Modified Homotopy Analysis Method for Zakharov-Kuznetsov Equations

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Received: 23 April 2012, Revised: 15 May 2012, Accepted: 8 July 2013

Abstract

In this paper, we apply Modified Homotopy Analysis Method (MHAM) to find appropriate solutions to Zakharov-Kuznetsov equations, which are of utmost importance in applied and engineering sciences. The proposed modification is an elegant coupling of the Homotopy Analysis Method (HAM) and Taylor’s series. Numerical results, coupled with graphical representation, explicitly reveal the complete reliability of the proposed algorithm.

Keywords: Homotopy analysis method, Taylor’s series, exact solutions, MAPLE, Zakharov-Kuznetsov equations

Introduction

The rapid development of nonlinear sciences witnesses a wide range of analytical and numerical techniques by various scientists [1-19]. Most of the developed schemes have their limitations, like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems [1-11]. In a similar context, Liao [7-9] developed the Homotopy Analysis Method (HAM) which has been applied to a wide range of nonlinear problems of a physical nature; see [1-19] and the references therein. The basic motivation of the present study is the modification of the traditional HAM to tackle Zakharov-Kuznetsov equations. The proposed modification is made by combining the traditional HAM with Taylor’s series. It is observed that the proposed modification is highly effective and absorbs some of the basic deficiencies of the original version of HAM. Moreover, this modified approach (MHAM) is more user-friendly and overcomes the complexities of selection of initial value. Several examples are given which reveal the efficiency and reliability of the proposed algorithm.

Analysis of homotopy analysis method (HAM)

The following differential equation

\[ N[u(\tau)] = 0 \] (1)

is considered where \( N \) is a nonlinear operator, \( \tau \) denotes independent variables, and \( u(\tau) \) is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional Homotopy method, Liao constructs a so called zero - order deformation equation.
(1 − 𝑝)𝐿[𝜓(𝜏; 𝑝) − 𝑢₀(𝜏)] = 𝑝ℎ𝐻(𝜏)𝑁[𝜓(𝜏; 𝑝)],

(2)

where 𝑝 ∈ [0, 1] is the embedding parameter, 𝑡 ≠ 0 is a nonzero parameter, 𝐻(𝜏) ≠ 0 is an auxiliary function, 𝐿 is an auxiliary linear operator, 𝑢₀(𝜏) is an initial guess of 𝑢(𝜏), and 𝑢(𝜏; 𝑝) is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary components in HAM. Obviously, when, 𝑝 = 0 and 𝑝 = 1, it holds;

𝜓(𝜏; 0) = 𝑢₀(𝜏), 𝜓(𝜏; 1) = 𝑢(𝜏).

(3)

Thus, as 𝑝 increases from 0 to 1, the solution 𝜓(𝜏; 𝑝) varies from the initial guesses 𝑢₀(𝜏) to the solution 𝑢(𝜏). Expanding in the Taylor series with respect to 𝑝,

𝜓(𝜏; 𝑝) = 𝑢₀(𝜏) + ∑∞ 𝑚=1 𝑢ₘ(𝜏)𝑝ᵐ,

(4)

where 𝑢ₘ(𝜏) = 1 𝑚! 𝜕𝑚−1𝜓(𝜏; 𝑝) 𝜕𝑝𝑚 at 𝑝 = 0,

(5)

is obtained. If the auxiliary linear operator, the initial guess, the auxiliary h, and the auxiliary function are so properly chosen, the above series converges at 𝑝 = 1.

𝑢(𝜏) = 𝑢₀(𝜏) + ∑∞ 𝑚=1 𝑢ₘ(𝜏),

(6)

is then obtained. The vector is defined as;

\( \vec{u} = \{u_0(\tau), u_1(\tau), u_2(\tau), \ldots, u_0(\tau)\} \)

(7)

Differentiating Eq. (2) 𝑚 times with respect to the embedding parameter 𝑝 and then setting 𝑝 = 0 and finally dividing them by 𝑚!, the 𝑚𝑡ℎ-order deformation equation is obtained;

\[ L[\tilde{u}_m(\tau) − \chi_m u_{m-1}(\tau)] = hH(\tau)R_m(\vec{u}_{m-1}) \]

