (39), and

$$\begin{aligned} \delta u_y^- &= \delta u_{xx}^- - 16y\delta u^+, \\ \delta u_y^+ &= \delta u_{xx}^+ - 16y\delta u^-, \end{aligned} \tag{44}$$

with the initial conditions (40). The VIM is applied to Eqs. (43) - (44), thus, u_k^-, u_k^+, δu_k^- and δu_k^+ are obtained for $k = 1,2,\ldots,30$ and u_{30}^-, u_{30}^+, δu_{30}^- and δu_{30}^+ are considered as approximations of exact solutions (42). **Figure 4** presents the error function and **Table 4** shows the numerical results.

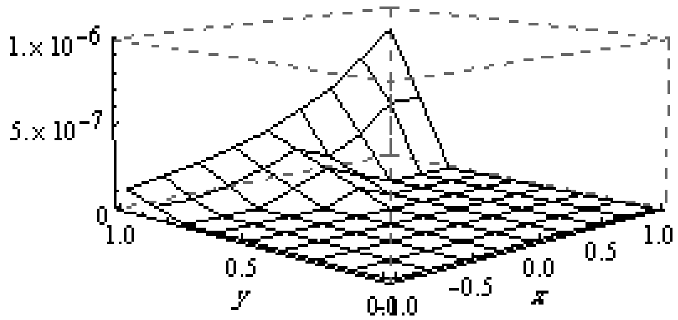

Figure 3 Plot of the error function by *D* distance in Example 2.

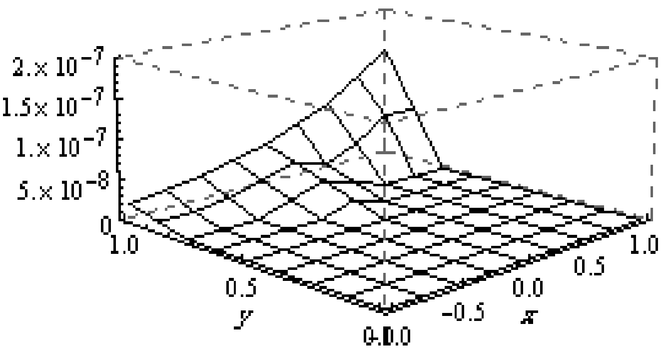

Figure 4 Plot of the error function by D_E distance in Example 2.

Table 3 The error and approximate values with the PR in Example 2.

x	y	Approximate values by VIM ($k = 30, \alpha = 0.5$)	Error (D)
-1	0	(-0.00091 ,0.00091)	0
-0.5	0.25	(-0.00318,0.00318)	1.73472×10^{-18}
0	0.5	(-0.0302,0.0302)	1.38778×10^{-17}
0.5	0.75	(-0.7788, 0.7788)	2.53131×10^{-14}
1	1	(-54.5982, 54.5982)	8.66096×10^{-7}

Table 4 The error and approximate values with the LUFR in Example 2.

x	y	Approximate values by VIM ($k = 30, \alpha = 0.5$)	Error (D_E)
-1	0	(-0.00091,0.00182,0.00091,-0.00182)	0
-0.5	0.25	(-0.00318,0.00637,0.00318,-0.00637)	4.5832×10^{-20}
0	0.5	(-0.0302,0.06039,0.0302,-0.06039)	6.0595×10^{-19}
0.5	0.75	(-0.7788,1.5576,0.7788,-1.5576)	4.2725×10^{-15}
1	1	(-54.5982,109.196,54.5982,-109.196)	1.4614×10^{-7}

Tables show the approximate values are close to the exact values in both representations of fuzzy number. Figures denote the well error function by arbitrary selection of 30 iterations of VIM. These comments denote convergence of the method.

Conclusions

In this paper, an UCCP has been converted to a system of equations in the crisp case by means of 2 representations of fuzzy number. By the PR and the LUFR, 2 representations of approximate fuzzy solution of the UCCP have been introduced. Furthermore, it can be seen that both the PR and LUFR are applied as well as for solving an UCCP, but it is trivial that the LUFR has more computations than the PR. It has also been shown how the VIM as a numerical algorithm can be used for solving an UCCP.

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