(8)

where

\[ R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \psi(\tau; p)}{\partial p^{m-1}} \text{ at } p = 0 \]

and

\[ \chi_m = 0, m \leq 1, \]

\[ = 1, m > 1, \]

Applying \( L^{-1} \) both sides of (8),

\[ u_m(\tau) = \chi_m u_{m-1}(\tau) + hL^{-1}[H(\tau)R_m(\vec{u}_{m-1})] \]

is obtained. This way, it is easy to obtain \( u_m \) for \( m \geq 1 \), at \( m𝑡ℎ \)-order;

\[ u(\tau) = \sum_{m=0}^{M} u_m(\tau). \]

(9)

when \( M \to \infty \), an accurate approximation of the original Eq. (1) is considered. For the convergence of the above method the reader is referred to is referred to Liao’s work. If Eq. (1) admits a unique solution, then this method will produce a unique solution. If Eq. (1) does not possess a unique solution, the HAM will give a solution among many other (possible) solutions.
**Numerical application**

In this section, MHAM is applied to find appropriate solutions of Zakharov-Kuznetsov equations. The numerical results are very encouraging.

**Example 1** Consider the ZK(2,2,2) equation;

\[
\begin{align*}
    u_t + (u^2)_x + \frac{1}{8} (u^2)_{xxx} + \frac{1}{8} (u^2)_{yyx} &= 0, \\
    u(x, y, 0) &= \frac{4}{5} \lambda \sinh^2(x + y),
\end{align*}
\]

(10a) \hspace{1cm} (10b)

To solve the given Equation by HAM the linear operator;

\[
L[\phi(x, y; t; q)] = \frac{\partial}{\partial t} (\phi(x, y; t; q)),
\]

(11)

is chosen, with the property;

\[
L[c_1 + tc_2] = 0,
\]

(12)

where \(c_1\) and \(c_2\) are the integral constants. The inverse operator \(L^{-1}\) is given by;

\[
L^{-1} = \int_0^t (\cdot) dt,
\]

(13)

and a nonlinear operator is defined as;

\[
N[\phi(x, y; t; q)] = \phi(x, y; t; q)_t + (\phi(x, y; t; q)^2)_x + \frac{1}{8} (\phi(x, y; t; q)^3)_{xxx} + \frac{1}{8} (\phi(x, y; t; q)^2)_{yyx},
\]

(14)

Using the above definition, the zeroth-order deformation equation is constructed.

\[
(1 - q)L[\phi(x, y; t; q) - u_0(x, y, t)] = qH(x, y, t) N[\phi(x, y; t; q)].
\]

(15)

For \(q = 0\) and \(q = 1\),

\[
\phi(x, y; t; 0) = u_0(x, y, t), \quad \phi(x, y; t; 1) = u(x, y, t).
\]

(16)

can be written. Thus, the \(m\)th-order deformation equation is obtained.

\[
L[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = hH(x, y, t) R_m(\overline{u}_{m-1}),
\]

(17)

with initial condition \(u_m(x, y, 0) = 0\).

\[
\text{where} \quad R_m(\overline{u}_{m-1}) = h \begin{bmatrix}
(u_{m-1}(x, y, t))_t + (\sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t))_x \\
+ \frac{1}{8} (\sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t))_{xxx} \\
+ \frac{1}{8} (\sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t))_{yyx}
\end{bmatrix}.
\]

(18)

After applying Taylor’s series, the result is;
\[
R_m(\vec{u}_{m-1}) = h \left[ (u_{m-1}(x,y,t) + f_m), + \left( \sum_{r=0}^{m-1} (u_r(x,y,t) + f_{r+1}) (u_{m-1-r}(x,y,t) + f_{m-r}) \right) \right] + \\
\frac{1}{8} \left( \sum_{r=0}^{m-1} (u_r(x,y,t) + f_{r+1}) (u_{m-1-r}(x,y,t) + f_{m-r}) \right)_{,xx} + \\
\frac{1}{8} \left( \sum_{r=0}^{m-1} (u_r(x,y,t) + f_{r+1}) (u_{m-1-r}(x,y,t) + f_{m-r}) \right)_{,yy}.
\]

(19)

Now the solutions of the \(m\)th-order deformation equation are;

\[
u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + L^{-1}[hH(x,y,t)R_m(\vec{u}_{m-1})], \quad m \geq 1,
\]

(20)

an initial approximation is used at the start;

\[
u_0(x,t) = \frac{4}{3} \sinh^2(y),
\]

(21)

and by means of the iteration formula as discussed above, if \(h = -1, H = 1\), the others components can be directly as;

\[
u_1 = -\frac{32}{9} \lambda^2 (-14 xcosh^2(y) + 2x + 10x sinh(y) cosh(y) sinh(x+y) cosh(x+y) - \\
8 sinh(x+y) cosh(x+y) - 5 sinh(y) cosh(y) - 2xcosh^2(x+y) + \\
10 sinh(x+y) cosh^3(x+y) + 12xcosh^4(x+y) + 7 sinh(y) cosh(y) cosh^2(x+y) + \\
2 sinh(x+y) cosh(x+y) cosh^2(y) + 4xcosh^2(y) cosh^2(x+y)t),
\]

(22)

\[
u_2 = \frac{16}{27} \lambda^2 t (-84xcosh^2(y)cosh^2(x+y) - 18 sinh(x+y) cosh(x+y) cosh^2(y) - \\
24x sinh(y) cosh(y) sinh(x+y) cosh(x+y) + \cdots,
\]

\[
\vdots
\]

The series form solution is given by;

\[
u(x,t) = \frac{4}{3} \sinh^2(y) + -\frac{32}{9} \lambda^2 (-14 xcosh^2(y) + 2x + 10x sinh(y) cosh(y) sinh(x+y) cosh(x+y) - \\
8 sinh(x+y) cosh(x+y) - 5 sinh(y) cosh(y) - 2xcosh^2(x+y) + 10 sinh(x+y) cosh^3(x+y) + \cdots
\]

(23)

The rest of the components of the iteration formulae can be obtained using MAPLE.

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Example 2 Consider the following ZK nonlinear PDE;

\[ u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yxx} = 0, \tag{24a} \]

\[ u(x, y, 0) = -\frac{4}{3}\lambda \cosh^2(x + y), \tag{24b} \]

To solve the given Equation by HAM the linear operator;

\[ L[\varphi(x, y, t; q)] = \frac{\partial}{\partial t}(\varphi(x, y, t; q)), \tag{25} \]

is chosen, with the property;

\[ L[c_1 + tc_2] = 0, \tag{26} \]

where \( c_1 \) and \( c_2 \) are the integral constants. The inverse operator \( L^{-1} \) is given by;

\[ L^{-1} = \int_0^t (\cdot) dt, \tag{27} \]

A nonlinear operator can now be defined as;

\[ N[\varphi(x, y, t; q)] = \varphi(x, y, t; q)_t + (\varphi(x, y, t; q))_x + \frac{1}{8}(\varphi(x, y, t; q)_x^2)_{xxx} + \frac{1}{8}(\varphi(x, y, t; q)_y^2)_{yxx}, \tag{28} \]

Using the above definition, zeroth-order deformation equation can be constructed;

\[ (1 - q)L[\varphi(x, y, t; q) - u_0(x, y, t)] = qH(x, y, t)N[\varphi(x, y, t; q)]. \tag{29} \]

for \( q = 0 \) and \( q = 1, \)

\[ \varphi(x, y, t; 0) = u_0(x, y, t), \quad \varphi(x, y, t; 1) = u(x, y, t). \tag{30} \]
can be written. Thus, the $m$th-order deformation equation is obtained;

\[ L[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = hH(x, y, t)R_m(\vec{u}_{m-1}) \]

with initial condition $u_m(x, y, 0) = 0$

\[
R_m(\vec{u}_{m-1}) = h \left[ \left( u_{m-1}(x, y, t) \right)_t + \left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_x ight] \\
+ \frac{1}{8} \left[ \left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_{xxx} \\
+ \frac{1}{8} \left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_{yyx} \right].
\]

After applying Taylor’s series, the result is;

\[
R_m(\vec{u}_{m-1}) = h \left[ \left( u_{m-1}(x, y, t) + f_m \right)_t + \left( \sum_{r=0}^{m-1} (u_r(x, y, t) + f_{r+1}) (u_{m-1-r}(x, y, t) + f_{m-r}) \right)_x \\
+ \frac{1}{8} \left( \sum_{r=0}^{m-1} (u_r(x, y, t) + f_{r+1})(u_{m-1-r}(x, y, t) + f_{m-r}) \right)_{xxx} \\
+ \frac{1}{8} \left( \sum_{r=0}^{m-1} (u_r(x, y, t) + f_{r+1})(u_{m-1-r}(x, y, t) + f_{m-r}) \right)_{yyx} \right].
\]

Now the solutions of the $m$th-order deformation equation are;

\[ u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + L^{-1}[hH(x, y, t)R_m(\vec{u}_{m-1})], m \geq 1, \]

an initial approximation is used at the start;

\[ u_0(x, t) = -\frac{4}{3} \lambda \cosh^2(y), \]

and by means of the iteration formula as discuss above, if $h = -1, H = 1$, the others components can be directly obtained as;

\[ u_1 = -\frac{16}{9} \lambda^2 (-3 \cosh(y) \sinh(y) + 12 \cosh^3(y) \sinh(y) + 2x + 24 \cosh^4(y)x - 24 \cosh^2(y)x t), \]

\[ u_2 = -\frac{16}{27} \lambda^2 t (-9 \cosh(y) \sinh(y) + 18 \cosh^3(y) \sinh(y) + 3x + 72 \cosh^4(y)x - 54 \cosh^2(y)x + 336 \lambda \cosh(y) \sinh(y) t x + \cdots, \]

\[ u_3 = \cdots \]

The series form solution is given by;

\[ u(x, t) = \left[ -\frac{4}{3} \cosh^2(y) - \frac{16}{9} \lambda^2 (-3 \cosh(y) \sinh(y) + 12 \cosh^3(y) \sinh(y) + 2x + 24 \cosh^4(y)x - 24 \cosh^2(y)x t) - \frac{16}{27} \lambda^2 t (-9 \cosh(y) \sinh(y) - \frac{16}{27} \lambda^2 (-9 \cosh(y) \sinh(y) + \cdots. \right] \]

The rest of the components of the iteration formulae can be obtained using MAPLE.
Example 3 Consider the following nonlinear PDE:

\[ u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyy} = 0, \]  
\[ u(x, y, 0) = \frac{3}{2} \lambda \sinh \left( \frac{x+y}{\lambda} \right), \]  

To solve the given equation by HAM the linear operator;

\[ L[\varphi(x, y, t; q)] = \frac{\partial}{\partial t} (\varphi(x, y, t; q)), \]  

is chosen with the property;

\[ L[c_1 + t c_2] = 0, \]  

where \( c_1 \) and \( c_2 \) are the integral constants. The inverse operator \( L^{-1} \) is given by;

\[ L^{-1} = \int_0^t (\cdot) dt, \]  

a nonlinear operator can be defined as;

\[ N[\varphi(x, y, t; q)] = \varphi(x, y, t; q)_t + (\varphi(x, y, t; q)^2)_x + 2(\varphi(x, y, t; q)^2)_{xxx} + 2(\varphi(x, y, t; q)^2)_{yyy}, \]  

using the above definition, the zeroth-order deformation equation is constructed;

\[ (1 - q)L[\varphi(x, y, t; q) - u_0(x, y, t)] = q h H(x, y, t) N[\varphi(x, y, t; q)]. \]  

for \( q = 0 \) and \( q = 1, \)

\[ \varphi(x, y, t; 0) = u_0(x, y, t), \quad \varphi(x, y, t; 1) = u(x, y, t). \]
can be written. Thus, we obtain the $m$th—order deformation equation;

$$L[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = hH(x, y, t)R_m(\bar{u}_{m-1}).$$

(46)

with the initial condition $u_m(x, y, 0) = 0$

where

$$R_m(\bar{u}_{m-1}) = h \left[ \left( u_{m-1}(x, y, t) \right)_t + \left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_x \right] + 2\left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_{xxx} + 2\left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_{yyy}.$$  

(47)

After applying Taylor's series the result is;

$$R_m(\bar{u}_{m-1}) = h \left[ \left( u_{m-1}(x, y, t) + f_m \right) + \left( \sum_{r=0}^{m-1} u_r(x, y, t) + f_{r+1} \right) (u_{m-1-r}(x, y, t) + f_{m-r}) \right]_{xxx} + 2\left( \sum_{r=0}^{m-1} u_r(x, y, t) + f_{r+1} \right) (u_{m-1-r}(x, y, t) + f_{m-r})_{yyy}.$$  

(48)

Now the solutions of the $m$th—order deformation equation are;

$$u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + L^{-1}[hH(x, y, t)R_m(\bar{u}_{m-1})], \ m \geq 1,$$

(49)

an initial approximation can be used at the start;

$$u_0(x, t) = \frac{3}{2} \lambda \sinh \left( \frac{1}{6} y \right).$$

(50)

by means of the iteration formula as discuss above, if $h = -1, H = 1$, the other components can be directly obtained as;

$$u_1 = \frac{1}{128} \lambda^3 \left( -8 \sinh \left( \frac{1}{6} y \right) x + 108 \sinh \left( \frac{1}{6} y \right) \cosh^2 \left( \frac{1}{6} y \right) x \right)$$

$$-300 \cosh \left( \frac{1}{6} y \right) + 9x^2 \cos h^3 \left( \frac{1}{6} y \right) + 348 \cos h^3 \left( \frac{1}{6} y \right) - 2 \cosh \left( \frac{1}{6} y \right) x^2 \right)$$

(51)

$$u_2 = \frac{1}{10432} \lambda^3 t \left( 6048 \sinh \left( \frac{1}{6} y \right) x - 11232 \sinh \left( \frac{1}{6} y \right) \cosh^2 \left( \frac{1}{6} y \right) x + 10368 \cosh \left( \frac{1}{6} y \right) - 3888x^2 - 10368 \cosh^3 \left( \frac{1}{6} y \right) + 3600 \cosh \left( \frac{1}{6} y \right) x^2 + \cdots \right),$$

$$\cdots$$

The series form solution is given by;

$$u(x, t) = \frac{3}{2} \lambda \sinh \left( \frac{1}{6} y \right) - \frac{1}{128} \lambda^3 \left( -8 \sinh \left( \frac{1}{6} y \right) x + 108 \sinh \left( \frac{1}{6} y \right) \cosh^2 \left( \frac{1}{6} y \right) - 300 \cosh \left( \frac{1}{6} y \right) + 9x^2 \cos h^3 \left( \frac{1}{6} y \right) + 348 \cosh^3 \left( \frac{1}{6} y \right) - 2 \cosh \left( \frac{1}{6} y \right) x^2 \right) + \frac{1}{10432} \lambda^3 t \left( 6048 \sinh \left( \frac{1}{6} y \right) x - \cdots \right).$$

(53)
The rest of the components of the iteration formulae can be obtained using MAPLE.

Figure 3 Graphical representation of exact and approximate solutions of Eq. (39) for different values of $x$ and $t$.

Example 4 Consider the following nonlinear PDE;

$$u_t + (u^3)_x + \frac{1}{6}(u^3)_{xxx} + \frac{1}{8}(u^3)_{yxx} = 0,$$

(54a)

$$u(x, y, 0) = \frac{3}{2} \lambda \cosh \left(\frac{1}{6}(x + y)\right).$$

(54b)

To solve the given Equation by HAM the linear operator;

$$L[\varphi(x, y, t; q)] = \frac{\partial}{\partial t} (\varphi(x, y, t; q)),$$

(55)

is chosen with the property;

$$L[c_1 + tc_2] = 0,$$

(56)

where $c_1$ and $c_2$ are the integral constants. The inverse operator $L^{-1}$ is given by;

$$L^{-1} = \int_0^t (\cdot) dt,$$

(57)

a nonlinear operator can be defined as;

$$N[\varphi(x, y, t; q)] = \varphi(x, y, t; q)_t + (\varphi(x, y, t; q)^2)_x + \frac{1}{6}(\varphi(x, y, t; q)^2)_{xxx} + \frac{1}{8}(\varphi(x, y, t; q)^2)_{yxx},$$

(58)

using the above definition, the zeroth-order deformation equation can be constructed;

$$(1 - q)L[\varphi(x, y, t; q) - u_0(x, y, t)] = qhH(x, y, t) N[\varphi(x, y, t; q)].$$

(59)

for $q = 0$ and $q = 1$,
\[ \varphi(x, y, t; 0) = u_0(x, y, t), \quad \varphi(x, y, t; 1) = u(x, y, t). \quad (60) \]

can be written. Thus, we obtain the \( m \)th-order deformation equation;

\[ L[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = hH(x, y, t)R_m(\vec{u}_{m-1}). \quad (61) \]

with the initial condition \( u_m(x, y, 0) = 0 \)

where

\[ R_m(\vec{u}_{m-1}) = \left[ \left(u_{m-1}(x, y, t) \right)_t + \left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_x \right] + \frac{1}{8} \left( \sum_{r=0}^{m-1} u_r(x, y, t) u_{m-1-r}(x, y, t) \right)_{xxx}, \quad (62) \]

After applying Taylor’s series the result is;

\[ R_m(\vec{u}_{m-1}) = \left[ \left(u_{m-1}(x, y, t) + f_m \right)_t + \left( \sum_{r=0}^{m-1} u_r(x, y, t) + f_{r+1} \right) \left( u_{m-1-r}(x, y, t) + f_{m-r} \right)_x \right] + \frac{1}{8} \left( \sum_{r=0}^{m-1} u_r(x, y, t) + f_{r+1} \right) \left( u_{m-1-r}(x, y, t) + f_{m-r} \right)_{xxx}, \quad (63) \]

Now the solutions of the \( m \)th-order deformation equation are;

\[ u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + L^{-1}[hH(x, y, t)R_m(\vec{u}_{m-1})], \quad m \geq 1, \quad (64) \]

an initial approximation can be used at the start;

\[ u_0(x, t) = \frac{3}{2} \lambda \cosh \left( \frac{1}{2} y \right), \quad (65) \]

by means of the iteration formula as discuss above, if \( h = -1, H = 1 \), the other components can be directly obtained can be directly obtained as;

\[ u_1 = -\frac{1}{2048} \lambda^3 \left( 3588 \sinh \left( \frac{1}{6} y \right) \cosh^2 \left( \frac{1}{2} y \right) - 97 \sinh \left( \frac{1}{2} y \right) x^2 - 48 \sinh \left( \frac{1}{3} y \right) + 99x^2 \sinh \left( \frac{1}{6} y \right) \cos h \left( \frac{1}{2} y \right) - 1180 \cos h \left( \frac{1}{2} y \right) x + 1188 \cosh \left( \frac{1}{6} y \right) x \right), \quad (66) \]

\[ u_2 = \frac{1}{4718992} \lambda^3 \left( -165888 \sinh \left( \frac{1}{6} y \right) \cosh^2 \left( \frac{1}{2} y \right) + 4608 \sinh \left( \frac{1}{3} y \right) x^2 - \cdots, \quad (67) \]

The series form solution is given by;
\[ u(x, t) = \frac{3}{2} \lambda\cosh\left(\frac{1}{6}y\right) - \frac{1}{2048} \lambda^3 \left(3588 \sinh\left(\frac{1}{6}y\right) \cosh^2\left(\frac{1}{6}y\right) - 97 \sinh\left(\frac{1}{6}y\right) x^2 - 48 \sinh\left(\frac{1}{6}y\right) + 99x^2 \sinh\left(\frac{1}{6}y\right) \cosh^2\left(\frac{1}{6}y\right) - 1180 \cos h\left(\frac{1}{6}y\right) x + 1188 \cosh^3\left(\frac{1}{6}y\right) x \right) + \frac{1}{4718592} \lambda^3 t (-165888 \sinh\left(\frac{1}{6}y\right) \cosh^2\left(\frac{1}{6}y\right) + \cdots, \] (68)

The rest of the components of the iteration formulae can be obtained using MAPLE.

![Graphical representation of exact and approximate solutions of Eq. (4) for different values of x and t.](image)

**Figure 4** Graphical representation of exact and approximate solutions of Eq. (4) for different values of \(x\) and \(t\).

**Conclusions**

MHAM is applied to find appropriate solutions of nonlinear partial differential equations (PDEs). The proposed modified version is fully capable to cope with the nonlinearity of the physical problems. The suggested technique can be a nice addition in the existing techniques of solving nonlinear problems of a versatile physical nature.

**References**


